IMPRIMITIVE IRREDUCIBLE REPRESENTATIONS OF FINITE QUASISIMPLE GROUPS

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Group Theory and Computational Methods
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   Equivalently, $V \cong \mathrm{Ind}_H^G(V_1) := kG \otimes_{kH} V_1$ *as* kG*-modules, where* $H := \mathrm{Stab}_G(V_1)$.

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In Cases 2–5, the group X is the stabilizer of a non-degenerate form (symplectic, quadratic or hermitian) on V.

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Kleidman-Liebeck and Bray-Holt-Roney-Dougal: Determine the maximal subgroups among the members of $\cup_{i=1}^{8} \mathcal{C}_i(X)$ (amot).

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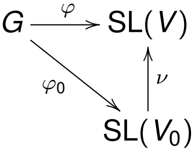

for some proper subfield $k_0 \lneq k$, a k_0-vector space V_0 with
$V = k \otimes_{k_0} V_0$, and a representation $\varphi_0 : G \to \mathrm{SL}(V_0)$.]

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Moreover, H/Z is almost simple, i.e. there is a nonabelian simple group S such that H/Z fits into a short exact sequence

$$1 \to S \to H/Z \to \mathrm{Aut}(S) \to 1$$

On the maximality of the elements of $\mathcal{S}(X)$

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By the definiton of the classes $\mathcal{C}_i(X)$ and $\mathcal{S}(X)$, we have

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In this talk we investigate the possibility $K \in \mathcal{C}_2(X)$, which we call a *$\mathcal{C}_2$-obstruction* to the maximality of H.

The subgroup class $\mathcal{C}_2(X)'$

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The group H belongs to $\mathcal{C}_2(X)$, if H is the **full** stabilizer of a decomposition (1) satisfying (a)–(c).

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AN EXAMPLE: THE MATHIEU GROUP M_{11}

Let X be a finite classical group.

Let $\varphi : M_{11} \to X$ be absolutely irreducible, faithful, and not realizable over a smaller field. (All such (φ, X) are known.)

Put $G := \varphi(M_{11})$. Then $N_X(G) = Z(X) \times G$.

Is $Z(X) \times G$ maximal in X?

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What about $\varphi : M \to \Omega^-_{196882}(2)$? ($M$: Monster)

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The case $\text{char}(\mathbf{k}) = 0$ is included as a model for the desired classification; it has provided most of the ideas for an approach to the general case.

The finite quasisimple groups

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A finite quasisimple group is one of

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4. *a quotient of a quasisimple finite reductive group.*

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Let **G** denote a reductive algebraic group over $\mathbb{F}$, the algebraic closure of the prime field $\mathbb{F}_p$.

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Such a pair $(\mathbf{L}, \mathbf{P})$ gives rise to a parabolic subgroup $P = \mathbf{P}^F$ of G with Levi complement $L = \mathbf{L}^F$.

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- G *is a finite quasisimple group,*
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are known if $char(\mathbf{k}) = 0$.

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Thus it remains to study finite reductive groups in non-defining characteristics.

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It remains to consider alternating groups or finite reductive groups in case $p \neq \mathrm{char}(\mathbf{k}) > 0$.

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Then $H = P$ is a parabolic subgroup of G.

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This allows to apply Harish-Chandra theory to our classification problem, reducing certain aspects to Weyl groups or Iwahori-Hecke algebras.

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Theorem (H.-Magaard, '16+)

Let $\chi \in \mathcal{E}(G,[s])$ *and* $\lambda \in \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F,[1])$ *with* $[\chi] \mapsto [\lambda]$.

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Let $\chi \in \mathcal{E}(G,[s])$ *and* $\lambda \in \mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^F,[1])$ *with* $[\chi] \mapsto [\lambda]$.
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Thank you for listening!