Commutativity preserving extensions of groups

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CP extensions

An **extension of a group N by a group Q** is an exact sequence $e = (\chi, G, \pi)$ of groups

$$1 \longrightarrow N \xrightarrow{\chi} G \xrightarrow{\pi} Q \longrightarrow 1 \,.$$

We (almost) always assume that N is abelian, hence a Q-module.

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Commutativity preserving extensions

The extension $e = (\chi, G, \pi)$ of N by Q is **commutativity preserving** (**CP**) if every commuting pair of elements of Q has a commuting lift in G.

Example

Let Q be a group in which for every commuting pair x, y the subgroup $\langle x, y \rangle$ is cyclic. Then every commuting pair of elements in Q has a commuting lift. Thus every extension of Q is CP.

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The above condition is equivalent to Q having all abelian subgroups cyclic.

- In the case of finite groups, it is known that such groups are precisely the groups with periodic cohomology, and this further amounts to Q having cyclic Sylow p-subgroups for p odd, and cyclic or quaternion Sylow p-subgroups for $p = 2$.
- Infinite groups with this property include free products of cyclic groups.

Special case

Example

Taking the simplest case $Q = C_p$ in the previous example, we see that every extension of a group by C_p is CP. Thus in particular, every finite p-group can be viewed as being composed from a sequence of CP extensions.

The same argument shows that every polycyclic group can be obtained by a sequence of CP extensions.

Example of an extension which is not CP

Example

Take

$$Q = \langle x_1 \rangle \times \langle x_2 \rangle \cong C_p \times C_p,$$
$$N = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle \cong C_p \times C_p \times C_p.$$

Let Q act on N via the following rules:

$$a_1^{x_1} = a_1,\ a_2^{x_1} = a_2,\ a_3^{x_1} = a_3,\ a_1^{x_2} = a_2,\ a_2^{x_2} = a_1,\ a_3^{x_2} = a_3.$$

Thus N is a Q-module. Now construct an extension G corresponding to this action by specifying

$$x_2^{x_1} = x_2 a_3.$$

This extension is not CP because the commuting pair x_1, x_2 in Q does not have a commuting lift in G.

CP extensions and equivalence

Extensions (χ, G, π) and $(\bar{\chi}, \bar{G}, \bar{\pi})$ of N by Q are **equivalent** if there exists a homomorphism $\beta : G \to \bar{G}$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\chi} & G & \xrightarrow{\pi} & Q & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle 1} & & \\
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0 & \longrightarrow & N & \xrightarrow{\bar{\chi}} & \bar{G} & \xrightarrow{\bar{\pi}} & Q & \longrightarrow & 1
\end{array}$$

Lemma

The class of CP extensions is closed under equivalence of extensions.

Cohomology

Let N be a Q-module. A map $\omega : Q \times Q \to N$ is a **2-cocycle** if

$$x\omega(y,z) + \omega(x,yz) = \omega(xy,z) + \omega(x,y) \quad \forall x,y,z \in Q.$$

Let $Z^2(Q,N)$ denote the set of all such 2-cocycles.

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A map $\omega : Q \times Q \to N$ is a **2-coboundary** if there exists a function $\varphi : Q \to N$ such that

$$\omega(x,y) = x\varphi(y) - \varphi(xy) + \varphi(x) \quad \forall x,y \in Q.$$

Let $\mathrm{B}^2(Q,N)$ denote the set of all such 2-coboundaries. Put

$$\mathrm{H}^2(Q,N) = \mathrm{Z}^2(Q,N)/\,\mathrm{B}^2(Q,N).$$

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Equivalence classes of extensions with abelian kernel are determined by the elements of $\mathrm{H}^2(Q,N)$.

CP cocycles

Definition

A cocycle $\omega \in \mathrm{Z}^2(Q, N)$ is said to be a **CP cocycle** if for all commuting pairs $x_1, x_2 \in Q$ there exist $a_1, a_2 \in N$ such that

$$\omega(x_1, x_2) - \omega(x_2, x_1) = (x_1 - 1)a_1 + (x_2 - 1)a_2$$

Denote by $\mathrm{Z}^2_{\mathrm{CP}}(Q, N)$ the set of all CP cocycles in $\mathrm{Z}^2(Q, N)$.

Define

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Example

Let Q be an abelian group and N a trivial Q-module. Then $\mathrm{H}^2_{\mathrm{CP}}(Q, N)$ coincides with $\mathrm{Ext}(Q, N)$.

Equivalence classes of CP extensions

Proposition

Let N be a Q-module. Then the equivalence classes of CP extensions of N by Q are in bijective correspondence with the elements of $\mathsf{H}^2_{\mathrm{CP}}(Q, N)$.

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Example

Let Q be a group with all abelian subgroups cyclic and N a Q-module. Then

$$\mathsf{H}^2_{\mathrm{CP}}(Q, N) = \mathsf{H}^2(Q, N).$$

Schur multiplier and B_0

Given a group G, let $K(G)$ denote the *set* of all commutators in G.

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By Hopf's formula, $M(G) \cong H_2(G, \mathbb{Z})$; this is the **Schur multiplier** of G.

Universal Coefficient Theorem

Theorem (Universal Coefficient Theorem)

Let N be a trivial Q-module. Then there is a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(Q^{\mathrm{ab}}, N) \xrightarrow{\psi} \mathrm{H}^2(Q, N) \xrightarrow{\varphi} \mathrm{Hom}(\mathrm{M}(Q), N) \longrightarrow 0\ .$$

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where the maps ψ and $\tilde{\varphi}$ are induced by the Universal Coefficient Theorem.

Special case: $N = \mathbb{Q}/\mathbb{Z}$

Let $\mathbb{Q}/\mathbb{Z}$ be trivial Q-module. As

$$\mathrm{Ext}(Q^{\mathrm{ab}}, \mathbb{Q}/\mathbb{Z}) = 0,$$

the "Universal Coefficient Theorem" implies

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The LHS group is easily seen to be isomorphic to

$$\mathrm{Bog}(Q) = \bigcap_{\substack{A \leq Q, \\ A \text{ abelian}}} \ker \mathrm{res}^Q_A,$$

where $\mathrm{res}^Q_A : \mathrm{H}^2(Q, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(A, \mathbb{Q}/\mathbb{Z})$ is the usual cohomological restriction map.

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If Q is finite, then $\mathrm{Bog}(Q)$ is isomorphic to the so-called ***unramified Brauer group*** *of $\mathbb{C}(V)^Q$ over $\mathbb{C}$, where V is a faithful finite dimensional complex representation of Q.*

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If Q is finite, then $\mathrm{Bog}(Q) \cong \mathrm{B}_0(Q)$.

Noether's problem

Let Q be a finite group and V a faithful finite dimensional representation of Q over $\mathbb{C}$.

Problem (Emmy Noether, 1916)

When is $\mathbb{C}(V)^Q/\mathbb{C}$ *is (stably) rational?*

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By Artin and Mumford (1972), the group $\mathrm{Bog}(Q)$, and hence also $\mathrm{B}_0(Q)$, is an obstruction to Noether's problem.

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- *If* G *is a* p*-group of order* $\leq p^4$*, then* $\mathrm{B}_0(G) = 0$*.*
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Theorem (Kunyavskiĭ, 2010)

If G is a finite simple group, then $\mathrm{B}_0(G) = 0$.

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If G is a finite simple group, then $\mathrm{B}_0(G) = 0$.

Theorem (Hoshi, Kang, Kunyavskiĭ (2012), M (2012))

Let $|G| = p^5$. Then

$$\mathrm{B}_0(G) \neq 0 \iff G \text{ is of maximal class.}$$

Further applications of B_0

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- Jaikin-Zapirain, Jezernik, Rodriguez (2016): Used groups with non-trivial Bogomolov multiplier to produce further counterexamples to the **Fake Degree Conjecture** (on character degrees of the so-called algebra groups, i.e., groups of the form $G = 1 + J$, where J is a finite dimensional nilpotent algebra over a finite field).

Calculations of B_0

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Ellis (2013). Algorithm for computing $\mathrm{B}_0(G)$, where G is an arbitrary finite group. Part of `HAP`. `BogomolovMultiplier(G)`

Isoclinism of central extensions

Let

$$e_1: \quad 1 \longrightarrow N_1 \xrightarrow{\chi_1} G_1 \xrightarrow{\pi_1} Q_1 \longrightarrow 1$$

and

$$e_2: \quad 1 \longrightarrow N_2 \xrightarrow{\chi_2} G_2 \xrightarrow{\pi_2} Q_2 \longrightarrow 1$$

be central extensions. We say that e_1 and e_2 are **isoclinic**, if there exist isomorphisms $\eta : Q_1 \to Q_2$ and $\xi : G_1' \to G_2'$ such that the diagram

$$\begin{array}{ccc} Q_1 \times Q_1 & \xrightarrow{c_1} & G_1' \\ \Big\downarrow{\scriptstyle \eta\times\eta} & & \Big\downarrow{\scriptstyle \xi} \\ Q_2 \times Q_2 & \xrightarrow{c_2} & G_2' \end{array}$$

commutes, where the maps c_i, $i = 1, 2$, are defined by the rules $c_i(\pi_i(x), \pi_i(y)) = [x, y]$.

Isoclinism and central CP extensions

Proposition

Let e_1 and e_2 be isoclinic central extensions. If e_1 is a CP extension, then so is e_2.

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Let e_1 and e_2 be isoclinic central extensions. If e_1 is a CP extension, then so is e_2.

Theorem

The isoclinism classes of central CP extensions with factor group isomorphic to Q correspond to the orbits of the action of $\operatorname{Aut} Q$ on the subgroups of $\mathrm{B}_0(Q)$ given by

$$(\varphi, U) \mapsto \mathrm{B}_0(\varphi)U,$$

where $\varphi \in \operatorname{Aut} Q$ and $U \leq \mathrm{B}_0(Q)$.

Maximal extensions

A central extension

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is termed to be **stem** whenever $\chi(N) \leq [G, G]$.

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Proposition

Every central CP extension is isoclinic to a stem central CP extension.

Definition

Given a group Q, any stem central CP extension of a group N by Q with $|N| = |\operatorname{B}_0(Q)|$ is called a **CP cover** of Q.

CP covers and maximality

Theorem

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2. *Let G be a CP cover of Q with kernel N. Then $N \cong \mathrm{B}_0(Q)$.*

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3. *CP covers of Q are precisely the stem central CP extensions of Q with kernel of maximal order.*

Existence of CP covers

Theorem

Let Q be a finite group given via a free presentation $Q = F/R$. Set $H = F/\langle \mathrm{K}(F) \cap R \rangle$ and $A = R/\langle K(F) \cap R \rangle$.

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2. *Let C be a complement to $T(A)$ in A. Then H/C is a CP cover of Q.*
3. *CP covers of Q are represented by the cocycles $\tilde{\varphi}^{-1}(1_{\mathrm{B}_0(Q)})$ in $\mathrm{H}^2(Q, \mathrm{B}_0(Q))$, where $\tilde{\varphi}$ is the mapping induced by the Universal Coefficient Theorem.*

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Corollary

The number of CP covers of a group Q is at most $|\operatorname{Ext}(Q^{\mathrm{ab}}, \mathsf{B}_0(Q))|$. In particular, perfect groups have a unique CP cover.

Some properties of CP covers

Theorem

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Let p be an arbitrary prime. The Schur cover of $C_p \times C_p$ is isomorphic to the unitriangular group $\mathrm{UT}_3(p)$. The Schur multiplier of the latter is $C_p \times C_p$.

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CP covers of isoclinic groups are isoclinic.

Example

Let p be an arbitrary prime. The Schur cover of C_{p^2} is C_{p^2}, and the Schur cover of $C_p \times C_p$ is isomorphic to $\mathrm{UT}_3(p)$. The two covers are not isoclinic.

Minimal CP extensions

Central CP extensions of a cyclic group of prime order by some given group Q are called **minimal** CP extensions.

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the action of Q on $\mathbb{F}_p$ being trivial.

Theorem

The group $\mathsf{H}^2_{\mathrm{CP}}(Q)$ *is elementary abelian of rank* $\mathsf{d}(Q) + \mathsf{d}(\mathsf{B}_0(Q))$.

Applications

Corollary

Let $Q = F/R$ *be a presentation with* $\mathrm{d}(Q) = \mathrm{d}(F)$. *Let* $\mathrm{r}(F, R)$ *be the minimal number of relators in* R *that generate* R *as a normal subgroup of* F*, and let* $\mathrm{r}_{\mathrm{K}}(F, R)$ *be the number of relators among these that belong to* $\mathrm{K}(F)$. *Then*

$$\mathrm{d}(\mathrm{B}_0(Q)) \leq \mathrm{r}(F, R) - \mathrm{r}_{\mathrm{K}}(F, R) - \mathrm{d}(Q).$$

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The corollary may be applied to show that the Bogomolov multiplier of a group is trivial. This works with classes of groups which may be given by a presentation with many simple commutators among relators.

Example: Unitriangular groups

The group of unitriangular matrices $\mathrm{UT}_n(p)$ has a minimal presentation with $n-1$ generators of order p, and all other relators are commutators (Biss, Dasgupta, 2001). Hence

$$\mathrm{B}_0(\mathrm{UT}_n(p)) = 0.$$

The same holds for lower central quotients of $\mathrm{UT}_n(p)$.

This was also proved by Michailov (2013) using different means. It is also implicit in the work of Fried and Völklein (1991) on the inverse Galois problem.

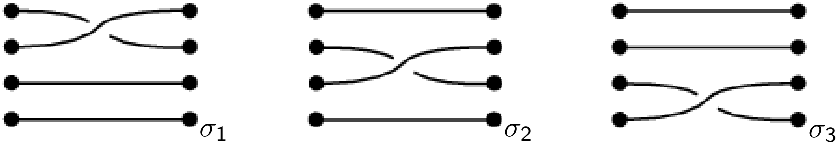
Example: Braid groups

The braid group B_n has a minimal presentation

$$B_n = \langle \sigma_1, \sigma_2, \dots \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \rangle,$$

where $i = 1, 2, \dots, n-2$ and $|i - j| \geq 2$.

Example (Generators of B_4)



This is a presentation with $n-1$ generators and $n-2$ braid relators that are not commutators, so $\mathrm{B}_0(B_n) = 0$.

Commuting probability

The **commuting probability** $\mathrm{cp}(G)$ of a finite group G is defined to be the probability that two randomly chosen elements of G commute, and is equal to

$$\mathrm{cp}(G) = \frac{|\{(x, y) \in G \times G \mid [x, y] = 1\}|}{|G|^2}.$$

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- Gustafson (1973). If $\mathrm{cp}(G) > 5/8$, then G is abelian.

Commuting probability and B_0

Theorem (Jezernik, M, 2013)

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Proposition

A central extension $0 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ is a CP extension if and only if $\mathrm{cp}(G) = \mathrm{cp}(Q)$.

Corollary

For every number p in the range of the commuting probability function, there exists a group G with $\mathrm{cp}(G) = p$ and $\mathrm{B}_0(G) = 0$.

Bounds

Theorem

Let $\epsilon > 0$, and let Q be a group with $\mathrm{cp}(Q) > \epsilon$.

- $|\mathrm{B}_0(Q)|$ *can be bounded in terms of a function of ϵ and*

$$\max\{\mathrm{d}(S) \mid S \text{ a Sylow subgroup of } Q\}.$$

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Corollary

Given $\epsilon > 0$, *there exists a constant* $C = C(\epsilon)$ *such that for every group* Q *with* $\mathrm{cp}(Q) > \epsilon$, *we have*

$$\exp M(Q) \leq C \cdot \exp Q.$$