Rationality and Morita equivalence for blocks of finite groups

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The σ-twist of A, is the k-algebra $A^{(\sigma)}$ such that $A^{(\sigma)} = A$ as a ring and with scalar multiplication defined by

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Let k_0 be a subfield of A. We say that A is defined over k_0 if there exists a k_0-algebra A_0 such that $A = k \otimes_{k_0} A_0$, i.e. if there exists a k-basis $\mathcal{B}$ of A containing 1_A such that for all $x, y \in \mathcal{B}$,

$$xy = \sum_{z \in \mathcal{B}} \alpha_{x,y,z} z \quad \text{with} \quad \alpha_{x,y,z} \in k_0.$$

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The Frobenius number, $f(A)$ of A is the least possible d such that $A^{(\sigma^d)} \cong A$ as k-algebras.

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Let $\zeta \in k^\times$ and let $A = k\langle x, y\rangle / \langle x^p, y^p, xy - \zeta yx\rangle$.

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- If $B = A \times \mathrm{Mat}_2(A^{(\sigma)})$, then $f(B) = 2$ and $mf(B) = 1$.

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- If $1_A \in \mathbb{F}_p G$, then $\tau(A) = A$ and hence $f(A) = 1$.
- If A has an ordinary irreducible character which is rational-valued, then $f(A) = 1$.

Example

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Let $p = 7$. Then kG has 3 blocks, with corresponding central idempotents:

$$1_{A_1} = \frac{1}{5}(1 + y + y^2 + y^3 + y^4)$$

$$1_{A_2} = \frac{1}{5}(2 + \theta y + \theta' y^2 + \theta' y^3 + \theta y^4)$$

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- $1_{A_1} \in \mathbb{F}_7 G$, hence by previous slide $f(A) = 1$.
- $\tau(A_2) = A_3$ but by using block theory can show

$$f(A_2) = f(A_3) = 1$$

Defect groups and Finiteness Conjectures

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- Defect groups form a conjugacy class of p-subgroups of G.
- If A is the principal block of kG, i,e. the block containing the 1-dimensional trivial kG-module, then the defect groups of A are the Sylow p-subgroups of G.

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- The largest elementary divisor of the Cartan matrix of A is $|P|$ (Brauer 1954).
- Hochschild cohomology of A is bounded by P: For any $n \in \mathbb{N}$, there exists an integer $h(P, n)$, depending only on P and n such that $\dim_k(HH^n(A)) \leq h(P, n)$ and there are only finitely many possibilities for the corresponding Hilbert series $\sum_{n=0}^{\infty} \dim_k(HH^n(A))t^n \in \mathbb{Z}[[t]]$ (K.-Linckelmann 2012).

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(a) *Donovan's conjecture holds for P-blocks.*

(b) *There exist integers $c(P)$ and $r(P)$ depending only on P such that for any P-block A,*
   - *the Cartan numbers of A are at most $c(P)$ and*
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The assertion that $c(P)$ exists is the "Weak Donovan Conjecture", and the assertion that $r(P)$ exists is the "Rationality Conjecture".

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- $p = 2$, $P = C_{2^m} \times C_{2^m}$, $m \geq 2$, then A is Morita equivalent to one of kP or $kP \rtimes C_3$. Consequently, $mf(A) = 1$ (Eaton-Hethelyi-K-Külshammer (2013))

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- $p = 2$ and $P = C_2 \times C_2 \times C_2$, then there are eight Morita equivalence classes of blocks with representatives the principal blocks of:
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  2. $kP \rtimes C_3$
  3. $kP \rtimes C_7$
  4. $k(P \rtimes (C_7 \rtimes C_3)$
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The above results rely on the classification of finite simple groups and the parametrisation of p-blocks of finite reductive groups through Lusztig's theory of irreducible characters.

Theorem (Farrell 2015)

Let G be a quasi-simple group, $\bar{G} = G/Z(G)$ and A a block of kG. Suppose that one of the following holds.

- *$\bar{G}$ is an alternating or sporadic group.*
- *$\bar{G}$ is a finite group of Lie type in characteristic p.*
- *$\bar{G}$ is $PSL_n(q)$ or $PSU_n(q)$.*
- *$\bar{G}$ is a finite group of Lie type in non-describing characteristic, not of type E_8, and A is a unipotent block.*

Then $mf(A) = 1$.

Theorem (Benson-K. 2007)

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- No known examples of blocks A with $mf(A) > 2$.
- No known examples of blocks A of quasi-simple groups with $mf(A) > 1$.
- Rationality Conjecture is open.

Brauer's first main theorem

Let G be a finite group and $P \leq G$ be a p-subgroup of G. There is a canonical bijection (Brauer Correpondence) between the set of blocks of kG with defect group P and the set of blocks of $kN_G(P)$ with defect group P.

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There are many known and conjectural connections between the module categories of Brauer correspondent blocks. These blocks are not in general Morita equivalent. However, if the relevant defect groups are abelian, then it is predicted that there is a weaker equivalence between their module categories.

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Abelian Defect Group Conjecture (Broué, 1990)

Let A be a block of kG with defect group P and B the block of $kN_G(P)$ in Brauer correspondence with A. If P is abelian, then there is an equivalence of derived bounded module categories $D^b(A\text{-mod}) \sim D^b(B\text{-mod})$.

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Chuang and Rouquier have identified a special type of derived equivalence called perverse equivalence which seems to be key for the Abelian defect group conjecture. A perverse equivalence is stronger than a derived equivalence, but weaker than a Morita equivalence, i.e. we have:

$$\text{Morita Equiv.} \implies \text{Perverse Equiv.} \implies \text{Derived Equiv.}$$

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Most derived equivalences which have been found in the context of the Abelian defect group conjecture are known to be compositions of perverse equivalences and it is believed that when the defect groups are abelian there always exists a derived equivalence between a block and its Brauer correspondent which is a composition of perverse equivalences.

Theorem (Chuang-K. 2016)

Let A be a block of kG with defect group P and let B be the block of $kN_G(P)$ in Brauer correspondence with A. Suppose that there exists a derived equivalence between A and B which is a composition of perverse equivalences. Then,

$$mf(A) \leq |P||\mathrm{Aut(P)}|^2.$$

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Corollary

An affirmative answer to the perverse form of the Abelian defect group conjecture would imply the rationality conjecture for abelian P.