Group embeddings of partial Latin squares

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joint work with **Ian Wanless** (Monash University)
based on work of Ian with **Bridget Webb** (The Open University)

Latin squares

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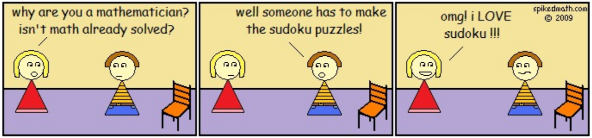

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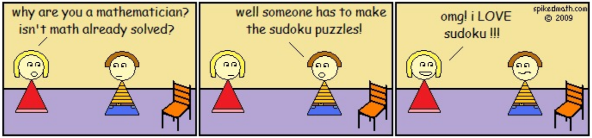

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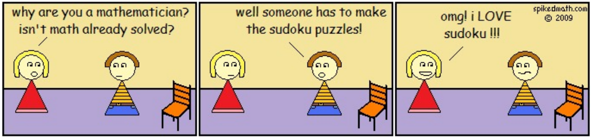

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- the Cayley table of a finite group is a LS

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Some (open) problems

- **Which PLS can be completed to a LS?**
  Andersen & Hilton (1983): $n \times n$ PLS is completable if size $\leq n-1$;
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- **Which PLS can be completed to a subtable of a Cayley table?**

Group embeddings

The PLS $\begin{bmatrix} 0 & \cdot & \cdot & 3 \\ \cdot & 2 & \cdot & 0 \\ 2 & 3 & 1 & \cdot \end{bmatrix}$ embeds in the Cayley table of $(\mathbb{Z}_4, +)$:

+	0	1	2	3
0	**0**	1	2	**3**
1	1	**2**	3	**0**
2	**2**	**3**	**1**	1
3	3	0	1	2

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Group embedding of PLS

An **embedding** of a PLS in a group G is a triple (ρ, κ, σ) of injective maps from respectively the rows, columns, and symbols, to G, such that for any symbol s in row r and column c, we have $\rho(r)\kappa(c) = \sigma(s)$.

Smallest PLS not embedding in group of order n

We'll discuss solutions to the following questions

1. **Dénes & Keedwell asked ("Open Problem 3.8", 1974):** given n, what is the largest number $m = \psi(n)$ such that *every* PLS of size m can be embedded in some group of order n.

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Clearly, $\psi_\circ(n) \leq \psi_+(n) \leq \psi(n)$.

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Note:

- (1), (2), and (3) are based on work of Wanless & Webb
- (4) is joint work of D. & Wanless

Finite Case

The solution to (1)–(3)

Theorem (Wanless & Webb 2015)

If n is a positive integer, then

$$\psi(n) = \begin{cases} 1 & (n = 1, 2) \\ 2 & (n = 3) \\ 3 & (n = 4,\ \text{or } n > 3 \text{ odd}) \\ 5 & (n = 6,\ \text{or } n \equiv 2, 4 \bmod 6 \text{ and } n > 4) \\ 6 & (n \equiv 0 \bmod 6 \text{ and } n > 6), \end{cases}$$

and

$$\psi_+(n) = \begin{cases} 1 & (n = 1, 2) \\ 2 & (n = 3) \\ 3 & (n = 4 \text{ or } n > 3 \text{ odd}) \\ 5 & (n > 4 \text{ even}), \end{cases}$$

and $\psi_\circ(n) = \psi_+(n)$.

Quadrangle Criterion

Quadrangle Criterion (QC)

A 4-tuple of entries of a matrix M is a **quadrangle** if the four elements are the corners of a rectangular block in M. The matrix M satisfies the **QC** if the any 3 entries of a quadrangle determine the 4th.

$$M = \begin{bmatrix} a & b \\ d & c \end{bmatrix}$$

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For distinct group elements i, j let $m_{i,j} = ij$ be the element in the Cayley table with row j and column j. Consider a quadrangle defined by rows i, k and columns j, l, that is, $m_{i,j} = ij$, $m_{i,l} = il$, $m_{k,l} = kl$, $m_{k,j} = kj$.

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$$m_{k,j} = kj = m_{k,l} l^{-1} i^{-1} m_{i,j} = m_{k,l} m_{i,l}^{-1} m_{i,j}.$$

An upper bound

Consider the PLS $P = \begin{bmatrix} a & b & \cdot \\ c & a & b \\ \cdot & c & d \end{bmatrix}$.

There are two quadrangles (a, b, a, c) and (a, b, d, c) which coincide in exactly 3 positions,

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This shows:

Lemma

For every n, we have $\psi(n) \leq 6$.

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For each $\ell \geq 2$ there is a PLS of size 2ℓ that can only be embedded in groups of order divisible by ℓ.

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Suppose the PLS

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is embedded in a group G; let the rows and columns of that embedding be labelled with group elements r_1, r_2 and $c_1, \ldots, c_\ell$, respectively.

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	c_1	c_2	$\cdots$	$c_{\ell-1}$	c_ℓ
r_1	a_1	a_2	$\cdots$	$a_{\ell-1}$	a_ℓ
r_2	a_2	a_3	$\cdots$	a_ℓ	a_1

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	1	a_2	$\cdots$	$a_{\ell-1}$	a_ℓ
1	1	a_2	$\cdots$	$a_{\ell-1}$	a_ℓ
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	1	r	$\cdots$	$r^{\ell-2}$	$r^{\ell-1}$
1	1	r	$\cdots$	$r^{\ell-2}$	$r^{\ell-1}$
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and $r \in G$ has order ℓ, which implies $\ell \mid |G|$.

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Clearly, P can be embedded in any group which has an element of order ℓ.

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Lemma

Upper bounds for ψ are given as follows:

- $\psi(1) = \psi(2) = 1,\quad \psi(3) = 2,\quad \psi(4) = 3,$ and $\psi(6) = 5$ (*direct comp.*)
- $\psi(n) = 3$ for $n \geq 5$ odd (*by previous lemma*)
- $\psi(n) \leq 5$ for $n > 6$ with $n \equiv 2, 4 \bmod 6$ (*by previous lemma*)
- $\psi(n) \leq 6$ for $n > 6$ with $n \equiv 0 \bmod 6$ (*QC showed* $\psi(n) \leq 6$ *for all* n)

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Good news: there are only finitely many PLS to consider!

The list of candidates

Number of species (Wanless 2007)

A **species** is an orbit of PLS (of order n) under the action of $\mathrm{Sym}(n) \wr \mathrm{Sym}(3)$, acting naturally on the PLS represented as sets of triples (r, c, s):

Two LS P and P' are **isotopic** if they are in the same $\mathrm{Sym}(n) \times \mathrm{Sym}(n) \times \mathrm{Sym}(n)$-orbit; they are **conjugate** if they are in the same $\mathrm{Sym}(3)$-orbit.

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Known: The 6 conjugates of a Cayley table are all isotopic; this implies:
If P and P' are PLS of the same species, then P can be embedded in a group if and only if P' can.

Reducing the list of candidates

Most PLS don't need to be considered because they contain entries which may be omitted without affecting embeddability, for example, if

$$P = \begin{bmatrix} a & \cdot & \cdot \\ \cdot & a & b \\ b & c & \cdot \end{bmatrix} \quad \text{and} \quad P' = \begin{bmatrix} a & \cdot & \cdot \\ \cdot & a & b \\ b & \cdot & \cdot \end{bmatrix}$$

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Reduced number of species

Getting rid of "obvious" candidates which do / do not embedd in groups of size $n \geq 6$, the following numbers of species remain:

size	1	2	3	4	5	6	7
# species	1	2	5	18	59	306	1861
# "reduced"	0	0	0	2	0	11	50

A lower bound

Recall: it remains to show that **for all even $n > 6$** we have

$$\psi(n) \geq 5 \text{ if } n \equiv 2, 4 \bmod 6,$$
$$\psi(n) \geq 6 \text{ if } n \equiv 0 \bmod 6;$$

in other words, that every PLS of size ≤ 5 can be embedded in some group of order n for all $n > 6$ with $n \equiv 2, 4 \bmod 6$; analogously for $n \equiv 0 \bmod 6$.

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Proof of lower bound

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- every PLS of size $s \leq 3$ can be embedded in any group of order $n > 6$.
- there are two species of PLS of size 4 to check, namely

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b & \cdot \\ \cdot & a & b \end{bmatrix};$$

The left one embeds in every group of even order; the right one in every group which has an element of order more than 2.

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Proof of lower bound

- every PLS of size $s \leq 3$ can be embedded in any group of order $n > 6$.
- there are two species of PLS of size 4 to check, namely

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b & \cdot \\ \cdot & a & b \end{bmatrix};$$

  The left one embeds in every group of even order; the right one in every group which has an element of order more than 2.
- there are no PLS of size 5 to check; **this proves $\psi(n) \geq 5$ for even $n > 6$**

A lower bound

Recall: it remains to show that **for all even $n > 6$** we have

$$\psi(n) \geq 5 \text{ if } n \equiv 2, 4 \bmod 6,$$
$$\psi(n) \geq 6 \text{ if } n \equiv 0 \bmod 6;$$

in other words, that every PLS of size ≤ 5 can be embedded in some group of order n for all $n > 6$ with $n \equiv 2, 4 \bmod 6$; analogously for $n \equiv 0 \bmod 6$.

Proof of lower bound

- every PLS of size $s \leq 3$ can be embedded in any group of order $n > 6$.
- there are two species of PLS of size 4 to check, namely

  $$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b & \cdot \\ \cdot & a & b \end{bmatrix};$$

  The left one embeds in every group of even order; the right one in every group which has an element of order more than 2.
- there are no PLS of size 5 to check; **this proves $\psi(n) \geq 5$ for even $n > 6$**
- it remains to consider $n \equiv 0 \bmod 6$ and 11 species of PLS of size 6

A lower bound

Proof of lower bound (cont.)

Let $n > 6$ with $n \equiv 0 \bmod 6$; consider the 11 candidates of size 6. Two of them are

$$\begin{bmatrix} a & b & c \\ b & c & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b & \cdot \\ c & \cdot & b \\ \cdot & c & d \end{bmatrix}.$$

A lower bound

Proof of lower bound (cont.)

Let $n > 6$ with $n \equiv 0 \bmod 6$; consider the 11 candidates of size 6. Two of them are

$$\begin{bmatrix} a & b & c \\ b & c & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b & \cdot \\ c & \cdot & b \\ \cdot & c & d \end{bmatrix}.$$

The left embeds in groups of order $n \equiv 0 \bmod 3$;

A lower bound

Proof of lower bound (cont.)

Let $n > 6$ with $n \equiv 0 \bmod 6$; consider the 11 candidates of size 6. Two of them are

$$\begin{bmatrix} a & b & c \\ b & c & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b & \cdot \\ c & \cdot & b \\ \cdot & c & d \end{bmatrix}.$$

The left embeds in groups of order $n \equiv 0 \bmod 3$; the right one embeds in the dihedral group $D_6 = \langle r, s \mid r^3 = s^2 = 1,\ sr = r^{-1}s \rangle$ via

	1	rs	s
1	1	rs	$\cdot$
r	r	$\cdot$	rs
r^2s	$\cdot$	r	r^2

A lower bound

Proof of lower bound (cont.)

Let $n > 6$ with $n \equiv 0 \bmod 6$; consider the 11 candidates of size 6. Two of them are

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The remaining 9 PLS can be embedded in cyclic groups of any order > 6.

A lower bound

Proof of lower bound (cont.)

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1	1	rs	$\cdot$
r	r	$\cdot$	rs
$r^2 s$	$\cdot$	r	r^2

The remaining 9 PLS can be embedded in cyclic groups of any order > 6.

This completes the proof for $\psi(n)$; cases $\psi_+(n)$ and $\psi_\circ(n)$ analogously.

Infinite Case

The infinite case

Is there a PLS which can be embedded in an infinite group, but in no finite?

The infinite case

Is there a PLS which can be embedded in an infinite group, but in no finite?

Hirsch & Jackson (2012), *Undecidability of representability as binary relations.*

*	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3		10					
3	3							13	
4	4			5		11			
5	5	10							
6	6					7		12	
7	7			11					
8	8	13						9	
9	9					12			

FIGURE 1. A partial group embeddable in a group but not into any finite group.

EXAMPLE 3.7. *The content of the table in Figure 1 gives a pattern that does not appear in any Latin square isotopic to the multiplication table of any finite group but that does appear in the multiplication table of an infinite group.*

An interesting combinatorial problem is to find the smallest number of entries such a partial Latin square may have. A careful analysis of the proof of [32, Lemma 1.2], shows that in the partial table of Figure 1 we do not need all of the entries resulting from products with the element 1 (5 entries may be dropped from the existing 29; again, we omit details).

The infinite case

ROBIN HIRSCH AND MARCEL JACKSON

*	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3		10					
3	3							13	
4	4			5		11			
5	5	10							
6	6					7		12	
7	7			11					
8	8	13						9	
9	9					12			

FIGURE 1. A partial group embeddable in a group but not into any finite group.

This PLS of size 29 can be embedded in Higman's (1951) group

$$\langle a, b, c, d \mid b^a = b^2,\ c^b = c^2,\ d^c = d^2,\ a^d = a^2 \rangle$$

which has no non-trivial finite quotients!

The infinite case

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Question

What is the smallest PLS which can be embedded in an infinite group but in no finite group?

From PLS to group

A PLS which is embedded in a group G yields a list of relations in G:
If s is in the row and column labelled r and c, respectively, then $rc = s$ in G.

Example

The embedding of

$$P = \begin{bmatrix} s_1 & s_2 & s_3 \\ \cdot & s_3 & s_1 \\ s_2 & s_1 & \cdot \end{bmatrix}$$

in a group via

	c_1	c_2	c_3
r_1	s_1	s_2	s_3
r_2	$\cdot$	s_3	s_1
r_3	s_2	s_1	$\cdot$

yields relations

$$\begin{aligned} \mathcal{R} \quad = \quad & \{r_1c_1 = s_1,\ r_1c_2 = s_2,\ r_1c_3 = s_3,\ r_2c_2 = s_3, \\ & r_2c_3 = s_1,\ r_3c_1 = s_2,\ r_3c_2 = s_1\}. \end{aligned}$$

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$$\mathcal{R} = \{r_1c_1 = s_1,\ r_1c_2 = s_2,\ r_1c_3 = s_3,\ r_2c_2 = s_3,\ r_2c_3 = s_1,\ r_3c_1 = s_2,\ r_3c_2 = s_1\}.$$

Every group in which P can be embedded must satisfy these relations;

From PLS to group

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Every group in which P can be embedded must satisfy these relations;
we define

$$\langle P \rangle = \langle r_1, r_2, r_3, c_1, c_2, c_3, s_1, s_2, s_3 \mid \mathcal{R} \rangle;$$

without loss of generality, we can add the relations $r_1 = c_1 = 1$.

The group $\langle P \rangle$

Definition

Let P be a PLS whose rows and columns are labelled $1, \ldots, n$ and $1, \ldots, m$, respectively, and whose entries are $1, \ldots, s$. The **group defined by** P is $\langle P \rangle = \langle X \mid \mathcal{R} \rangle$, where $X = \{r_1, \ldots, r_n, c_1, \ldots, c_m, e_1, \ldots, e_s\}$,

$$\mathcal{R} = \{r_i c_j = e_k \mid P \text{ has entry } k \text{ in row } i \text{ and } j\}.$$

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Lemma

Let $(\rho, \kappa, \sigma)\colon P \to G$ be a group embedding. Then P embeds in the subgroup H of G generated by $\mathrm{im}(\rho) \cup \mathrm{im}(\kappa) \cup \mathrm{im}(\sigma)$, and H is a quotient of $\langle P \rangle$.
In particular, if P can be embedded in some group, then also in $\langle P \rangle$.

Proof.

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Proof.
Clearly, P embeds in the group H, and the generators of H satisfy the relations of $\langle P \rangle$.

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Let P be a PLS whose rows and columns are labelled $1, \ldots, n$ and $1, \ldots, m$, respectively, and whose entries are $1, \ldots, s$. The **group defined by** P is $\langle P \rangle = \langle X \mid \mathcal{R} \rangle$, where $X = \{r_1, \ldots, r_n, c_1, \ldots, c_m, e_1, \ldots, e_s\}$,

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Based on this, we use GAP to study possible embeddings of PLS.

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However, it is easy to compute in abelianisations or in free groups.

Lemma

Let P be a PLS and $G = \langle P \rangle$.

a) If P can be embedded in G, and G is residually finite (for example, free), then P can be embedded in a finite group.

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However, it is easy to compute in abelianisations or in free groups.

Lemma

Let P be a PLS and $G = \langle P \rangle$.

a) If P can be embedded in G, and G is residually finite (for example, free), then P can be embedded in a finite group.
b) If P can be embedded in G/G', then also in a finite abelian group.

This is a **good first test** to see if P can be embedded in some finite group.

A smaller example

The PLS of the Hirsch & Jackson paper had size 29; too big for exhaustive search! We looked around for other non-”residually finite” groups: groups which admit nontrivial elements which are trivial in each finite quotient.

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Baumslag group (1969)

$\mathcal{B} = \langle a, b \mid b = [b, b^a] \rangle$ is infinite, non-cyclic, with $b = 1$ in every finite quotient.

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$\mathcal{B} = \langle a, b \mid b = [b, b^a] \rangle$ is infinite, non-cyclic, with $b = 1$ in every finite quotient.

After playing with this presentation a lot, we were able to find:

Theorem

$$\mathcal{P} = \begin{bmatrix} \cdot & b & \cdot & c & \cdot \\ a & \cdot & c & \cdot & \cdot \\ \cdot & a & b & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & d \\ c & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & c \end{bmatrix}$$ can be embedded in an infinite group but in no finite group.

Thus the smallest PLS which can be embedded in an infinite group, but in no finite one, has size ≤ 12.

A smaller example

Theorem

$\mathcal{P} = \begin{bmatrix} \cdot & b & \cdot & c & \cdot \\ a & \cdot & c & \cdot & \cdot \\ \cdot & a & b & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & d \\ c & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & c \end{bmatrix}$ can be embedded in an infinite group but in no finite group.

Proof.

A smaller example

Theorem

$$\mathcal{P} = \begin{bmatrix} \cdot & b & \cdot & c & \cdot \\ a & \cdot & c & \cdot & \cdot \\ \cdot & a & b & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & d \\ c & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & c \end{bmatrix}$$ can be embedded in an infinite group but in no finite group.

Proof.

From an embedding in a group G with row and column labels $1, r_2, \ldots, r_6$ and $1, c_2, \ldots, c_5$ we deduce that … $c = aba^{-1}b$ and $d = aba^{-1}b^2$, and

$$c_2 = b, \quad c_3 = ba^{-1}b, \quad c_4 = aba^{-1}b, \quad c_5 = ab^2a^{-1}b$$
$$r_2 = a, \quad r_3 = ab^{-1}, \quad r_4 = b, \quad r_5 = aba^{-1}b, \quad r_6 = ab^{-1}a^{-1},$$

and, eventually,

$$bab^2a^{-1}b = r_4c_5 = d = aba^{-1}b^2,$$

thus $b^2 = a^{-1}b^{-1}aba^{-1}ba$, and $b = b^{-1}a^{-1}b^{-1}aba^{-1}ba$, …**and so $b = [b, b^a]$**.

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$\mathcal{P} = \begin{bmatrix} \cdot & b & \cdot & c & \cdot \\ a & \cdot & c & \cdot & \cdot \\ \cdot & a & b & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & d \\ c & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & c \end{bmatrix}$ can be embedded in an infinite group but in no finite group.

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From an embedding in a group G with row and column labels $1, r_2, \ldots, r_6$ and $1, c_2, \ldots, c_5$ we deduce that ... $c = aba^{-1}b$ and $d = aba^{-1}b^2$, and

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and, eventually,

$$bab^2a^{-1}b = r_4c_5 = d = aba^{-1}b^2,$$

thus $b^2 = a^{-1}b^{-1}aba^{-1}ba$, and $b = b^{-1}a^{-1}b^{-1}aba^{-1}ba$, ...**and so $b = [b, b^a]$**.
By von Dyck's Theorem, $K = \langle a, b \rangle \leq G$ is a quotient of the Baumslag group.
If G is finite, then so is K, hence $b = 1$, and $r_4 = 1 = r_1$ yields a contradiction.

A smaller example

Theorem

$\mathcal{P} = \begin{bmatrix} \cdot & b & \cdot & c & \cdot \\ a & \cdot & c & \cdot & \cdot \\ \cdot & a & b & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & d \\ c & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & c \end{bmatrix}$ can be embedded in an infinite group but in no finite group.

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A similar computation shows that $\mathcal{P}$ does indeed embed in $\langle\mathcal{P}\rangle \cong \mathcal{B}$:
if two labels or symbols coincide, then $\mathcal{B}$ would be cyclic, which is not possible.

The smallest example!

Theorem

The PLS $\mathcal{P}$ is a smallest possible example.

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Serious problems:

- Constructing *all* PLS of size ≤ 12 is just feasible.

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- If P can be embedded in $\langle P \rangle$, how to prove that it can or cannot be embedded in any finite group?

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Serious problems:

- Constructing *all* PLS of size ≤ 12 is just feasible.
- How to decide whether P can be embedded in $\langle P \rangle$?
- If P can be embedded in $\langle P \rangle$, how to prove that it can or cannot be embedded in any finite group?

In the end, we were just lucky...

How many candidates?

We only have to test species of connected PLS, since otherwise we simply embed each piece and use direct products:

size	1	2	3	4	5	6	7	8	9	10
all	1	2	5	18	59	306	1861	15097	146893	1693416
conn.	1	1	3	11	36	213	1405	12274	125235	1490851
red.	0	0	0	**2**	0	**11**	**50**	**489**	**6057**	**92533**

size	11	12
all	22239872	327670703
conn.	20003121	299274006
red.	**1517293**	**27056665**.

Proving non-embeddability

Let P be a PLS and $G = \langle P \rangle = \langle X \mid \mathcal{R} \rangle$, where

$$X = \{r_1, \ldots, r_n, c_1, \ldots, c_m, e_1, \ldots, e_s\}.$$

In GAP:

- Use Tietze transformations to obtain $G \cong \langle \hat{X} \mid \hat{\mathcal{R}} \rangle$ with significantly fewer generators and relations. For our PLS of size ≤ 12, usually $|\hat{X}|, |\hat{\mathcal{R}}| \leq 3$.

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- Construct isomorphism $\langle \hat{X} \mid \hat{\mathcal{R}} \rangle \to G$, and write the original generators X as words in $\hat{X}$.
- If we observe duplicates in the labels or symbols (either as elements of the free group, or by using a Knuth-Bendix completion algorithm), then we have proved that P *cannot be embedded* in $\langle P \rangle$, thus P cannot be embedded in any group.

Proving non-embeddability

Example

The PLS

$$P = \begin{bmatrix} a & d & \cdot & \cdot \\ \cdot & a & d & \cdot \\ \cdot & b & \cdot & c \\ c & \cdot & b & a \end{bmatrix}$$

yields $G = \langle P \rangle = \langle X \mid \mathcal{R} \rangle$ where $X = \{r_2, r_3, r_4, c_2, c_3, c_4, a, b, c, d\}$ and

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note that we assume $r_1 = c_1 = 1$. After applying Tietze transformations, we see that G is a one-generator group with no relations, that is, $G \cong (\mathbb{Z}, +)$. Expressing X in terms of the new generator, we obtain that $c = d$ in G; this shows that P cannot be embedded in any group.

Other computational methods

Computational approaches to find embeddings in finite groups:

Let P be a PLS and $G = \langle P \rangle$.

- Lemma: If P embeds in G/G' or in a free G, then P embeds in a finite group

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- brute-force: try if some random homomorphisms in some small groups give an embedding
- use GAP to construct set $\mathcal{U}$ of subgroups of G of low index;
  let $U = \cap_{H \in \mathcal{U}} H$ and consider permutation action of G on cosets of U in G;
  check if P embeds under this homomorphism

Statistics

size	NE	abelian	nonabelian	infNotFin
4	0	2	0	0
6	0	10	1	0
7	2	44	4	0
8	16	435	38	0
9	147	5447	463	0
10	2402	82555	7576	0
11	42884	1338816	135593	0
12	854559	23520406	2681650	**50**

NE : cannot be embedded in any group;
abelian : can be embedded in a finite abelian group;
nonabelian : can be embedded in a finite nonabelian group, but not in any abelian group;
infNotFin : can be embedded in an infinite group, but not in any finite group.

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- That each of the 50 PLS embeds in $\langle P \rangle$ can be proved as for $\mathcal{P}$ by showing that there are no duplicates among the list of symbols, row labels, and columns labels, respectively. The structure of the embedded PLS implies that no embedding in a finite group exists.

Conclusion

Thus the answer to Hirsch & Jackson's question is:

Theorem

The smallest PLS which can be embedded into an infinite group but in no finite group has size 12; an example of such a PLS is

$$\mathcal{P} = \begin{bmatrix} \cdot & b & \cdot & c & \cdot \\ a & \cdot & c & \cdot & \cdot \\ \cdot & a & b & \cdot & \cdot \\ b & \cdot & \cdot & \cdot & d \\ c & d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & c \end{bmatrix}.$$

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$$\begin{bmatrix} \cdot & T & \cdot & H & \cdot & N & \cdot \\ \cdot & \cdot & \cdot & A & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & K \\ \cdot & \cdot & Y & \cdot & O & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & U & ! \end{bmatrix}$$