Computational Group Cohomology

Bangalore, November 2016

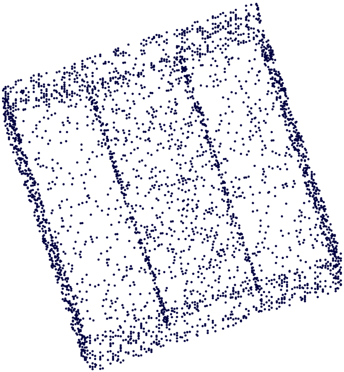
Graham Ellis
NUI Galway, Ireland

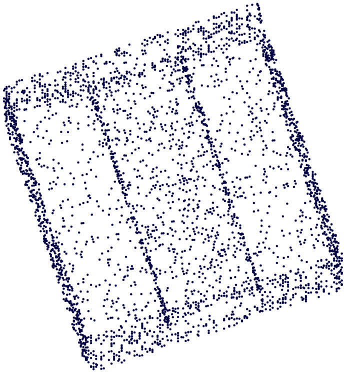
Slides available at **http://hamilton.nuigalway.ie/Bangalore**

Password: **Cork**

Outline

- Lecture 1: CW spaces and their (co)homology
- Lecture 2: Algorithms for classifying spaces of groups
- Lecture 3: Homotopy 2-types
- **Lecture 4: Steenrod algebra**
- Lecture 5: Curvature and classifying spaces of groups





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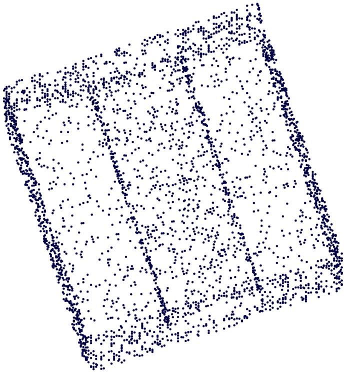

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$$X=\mathbb{S}^2\vee\mathbb{S}^1\vee\mathbb{S}^1 \quad \text{or} \quad X=\mathbb{T}\ ?$$

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$$H^{p+q}(X, \mathbb{Z}) \leftarrow H^{p+q}(X \times X, \mathbb{Z}) \leftarrow H^p(X, \mathbb{Z}) \times H^q(X, \mathbb{Z})$$

$$x \cup y \leftarrow (x, y)$$

The cohomology ring

$$H^*(X, \mathbb{Z}) = \bigoplus_{n>0} H^n(X, \mathbb{Z})$$

is a **graded commutative** ring:

$$x \cup y = (-1)^{pq} y \cup x$$

for $x \in H^p(X, \mathbb{Z})$, $y \in H^q(X, \mathbb{Z})$.

We'll consider the special case: $X = EG/G$

For $BG = EG/G$ the first n degrees of

$$H^*(BG, \mathbb{Z})$$

are calculated from $n + 1$ terms of a free $\mathbb{Z}G$-resolution with contracting homotopy.

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$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
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and $\Delta\colon C_*(EG) \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_*(EG \times EG) \otimes_{\mathbb{Z}G} \mathbb{Z}$

Cartan-Eilenberg double coset formula

For finite G and $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$

$$H^*(Syl_p(G), \mathbb{F})$$

is the sub**algebra** of

$$H^*(G, \mathbb{F})$$

consisting of the 'stable elements'.

Poincaré series

For a finite group G and $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ define

$$P(t) = \sum_{n \geq 0} a_n t^n$$

with

$$a_n = \dim(H^n(G, \mathbb{F})) \ .$$

Example (E, Green, King)

For the third Conway group Co_3 the cohomology ring

$$H^*(Co_3, \mathbb{F}_2)$$

has Poincaré series

$$P(t) = \frac{f(t)}{(1-t^8)(1-t^{12})(1-t^{14})(1-t^{15})}$$

where $f(t)$ is the monic polynomial of degree 45 with coefficients 1, 1, 1, 1, 2, 3, 3, 4, 4, 6, 7, 8, 9, 10, 10, 11, 13, 12, 14, 15, 13, 13, 15, 14, 12, 13, 11, 10, 10, 9, 8, 7, 6, 4, 4, 3, 3, 2, 1, 1, 1, 1.

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$$H^*(Co_3, \mathbb{F}_2) \cong \frac{\mathbb{F}_2[x_1, \cdots, x_{16}]}{(r_1 = 0, \cdots, r_{71} = 0)}$$

where $3 \leq \deg(x_i) \leq 13$ and $deg(r_i) \leq 33$.

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Yes: it has Krull dimension 4 and depth 4.

Simpler Example

$$H^*(D_{256}, \mathbb{F}) = \frac{\mathbb{F}[x, y, z]}{(xy = y^2)}$$

where $\deg(x) = \deg(y) = 1, \deg(z) = 2$

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```
gap> Mod2CohomologyRingPresentation(DihedralGroup(512));
Graded algebra GF(2)[ x_1, x_2, x_3 ] / [ x_1*x_2+x_2^2
 ] with indeterminate degrees [ 1, 1, 2 ]
```

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1. It was shown by Len Evens and others that $H^*(G, \mathbb{F}_2)$ is a finitely presented algebra. So we can compute a presentation from sufficiently many terms of a free $\mathbb{F}G$-resolution of $\mathbb{F}$.

2. But we don't have realistic bounds on the number of terms of the resolution that we need.

Computing a free $\mathbb{F}G$-resolution for a finite p-group G

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Computing generators for an $\mathbb{F}G$-module $M \subseteq \mathbb{F}G$

The **Radical** of M is the submodule

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Any basis for the vector space $M/\mathrm{Rad}(M)$ corresponds to a minimal generating set of the module M.

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- degrees of ring generators and relations

Complete set of generators and relators for the ring $H^*(G, \mathbb{F})$

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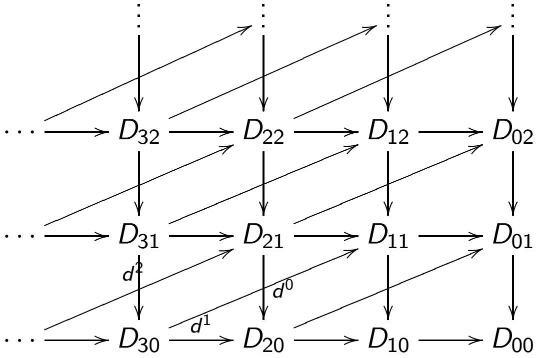
$$N \rightarrowtail Q_8 \twoheadrightarrow Q$$

$$N = Z(G) = C_2, \quad Q = C_2 \times C_2$$

Wall's resolution

$$R_n^G = \bigoplus_{p+q=n} D_{pq}, \quad D_{pq} = R_p^N \otimes_{\mathbb{Z}} R_q^G$$



Lyndon-Hochschild-Serre spectral sequence

$$E^0_{pq} = D_{pq} \otimes_{\mathbb{F}G} \mathbb{F}, \quad E^0 = \bigoplus E^0_{pq}$$

$$d^0 \colon E^0 \longrightarrow E^0$$

$$\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \vdots \\
 & \downarrow & \downarrow & \downarrow & \downarrow \\
\cdots & E^0_{32} & E^0_{22} & E^0_{12} & E^0_{02} \\
 & \downarrow & \downarrow & \downarrow & \downarrow \\
\cdots & E^0_{31} & E^0_{21} & E^0_{11} & E^0_{01} \\
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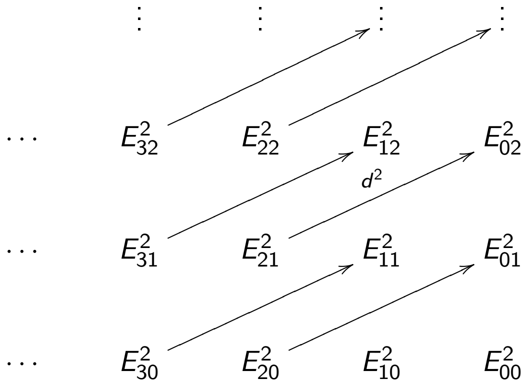
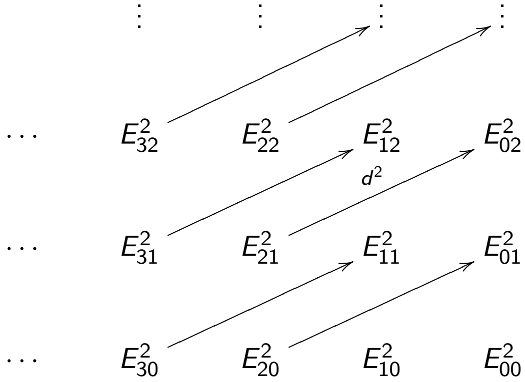


Lyndon-Hochschild-Serre spectral sequence

$$E^2 = H(E^1) \;\cong H^*(Q,\mathbb{F}) \otimes H^*(N,\mathbb{F})$$

$$d^2\colon E^2 \longrightarrow E^2 \quad \textbf{a derivation}$$



Lyndon-Hochschild-Serre spectral sequence

$$E^n = H(E^{n-1})$$

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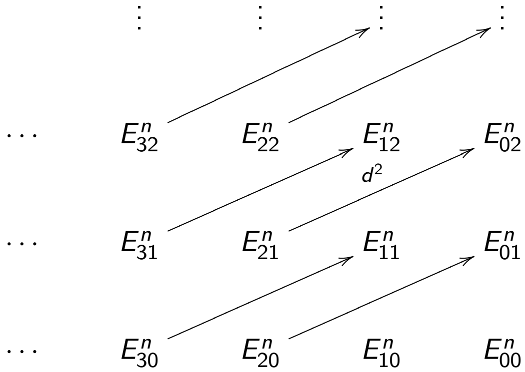

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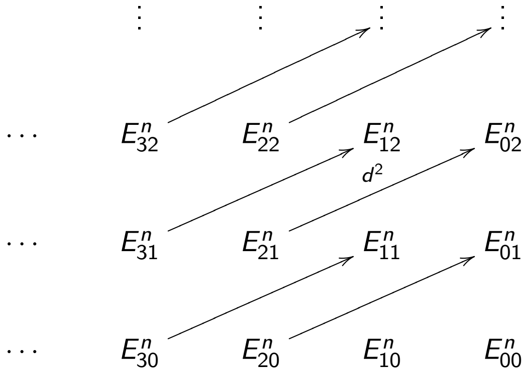
Gröbner bases yield $H^{n+1}(E^n)$.



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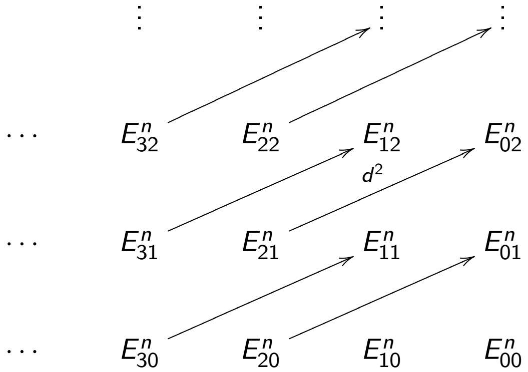
Gröbner bases yield $H^{n+1}(E^n)$. At some point $E^n \cong E^{n+1}$; minimal generators/relations for E^n and $H^*(G, \mathbb{F})$ have same degrees



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A presentation for $H^*(G, \mathbb{F})$ can be read from five terms of a free $\mathbb{F}G$-resolution of $\mathbb{F}$.

Steenrod operations ($\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$)

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5. $Sq^i(x + y) = Sq^i(x) + Sq^i(y)$.

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1. Sq^0 is the identity homomorphism.
2. $Sq^1 = \beta$ is equal to the Bockstein homomorphism.
3. $Sq^i(x) = x \cup x$ if x is a homogeneous element of degree i.
4. $Sq^i(x) = 0$ if x is a homogeneous element of degree $< i$.
5. $Sq^i(x + y) = Sq^i(x) + Sq^i(y)$.
6. $Sq^i(x \cup y) = \sum_{j=0}^{i} Sq^j(x) \cup Sq^{i-j}(y)$.

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8. $$Sq^a Sq^b = \sum_{c=0}^{[a/2]} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

Example: $X = BG$ with $G = D_{32}$

```
gap> CohomologicalData(DihedralGroup(64),9);

Group order: 64
Group number: 52
Group description: D64

Cohomology generators
Degree 1: a, b
Degree 2: c

Cohomology relations
1: a*b+b^2

Poincare series
(1)/(x^2-2*x+1)

Steenrod squares
Sq^1(c)=c*a
```

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$$Hom_{\mathbb{Z}G}(R_p^G,\mathbb{Z}) \otimes Hom_{\mathbb{Z}G}(R_q^G,\mathbb{Z}) \longrightarrow Hom_{\mathbb{Z}G}(R_{p+q}^G,\mathbb{Z}), \overline{u}\otimes\overline{v} \mapsto \overline{u}\cup_i\overline{v}$$

$$\overline{u} \cup_i \overline{v}\,(c) = \overline{u\otimes v}\,\phi_{p+q}(f^i \otimes c)$$

Theorem

The operation

$$Hom_{\mathbb{Z}G}(R_n^G) \to Hom_{\mathbb{Z}G}(R_{2n-1}^G), \overline{u} \mapsto \overline{u} \cup_i \overline{u}$$

induces a homomorphism

$$Sq_i \colon H^n(G, \mathbb{Z}_2) \to H^{2n-i}(G, \mathbb{Z}_2).$$

The homomorphism

$$Sq^i = Sq_{n-i} \colon H^n(G, \mathbb{Z}_2) \to H^{n+i}(G, \mathbb{Z}_2)$$

is independent of the choices in ϕ_* and satisfies the required properties.