Computational Representation Theory – Lecture IV

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Group Theory and Computational Methods
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Contents

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To overcome this problem, Condensation is used (Thackray, Parker, ca. 1980).

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As the Condensation functor is exact, it sends a composition series of $V \in$ mod-$\mathfrak{A}$ to a composition series of $Ve \in$ mod-$e\mathfrak{A}e$.

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Morita equivalent algebras have "the same" representations.

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Other idempotents can be used, e.g.,

$$e = \frac{1}{|H|} \sum_{x \in H} \lambda(x^{-1}) x \in FG,$$

where $\lambda : H \to F^*$ is a homomorphism (Noeske, 2005).

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W.r.t. this basis, the (i, j)-entry a_{ij} of the matrix of ege on Ve equals

$$a_{ij} = \frac{1}{|\Omega_j|} \, |\Omega_i g \cap \Omega_j|.$$

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Details depend on the realisation of the action on Ω.

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Homomorphism spaces

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Lux, Neunhöffer, Noeske develop general Condensation programs for such homomorphism spaces.

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More applications later.

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Definition

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$$\mathfrak{B}_{\mathfrak{B}} = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n$$

with PIMs Q_i such that $Q_i \not\cong Q_j$ for $1 \leq i \neq j \leq n$.

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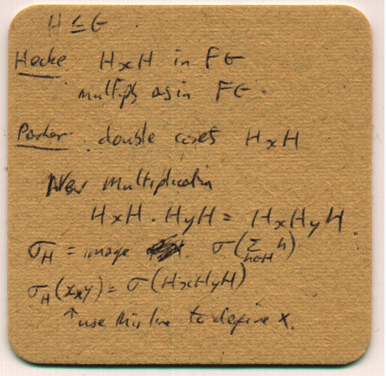
Applications: Cartan matrices for group algebras, cohomology computations.

CONDENSATION: HISTORY



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- We can draw conclusions on V from Ve, but not from $Ve|_{\mathfrak{C}}$.

Generation and Matching

Theorem (F. Noeske, 2005)

Let $H \trianglelefteq N \leq G$. If $\mathcal{T}$ is a set of double coset representatives of $N \backslash G / N$ and $\mathcal{N}$ a set of generators of N, then we have for $e = e_H$:

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Matching Problem: Let $e, e' \in FG$ be idempotents. Suppose $S, S' \in$ mod-FG are simple, and we know Se and $S'e'$. Can we decide if $S \cong S'$? Yes! (Noeske, 2008)

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   $P = e_H FG$, *for* $H = 3xG_2(3)$, $\dim(P) = 7\,124\,544\,000$,
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2. *(Envisaged):*
   $\dim(Q) = 43\,957\,879\,875$, $\dim_F(\mathrm{End}_{FG}(Q)) = 21\,530$
   $\rightsquigarrow$ *almost finish Th modulo* 5

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In joint work with Max Neunhöffer and F. Noeske we have computed the 2-modular character table of G.

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One such matrix over $\mathbb{F}_2$ would need $\approx 18\,403\,938$ GB.

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4. Every irreducible FG-module (of the principal 2-block) occurs in $19\,940 \otimes 19\,940$.

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1 951 872,	1 997 872,	1 997 872,	5 812 860,
7 821 240,	8 280 208,	17 276 520,	34 744 192,
73 531 392,	97 976 320,	166 559 744,	504 627 200,
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Using similar methods, Görgen and Lux have recently computed the irreducible characters of Fi_{23} over $\mathbb{F}_3$.
(Largest condensed module:184 644,
largest module found: 34 753 159.)

References

1. D. F. Holt, B. Eick and E. A. O'Brien, Handbook of Computational Group Theory, Chapman & Hall/CRC, 2005.
2. K. Lux, Algorithmic methods in modular representation theory, Habilitationsschrift, RWTH Aachen University, 1997.
3. K. Lux and H. Pahlings, Representations of Groups. A computational approach. Cambridge University Press, 2010.
4. J. Müller, M. Neunhöffer and R. Wilson, Enumerating big orbits and an application: B acting on the cosets of Fi_{23}, *J. Algebra* **314**, 2007, 75–96.
5. F. Noeske, Morita-Äquivalenzen in der algorithmischen Darstellungstheorie, Dissertation, RWTH Aachen University, 2005.

Thank you for your attention!