Computational Group Cohomology

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Graham Ellis
NUI Galway, Ireland

Slides available at **http://hamilton.nuigalway.ie/Bangalore**

Password: **Belfast**

Outline

- Lecture 1: CW spaces and their (co)homology
- Lecture 2: Algorithms for classifying spaces of groups
- **Lecture 3: Homotopy 2-types**
- Lecture 4: Steenrod algebra
- Lecture 5: Curvature and classifying spaces of groups

A filtered chain complex over a **field**

$$C_{*\,1} \hookrightarrow C_{*\,2} \hookrightarrow C_{*\,3} \hookrightarrow C_{*\,4} \hookrightarrow \cdots$$

induces

$$\iota_n^{s\,t} \colon H_n(C_{*\,s}) \longrightarrow H_n(C_{*\,t})$$

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The **Persistent Betti numbers** are

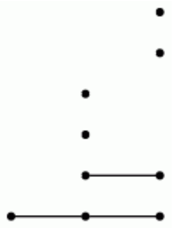
$$\beta_n^{s\,t} = \operatorname{rank}(\iota_n^{s\,t}) \qquad s \leq t$$

$$\beta_n^{s\,t} = 0 \qquad s > t.$$

β_n **bar code** has

$\beta_n^{s,t}$ horizontal lines from column s to column t

$$(\beta_2^{s\,t}) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$



Example $v_1, v_2, \ldots, v_{72} \subset \mathbb{R}^{262144}$

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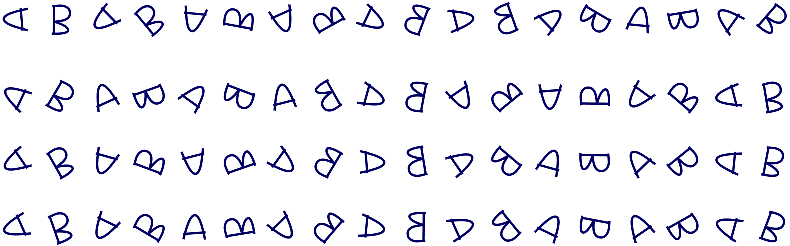
Toy data points from

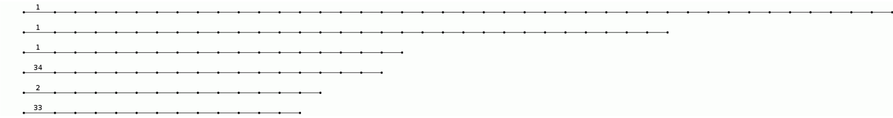


Fix a sequence of real numbers $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_T$.

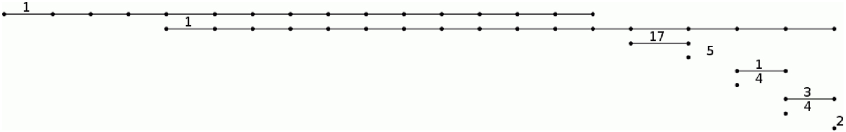
The **Rips simplicial complex** X_t has with

- vertex set $V = \{v_1, \ldots, v_{72}\}$.
- n-simplices the subsets $\sigma \subseteq V$ with $n + 1$ vertices and $||v - v'|| \leq \epsilon_t$ for all $v, v' \in \sigma$.

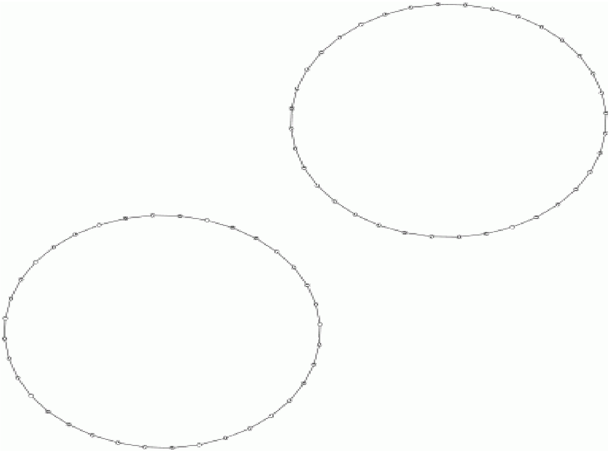
Persistent β_0 for $C_*(X_*)$:



Persistent β_1 for $C_*(X_*)$:



Data Model: A homotopy retract $Y \subset X_{20}$



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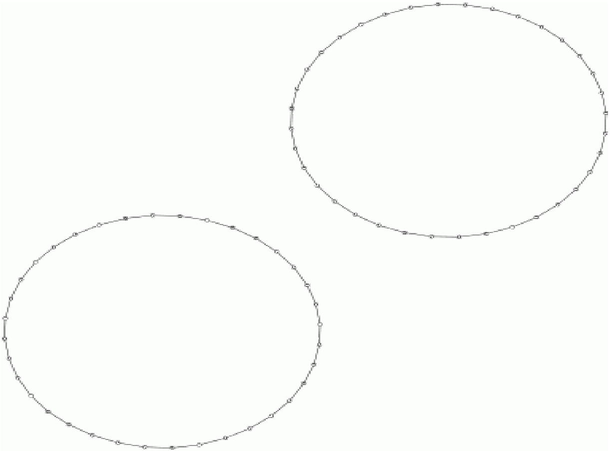
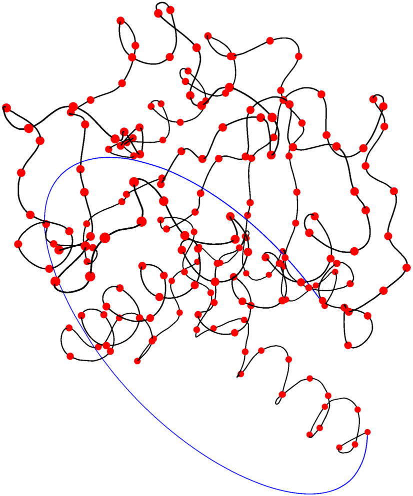


$$\mathbf{Y \simeq S^1 \sqcup S^1}$$

Second example: 1V2X protein backbone



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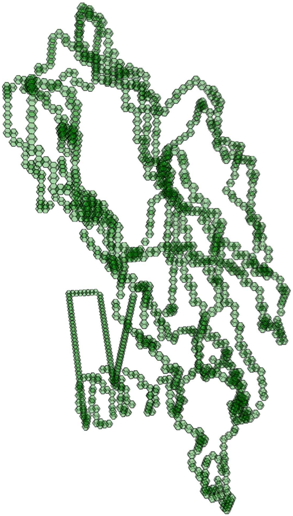

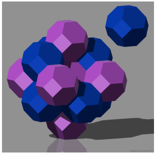

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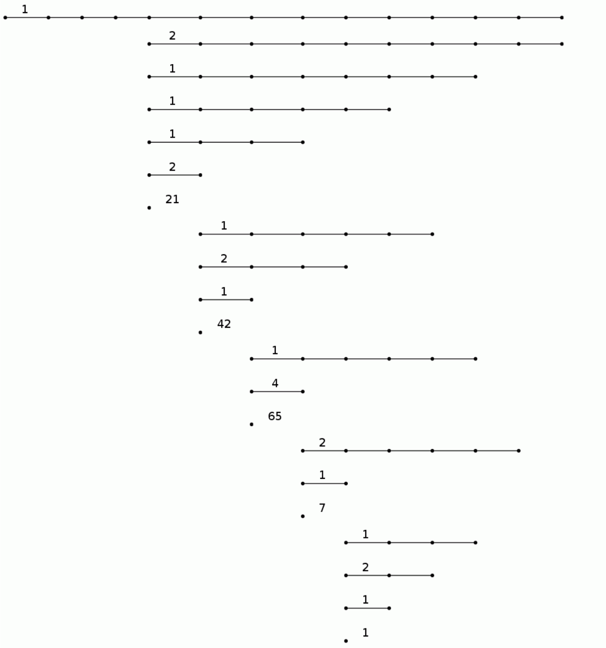
Persistent β_1



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By computing a discrete vector field on a finite region of $\mathbb{R}^3 \setminus K$ we find

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But what good is (a presentation of) the fundamental group?

Theorem (Gordon-Luecke, . J. Amer. Math. Soc. 1989)
Two knots $K, K' \subset \mathbb{R}^3$ are equivalent (up to mirror image) if and only if their complements are homeomorphic.

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Theorem (Whitten 1987, Gordon-Luecke 1989)
Prime knots are determined, up to mirror image, by their fundamental group.

Proposition: *The alpha carbon atoms of the Thermus Thermophilus protein determine a knot K with peripheral system*

$$\begin{array}{rcl} \pi_1(\partial K) \cong \langle a, b | aba^{-1}b^{-1} \rangle & \to & \pi_1(\mathbb{R}^3 \setminus K) \cong \langle x, y | xyx = yxy \rangle \\ a & \mapsto & x^{-2}yx^2y \\ b & \mapsto & x \end{array} .$$

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gap> phi:=FundamentalGroup(i,22495);
[ f1, f2 ] -> [ f1^-3*f2*f1^2*f2*f1, f1 ]
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min value of n needed to distinguish between knots on c crossings

c	3	4	5	6	7	8	9	10	11	12	13	14
n	2	2	3	3	3	3	5	5	6	6	7	7

Brendel, E., Juda, Mrozek

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The generators and relations correspond to the critical 1-cells and critical 2-cells in a discrete vector field on X with unique critical 0-cell.

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$$\pi_1(\partial) = G/\partial M \qquad\qquad \pi_2(\partial) = \ker \partial$$

J.H.C. Whitehead:

There is a functor

$$\Pi\colon (CW\ spaces) \longrightarrow (crossed\ modules)$$

which induces a bijection

$$\left\{ \begin{array}{l} \text{homotopy classes} \\ \text{of connected 2-} \\ \text{types} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \textit{weak equivalence} \\ \text{classes of crossed} \\ \text{modules} \end{array} \right\}.$$

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depending on the *weak equivalence* class of $M \xrightarrow{\partial} G$, which is **computable** if $M \xrightarrow{\partial} G$ is **finite**.

A **morphism** of crossed modules is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi_2} & M' \\ \big\downarrow{\scriptstyle \partial} & & \big\downarrow{\scriptstyle \partial'} \\ G & \xrightarrow{\phi_1} & G' \end{array}$$

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Two crossed modules ∂, ∂' are **weakly equivalent** if there exists a sequence of quasi-isomorphisms:

$$\partial \to \partial_1 \leftarrow \partial_2 \to \cdots \leftarrow \partial_k \to \partial'$$

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A useful weak equivalence invariant

$$H_n(M \xrightarrow{\partial} G, \mathbb{Z}) = H_n(X, \mathbb{Z})$$

where X is the corresponding homotopy 2-type.

Category $\mathcal{C}$ of a crossed module $M \xrightarrow{\partial} G$

$$\mathcal{C} = M \rtimes G$$

$$ob(\mathcal{C}) = G$$

$$s \colon \mathcal{C} \to \mathcal{C}, \qquad (m, g) \mapsto (1, g)$$

$$t \colon \mathcal{C} \to \mathcal{C}, \qquad (m, g) \mapsto (1, \partial(m)\, g)$$

$$\circ \colon \mathcal{C} \times_G \mathcal{C} \to \mathcal{C}, \quad ((m, g), (m', g') \mapsto (m, (\partial m)^{-1} g')$$

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s, t, $\circ$ are group homomorphisms.

Nerve construction

$$\mathcal{N}_n\mathcal{C} = \{X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_k\}$$

collection of composable sequences of n morphisms $f_i \in \mathcal{C}$.

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For the 2-type X corresponding to $M \xrightarrow{\partial} G$

$$C_nX \text{ has basis } \mathcal{N}_n\mathcal{N}_n\mathcal{C}.$$

Chain complex C_*X

$$\longrightarrow \mathbb{Z}\mathcal{N}_2\mathcal{N}_2\mathcal{C} \longrightarrow \mathbb{Z}\mathcal{N}_1\mathcal{N}_1\mathcal{C} \longrightarrow \mathbb{Z}\mathcal{N}_0\mathcal{N}_0\mathcal{C}$$

$C_*X \simeq B_*$ the total complex of

$$
\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & & \mathbb{Z}\mathcal{N}_2\mathcal{N}_2\mathcal{C} & \longrightarrow & \mathbb{Z}\mathcal{N}_2\mathcal{N}_1\mathcal{C} & \longrightarrow & \mathbb{Z}\mathcal{N}_2\mathcal{N}_0\mathcal{C} \\
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 & & \downarrow & & \downarrow & & \downarrow \\
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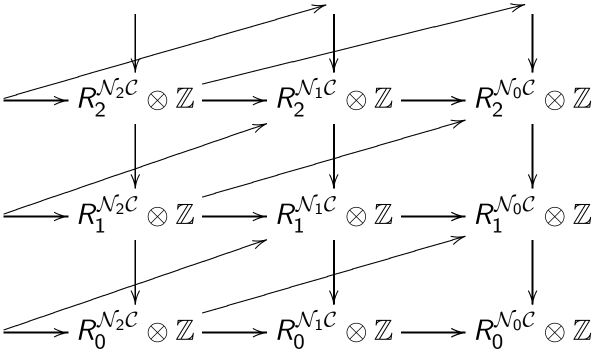
\end{array}
$$

$C_*X \simeq B_* \simeq R_*$ a filtered complex



A **homotopy equivalence data**

$$(R, d) \overset{p}{\underset{i}{\leftrightarrows}} (B, d), \ \ h \qquad (*)$$

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Perturbation Lemma: If $A = (1 - \epsilon h)^{-1}\epsilon$ exists then

$$(R,d') \overset{p'}{\underset{i'}{\leftrightarrows}} (B,d+\epsilon),\ \ h' \qquad (**)$$

is a homotopy equivalence data where

$$i' = i + hAi, \quad p' = p + pAh, \quad h' = h + hAh, \quad d' = d + pAi \quad .$$

$$
\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
\xrightarrow{0} & \mathbb{Z}\mathcal{N}_2\mathcal{N}_2\mathcal{C} & \xrightarrow{0} & \mathbb{Z}\mathcal{N}_2\mathcal{N}_1\mathcal{C} & \xrightarrow{0} & \mathbb{Z}\mathcal{N}_2\mathcal{N}_0\mathcal{C} & \\
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\xrightarrow{0} & \mathbb{Z}\mathcal{N}_1\mathcal{N}_2\mathcal{C} & \xrightarrow{0} & \mathbb{Z}\mathcal{N}_1\mathcal{N}_1\mathcal{C} & \xrightarrow{0} & \mathbb{Z}\mathcal{N}_1\mathcal{N}_0\mathcal{C} & (B, d) \\
 & \downarrow & & \downarrow & & \downarrow & \\
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\end{array}
$$

$$
\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & \downarrow & \\
\xrightarrow{0} & R_2^{\mathcal{N}_2\mathcal{C}} \otimes \mathbb{Z} & \xrightarrow{0} & R_2^{\mathcal{N}_1\mathcal{C}} \otimes \mathbb{Z} & \xrightarrow{0} & R_2^{\mathcal{N}_0\mathcal{C}} \otimes \mathbb{Z} & \\
 & \downarrow & & \downarrow & & \downarrow & \\
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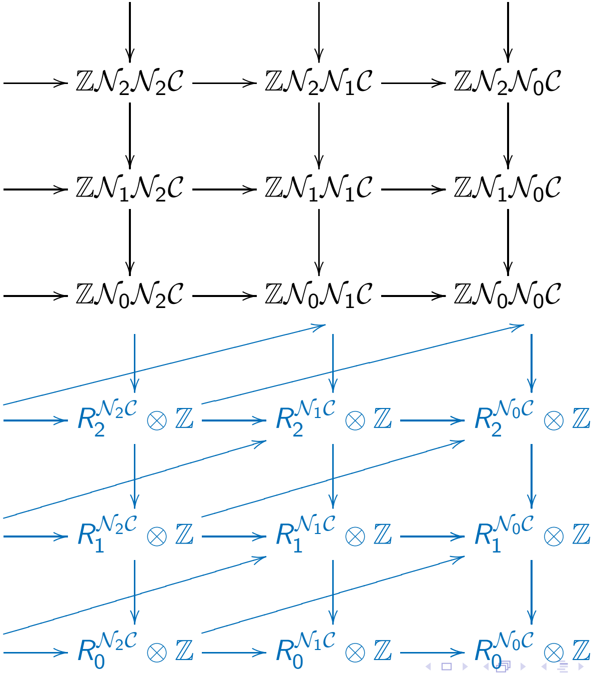
\xrightarrow{0} & R_0^{\mathcal{N}_2\mathcal{C}} \otimes \mathbb{Z} & \xrightarrow{0} & R_0^{\mathcal{N}_1\mathcal{C}} \otimes \mathbb{Z} & \xrightarrow{0} & R_0^{\mathcal{N}_0\mathcal{C}} \otimes \mathbb{Z} &
\end{array}
$$



$(B, d + \epsilon)$

(R, d')

$$Q \longrightarrow Aut(Q), a \mapsto \iota_a(x) = axa^{-1}$$

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gap> Q:=DihedralGroup(216);;
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gap> IdQuasiCrossedModule(G);
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gap> Homology(G,5);
[ 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 18 ]
```

A curiosity about vector field collapse

The crossed module

$$: \mathbb{Z}_2 \to 0$$

represents a homotopy 2-type X with

$$\pi_2(X) = \mathbb{Z}_2, \quad \pi_k(X) = 0 \text{ for } k \neq 2.$$

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gap> C:=ChainComplex(B);;
gap> List([0..11],CK!.dimension);
[ 1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024 ]
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gap> D:=CoreducedChainComplex(C);;
gap> List([0..10],D!.dimension);
[ 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34 ]
```