COMPUTATIONAL REPRESENTATION THEORY – LECTURE III

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Group Theory and Computational Methods
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Contents

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FG is a finite-dimensional F-algebra.

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From now on: representation = linear representation

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$\mathfrak{X} : G \to \mathrm{GL}(V)$ and $\mathfrak{Y} : G \to \mathrm{GL}(W)$ are equivalent, if and only if V and W are isomorphic as FG-modules.

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Irreducibility and equivalence are defined analogously to the case of group representations.

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A permutation representation of G on the finite set $\Omega = \{\omega_1, \ldots, \omega_n\}$ is a homomorphism

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Let V denote an F-vector space with basis Ω.

Replacing each $\kappa(g) \in S_\Omega$ by the linear map $\mathfrak{X}(g)$ of V, which permutes its basis like $\kappa(g)$, we obtain an F-representation $\mathfrak{X}$ of degree n of G.

Permutation Representations: Example

Let $G = S_3$, the symmetric group on three letters, acting (from the right) on $\Omega = \{1, 2, 3\}$. Then $G = \langle a, b \rangle$ with $a = (1, 2)$ and $b = (1, 2, 3)$.

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Take $F = \mathbb{Q}$; $V \cong \mathbb{Q}^{1 \times 3}$. Then G acts on V by $e_i.g = e_{i^g}$, where e_1, e_2, e_3 is the standard basis of V.

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Then

$$\mathfrak{X}(a) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathfrak{X}(b) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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Note: In order to describe (store) a representation $\mathfrak{X}$ of G, it suffices to give the matrices $\mathfrak{X}(a_i)$ for a generating set $\{a_1, \ldots, a_l\}$ of G.

Let $\mathfrak{X} : G \to \mathrm{GL}_d(F)$ and $\mathfrak{Y} : G \to \mathrm{GL}_e(F)$ be two matrix representations of G.

Kronecker Product of Representations

Let $\mathfrak{X} : G \to \mathrm{GL}_d(F)$ and $\mathfrak{Y} : G \to \mathrm{GL}_e(F)$ be two matrix representations of G.

Then $\mathfrak{X} \otimes \mathfrak{Y} : G \to \mathrm{GL}_{de}(F), \quad g \mapsto \mathfrak{X}(g) \otimes \mathfrak{Y}(g)$ is a matrix representation of G.

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Here, $\mathfrak{X}(g) \otimes \mathfrak{Y}(g)$ is the Kronecker product of the two matrices $\mathfrak{X}(g)$ and $\mathfrak{Y}(g)$, defined as follows:

$$A \otimes B := \begin{bmatrix} a_{11}\,B & \cdots & a_{1d}\,B \\ \vdots & \ddots & \vdots \\ a_{d1}\,B & \cdots & a_{dd}\,B \end{bmatrix}$$

for $A = [a_{ij}] \in F^{d \times d}$, and $B \in F^{e \times e}$.

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From this: $\chi_{\mathfrak{X} \otimes \mathfrak{Y}} = \chi_{\mathfrak{X}} \cdot \chi_{\mathfrak{Y}}$, i.e. the product of characters is a character.

Invariant Subspaces

Let $\mathfrak{X} : G \to \mathrm{GL}(V)$ be a representation of G on V.

Let W be a G-invariant subspace of V, i.e.:

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(W is an FG-submodule of V.)

We obtain F-representations

$$\mathfrak{X}_W : G \to \mathrm{GL}(W) \quad \text{and} \quad \mathfrak{X}_{V/W} : G \to \mathrm{GL}(V/W)$$

in the natural way.

Invariant Subspaces: Example

Let $G = S_3$ and V be as above.

$W := \langle e_1 + e_2 + e_3 \rangle$ is an invariant subspace.

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Choosing the basis $e_1 + e_2 + e_3, e_2, e_3$ of V, and transforming the matrices accordingly, we obtain a representation $\mathfrak{X}'$ with

$$\mathfrak{X}'(a) = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 1 & -1 & -1 \\ 0 & 0 & 1 \end{array}\right], \quad \mathfrak{X}'(b) = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 1 & -1 & -1 \end{array}\right].$$

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Moreover,

$$\mathfrak{X}'_{V/W}(a) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \mathfrak{X}'_{V/W}(b) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Iterating this process, we obtain a matrix representation $\mathfrak{X}^\infty$ of G, equivalent to $\mathfrak{X}$, s.t.:

$$\mathfrak{X}^\infty(g) = \begin{bmatrix} \mathfrak{X}_1(g) & 0 & \cdots & 0 \\ * & \mathfrak{X}_2(g) & \cdots & 0 \\ * & * & \ddots & 0 \\ * & * & \cdots & \mathfrak{X}_l(g) \end{bmatrix},$$

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They are unique up to equivalence (Jordan-Hölder theorem).

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one obtains all irreducible F-representations of G.

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Since then, it has been improved and enhanced by many people, including Derek Holt, Gábor Ivanyos, Klaus Lux, Jürgen Müller, Sarah Rees, Michael Ringe, and by Richard Parker himself.

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2. How does one prove that $\mathfrak{X}$ is irreducible?

Let $A_1, \ldots, A_l$ be $(d \times d)$-matrices over F.

Norton's Irreducibility Criterion

Let $A_1, \ldots, A_l$ be $(d \times d)$-matrices over F.

Put $\mathfrak{A} := F\langle A_1, \ldots, A_l\rangle$ and $\mathfrak{A}^{tr} := F\langle A_1^{tr}, \ldots, A_l^{tr}\rangle$ (algebra span).

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3. Every non-trivial vector in $\mathcal{N}(B^{tr})$ lies in a proper $\mathfrak{A}^{tr}$-invariant subspace.
4. $\mathfrak{A}$ acts irreducibly on $F^{1 \times d}$.

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THE MEATAXE: BASIC STRATEGY

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If YES, $\mathfrak{X}$ is irreducible.

Note: If $G = \langle g_1, \ldots, g_l \rangle$, and $\mathfrak{X} : G \to \mathrm{GL}_d(F)$ is an F-representation of G, take $A_i := \mathfrak{X}(g_i)$, $1 \leq i \leq l$.

The Spinning Algorithm

The space $w.\mathfrak{A}$ is computed with a linearized version of an orbit algorithm: $\mathrm{spin}(A_1, \ldots, A_l, w)$ returns F-basis of $w.\mathfrak{A}$.

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ALGORITHM ($\text{spin}(A_1, \ldots, A_l, w)$)

Input: $A_1, \ldots, A_l \in F^{d \times d}$, $0 \neq w \in F^{1 \times d}$
Output: *basis $\mathcal{B}$ of $w.\mathfrak{A}$ with $\mathfrak{A} = F\langle A_1, \ldots, A_l\rangle$*

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Output: *basis* $\mathcal{B}$ *of* $w.\mathfrak{A}$ *with* $\mathfrak{A} = F\langle A_1, \ldots, A_l\rangle$

$\mathcal{B} \leftarrow w$;
for v *in* $\mathcal{B}$ **do**
  for i *from* 1 *to* l **do**
    if $\{vA_i\} \cup \mathcal{B}$ *linearly independent* **then**
      append vA_i *to* $\mathcal{B}$;
    end if;
  end for;
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Let $\mathfrak{A}$ be an F-algebra, generated by $\mathfrak{a}_1, \ldots, \mathfrak{a}_l$.

The MeatAxe: Test for Equivalence

Let $\mathfrak{A}$ be an F-algebra, generated by $\mathfrak{a}_1, \ldots, \mathfrak{a}_l$.

Suppose $\mathfrak{X}, \mathfrak{Y} : \mathfrak{A} \to F^{d \times d}$ are irreducible representations of $\mathfrak{A}$.

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If nullity of $B := \mathfrak{w}(B_1, \ldots, B_l)$ is not 1, then $\mathfrak{X}$, $\mathfrak{Y}$ are not equivalent.

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Let $\mathfrak{w} \in F\langle X_1, \ldots, X_l \rangle$ such that $A := \mathfrak{w}(A_1, \ldots, A_l)$ has nullity 1.

If nullity of $B := \mathfrak{w}(B_1, \ldots, B_l)$ is not 1, then $\mathfrak{X}$, $\mathfrak{Y}$ are not equivalent. Otherwise, let $0 \neq v \in \mathcal{N}(A)$, $0 \neq w \in \mathcal{N}(B)$.

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The MeatAxe: Test for Equivalence

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$\mathfrak{X}$ and $\mathfrak{Y}$ are equivalent if and only if $\mathfrak{X}'$ and $\mathfrak{Y}'$ are equal.

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THE MEATAXE: REMARKS

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As F gets larger, it gets harder to find a suitable B by a random search.

Holt and Rees use characteristic polynomials of elements of $\mathfrak{A}$ to find suitable Bs and also to reduce the number of tests considerably.

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The Holt and Rees Irreducibility Criterion

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THEOREM (HOLT AND REES)

Suppose that p is a good factor of χ_B and let $0 \neq w \in \mathcal{N}(p(B))$, and $0 \neq w' \in \mathcal{N}(p(B)^{tr})$.
Then $\mathfrak{A}$ acts irreducibly on $F^{1\times d}$, if $w.\mathfrak{A} = F^{1\times d}$ and if $w'.\mathfrak{A}^{tr} = F^{1\times d}$.

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The MeatAxe according to Holt and Rees

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   if p is good, Return(“Irreducible”) fi;

5. Go back to Step 1.

Rob Wilson's Atlas

A huge collection of explicit representations of finite groups is contained in Rob Wilson's WWW *Atlas of Finite Group Representations:* (http://brauer.maths.qmul.ac.uk/Atlas/).

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Much of this information is also available through the GAP-package atlasrep (http://www.math.rwth-aachen.de/~Thomas.Breuer/atlasrep/).

References

1. D. F. Holt, B. Eick and E. A. O'Brien, Handbook of Computational Group Theory, Chapman & Hall/CRC, 2005.
2. D. F. Holt and S. Rees, Testing modules for irreducibility, *J. Austral. Math. Soc. Ser. A* **57** (1994), 1, 1–16.
3. K. Lux and H. Pahlings, Representations of Groups. A computational approach. Cambridge University Press, 2010.
4. R. A. Parker, The computer calculation of modular characters (the meat-axe), Computational group theory (Durham, 1982), 267–274, Academic Press, London, 1984.

Thank you for your attention!