COMPUTATIONAL REPRESENTATION THEORY – LECTURE II

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Group Theory and Computational Methods
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Contents

Notation

Throughout this lecture, G denotes a finite group and F a field.

Brauer Characters: Motivation

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Let $\mathfrak{X}$ be an F-representation of G of degree d.

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Instead one considers the Brauer character $\varphi_{\mathfrak{X}}$ of $\mathfrak{X}$.

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Choose a ring homomorphism $\alpha : R \to F$ sending ζ to a primitive m-th root of unity $\bar{\zeta} \in F$.

Notice that the restriction of α to $\langle\zeta\rangle$ is injective.

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Let $g \in G_{p'}$ and let $\bar{\zeta}^{i_1}, \ldots, \bar{\zeta}^{i_d}$ denote the eigenvalues of $\mathfrak{X}(g)$, counting multiplicities. Then $\varphi_{\mathfrak{X}}(g) := \sum_{j=1}^{d} \zeta^{i_j}$.

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Fact

Two irreducible F-representations of G are equivalent if and only if their Brauer characters are equal.

The Brauer Character Table

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The square matrix

$$\left[\varphi_i(g_j)\right]_{1 \leq i,j \leq l}$$

is called the Brauer character table or p-modular character table of G.

Example (The 3-Modular Character Table of M_{11}, (James, ’73))

	$1a$	$2a$	$4a$	$5a$	$8a$	$8b$	$11a$	$11b$
φ_1	1	1	1	1	1	1	1	1
φ_2	5	1	-1	.	α	$\bar{\alpha}$	γ	$\bar{\gamma}$
φ_3	5	1	-1	.	$\bar{\alpha}$	α	$\bar{\gamma}$	γ
φ_4	10	2	2	.	.	.	-1	-1
φ_5	10	-2	.	.	β	$-\beta$	-1	-1
φ_6	10	-2	.	.	$-\beta$	β	-1	-1
φ_7	24	.	.	-1	2	2	2	2
φ_8	45	-3	1	.	-1	-1	1	1

$(\alpha = -1 + \sqrt{-2}, \beta = \sqrt{-2}, \gamma = (-1 + \sqrt{-11})/2)$

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The matrix $D = [d_{ij}]$ is the decomposition matrix of G.

Remarks

$\mathrm{IBr}_p(G)$ is linearly independent (in $\mathrm{Maps}(G_{p'}, \mathbb{C})$) and so the decomposition numbers are uniquely determined.

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Knowing $\mathrm{Irr}(G)$ and D is equivalent to knowing $\mathrm{Irr}(G)$ and $\mathrm{IBr}_p(G)$.

If G is p-soluble, D has shape

$$D = \left[\begin{array}{c} I_l \\ \hline D' \end{array}\right],$$

where I_l is the $(l \times l)$ identity matrix (Fong-Swan theorem).

Example: Decomposition Numbers of M_{11}

		φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8
		1	5	5	10	10	10	24	45
χ_1	1	1	.	.	.	.	.	.	.
χ_2	10	.	.	.	1	.	.	.	.
χ_3	10	.	.	.	.	1	.	.	.
χ_4	10	.	.	.	.	.	1	.	.
χ_5	11	1	1	1	.	.	.	.	.
χ_6	16	1	1	.	.	.	1	.	.
χ_7	16	1	.	1	.	1	.	.	.
χ_8	44	.	1	1	1	.	.	1	.
χ_9	45	.	.	.	.	.	.	.	1
χ_{10}	55	1	1	1	.	1	1	1	.

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1. For alternating groups: complete up to A_{17}
2. For groups of Lie type: only partial results
3. For sporadic groups up to McL and other "small" groups (of order $\leq 10^9$): *An Atlas of Brauer Characters*, Jansen, Lux, Parker, Wilson, 1995

What is the Modular Atlas Project?

The program to compute the modular (= Brauer) character tables of the ATLAS groups.

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Methods: GAP, MOC, Meat-Axe, Condensation

The Players

Wilson
Waki
Thackray
Ryba
Parker
Noeske
Neunhöffer
Müller
Lux
Lübeck
Jansen
James
H.
and many others

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He, Ru, Suz, $O'N$, Co_3, Co_2, Fi_{22}, HN, Fi_{23} (9 groups)

various authors (1988 – 2016)

STATE OF THE ART, CONT.

Grp	Characteristic	
	Known	Not Completely Known
Ly	7, 11, 31, 37, 67	2, 3*, 5*
Th	19	2–7, $13^\dagger$, $31^\dagger$
Co_1	7–13, 23	2, 3, 5
J_4	5, 7, 37	2, 3, 11, $23^\dagger$, $29^\dagger$, $31^\dagger$, $43^\dagger$
Fi'_{24}	11, 23	2–7, $13^\dagger$, $17^\dagger$, $29^\dagger$
B	11, 23	2–7, 13°, $17^\dagger$, 19°, 31°, 47°
M	17, 19, 23, 31	2–13, 29°, 41°, 47°, 59°, 71°

*: Known “up to condensation” (mod 3: Thackray, mod 5: Lux & Ryba)

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$^\circ$: Cyclic defect, degrees unknown

Partial Character Tables

Remaining Problems for Th (neglecting $p = 13, 31$)

p	No. irr. char's	No. known char's	missing
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3	16	14	2
5	41	33	8
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These contain bounds for the degrees of the missing irreducibles.

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Developed: 1984 – 1987

Projective Characters: Definition

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Expanding a projective character in $\mathrm{Irr}(G)$ yields a sum of columns of the decomposition matrix.

The Orthogonality Relations

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The Orthogonality Relations

Put $\mathbb{Z}[\mathrm{IBr}_p(G)] := \{\sum_{i=1}^{l} z_i \varphi_i \mid z_i \in \mathbb{Z}, 1 \leq i \leq l\}$
(generalised Brauer characters) and
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Theorem (Orthogonality Relations)

$\langle \varphi_i, \Phi_j \rangle' = \delta_{ij}$.

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Proof. Let $X_1, X_2 \in \mathbb{N}^{l \times l}$ be the matrices expressing $B_\mathcal{B}$ in $\mathrm{IBr}_p(G)$ and $B_\mathcal{P}$ in $\mathrm{IPr}_p(G)$, respectively. Then, by the orthogonality relations, $U = X_1 X_2^{tr}$.

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Thank you for your attention!