Computational Group Cohomology

Bangalore, November 2016

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NUI Galway, Ireland

Slides available at **http://hamilton.nuigalway.ie/Bangalore**

Password: **Dublin**

Outline

- Lecture 1: CW spaces and their (co)homology
- **Lecture 2: Algorithms for classifying spaces of groups**
- Lecture 3: Homotopy 2-types
- Lecture 4: Steenrod algebra
- Lecture 5: Curvature and classifying spaces of groups

But first: answers to some questions on yesterday's material

$\mathcal{A}_p(G)$ = poset of non-trivial elementary abelian p-subgroups in G.
$\Delta\mathcal{A}_p(G)$ = order (simplicial) complex of $\mathcal{A}_p(G)$

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4. GAP

   $$\Delta\mathcal{A}_3(S_{10}) \simeq (\vee_1^{25200}\,\mathbb{S}^1) \vee K$$

   with K a 2-dimensional CW space having: 53 1-cells, 872 2-cells, nonabelian $\pi_1 K$, $(\pi_1 K)_{ab} = \mathbb{Z}^{42}$.

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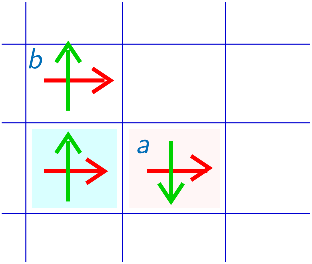

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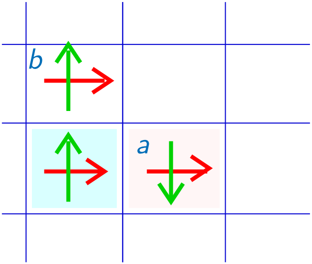

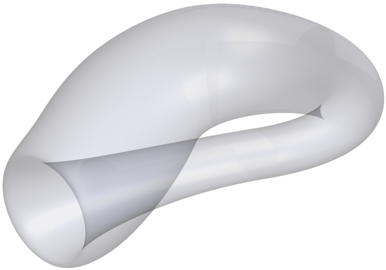

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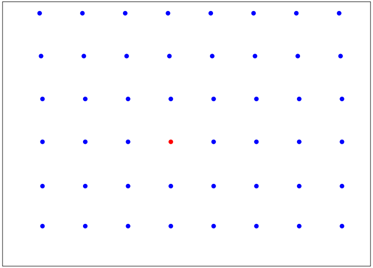

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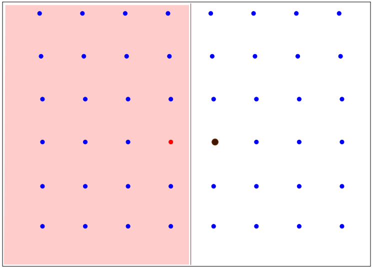

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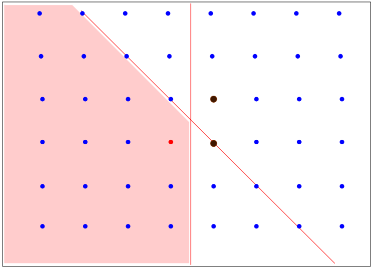

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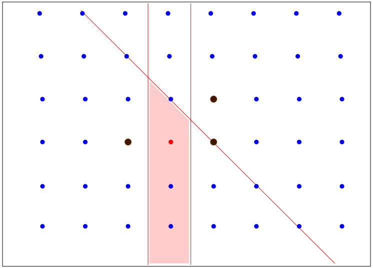

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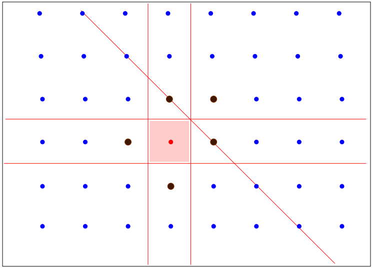

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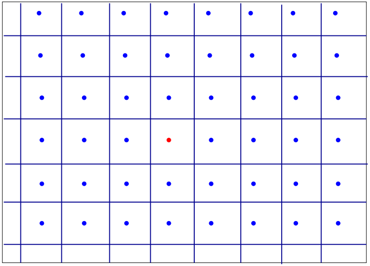

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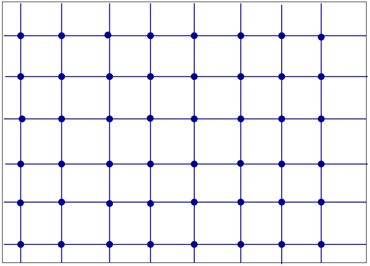

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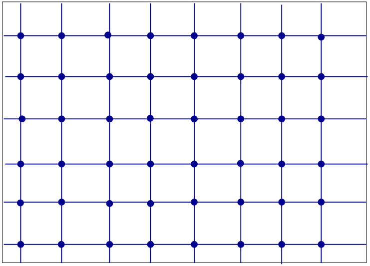

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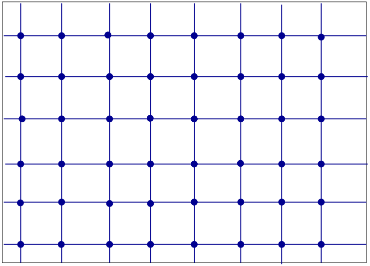

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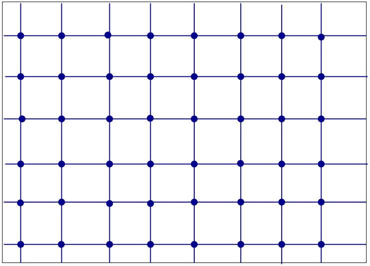

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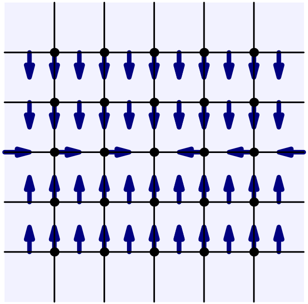

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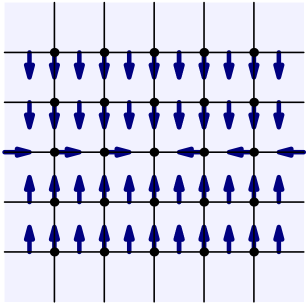

A GAP resolution $R_*^G = C_*(EG)$ stores

- $\mathrm{Rank}_{\mathbb{Z}G} R_n$
- $\partial_n(e_i^n)$ for free generators $e_i^n \in R_n^G$
- $h_n(g.e_i^n)$ for free generators $e_i^n \in R_n^G$ and $g \in G$

Crystallographic groups

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The resolution $R_*^G = C_*(EG)$ depends on $v \in \mathbb{R}^n$.

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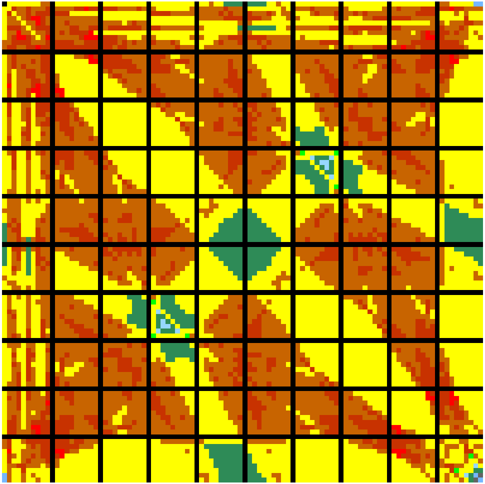

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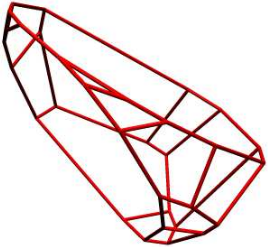

Finite groups

A representation $\rho\colon G \longrightarrow GL_n(\mathbb{R})$ and $v \in \mathbb{R}^n$ yields

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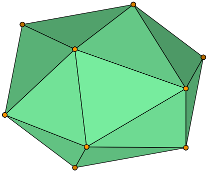
Example

Natural $\rho\colon A_4 \longrightarrow GL_4(\mathbb{R})$ and $v = (1, 2, 3, 4)$

$P =$ 

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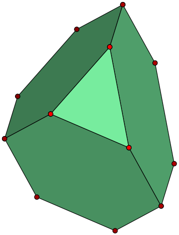
$P =$ 

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Example where C_*P is free

The group

$$G = \langle \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$$

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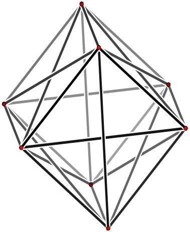

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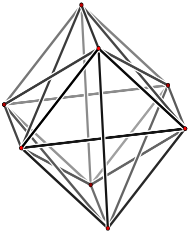

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$$\mathbf{Q} = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \colon \mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}, \mathbf{ikj} = \mathbf{1} \rangle.$$

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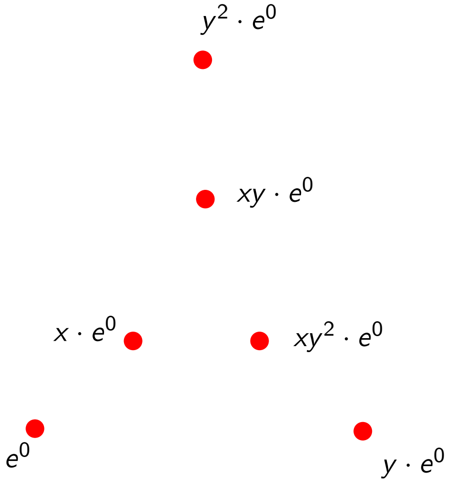
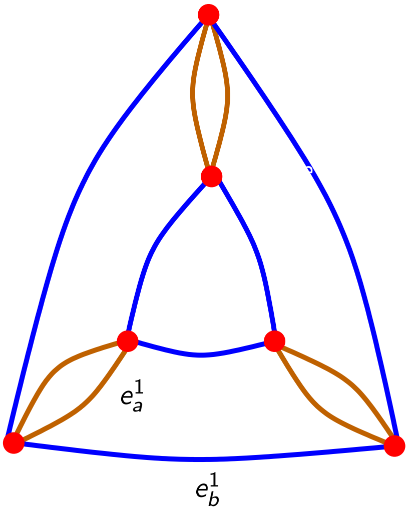
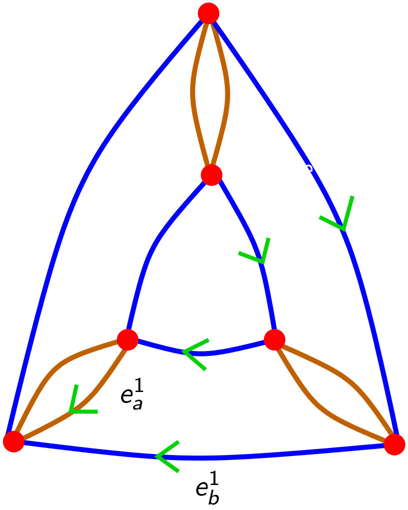
X^0 = one free orbit of vertices, $x = (1,2)$, $y = (1,2,3)$



$X^1 = X^0 \cup$ enough free orbits of edges to ensure $\pi_0(X^1) = 0$



Discrete vector field on X^1 ensures $\pi_0(X^1) = 0$



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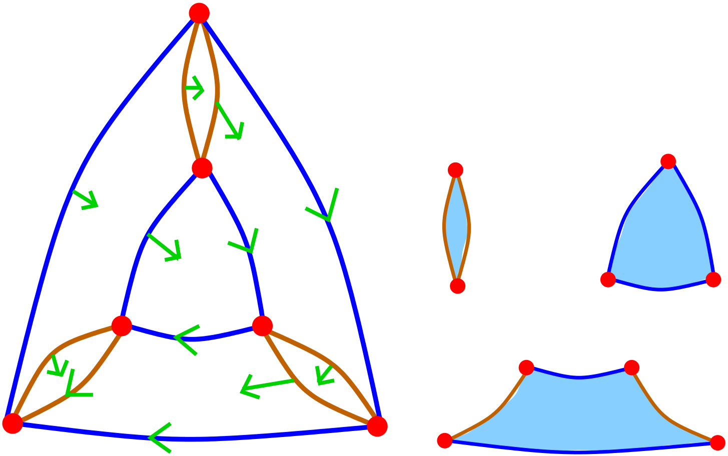

ensures that three orbits suffice.

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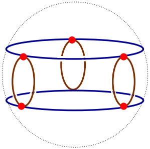

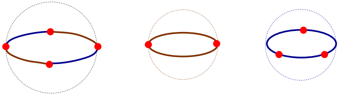

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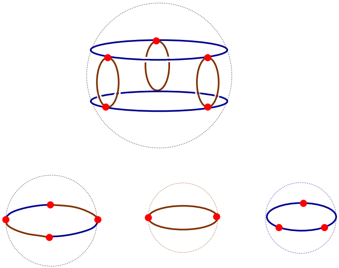

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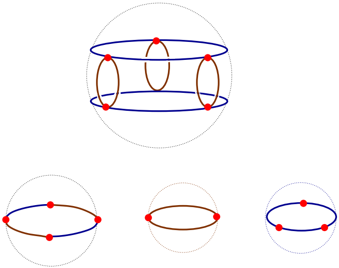

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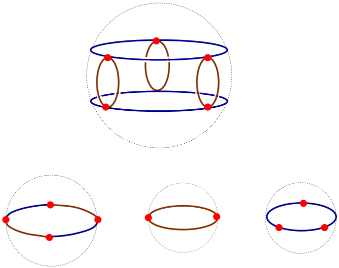

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gap> List([0..6],R!.dimension);
[ 1, 5, 15, 35, 70, 126, 196 ]
```

+ Cartan-Eilenberg double coset formula

For any finite group G there is a quotient

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Example

$$H_n(M_{24}, \mathbb{Z}) = \begin{cases} 0, & n = 1, 2 \\ \mathbb{Z}_{12}, & n = 3 \\ 0, & n = 4 \\ (\mathbb{Z}_2)^a \oplus (\mathbb{Z}_4)^b \oplus \mathbb{Z}_7, & n = 5 \end{cases}$$

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$$C_*\colon \to C_n \to C_{n-1} \to \cdots \to C_0$$

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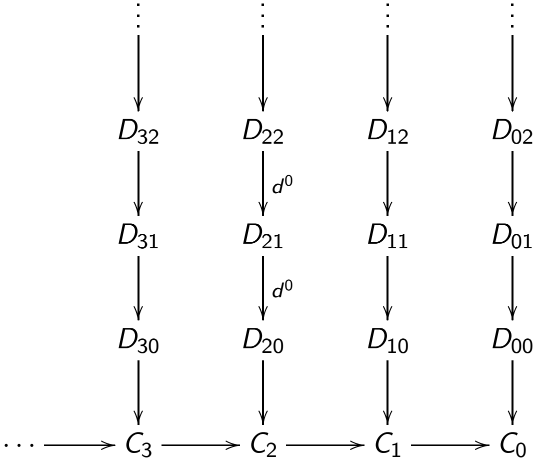
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$\cdots$ C_3 C_2 C_1 C_0

$$d^0 d^0 = 0$$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & D_{32} & \longrightarrow & D_{22} & \longrightarrow & D_{12} & \longrightarrow & D_{02} \\
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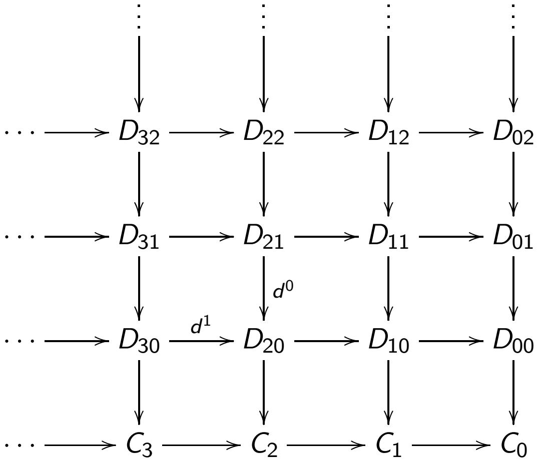

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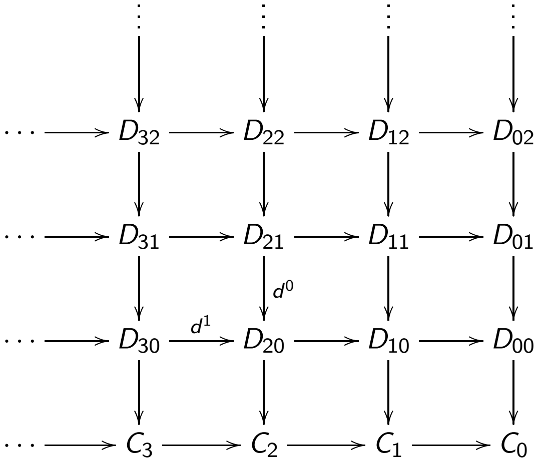

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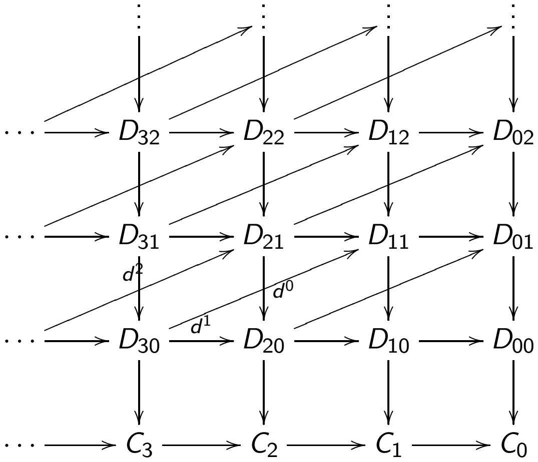
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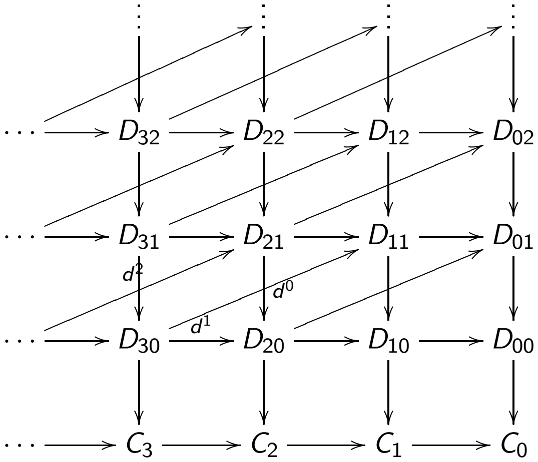

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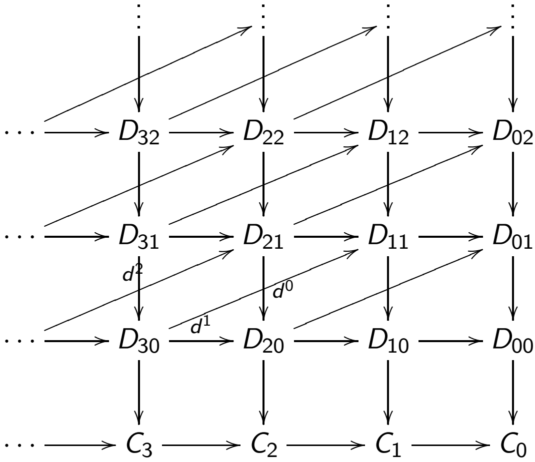

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A contracting homotopy on R_* can be constructed using homotopies on D_{p*} and C_*

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and hence a free $\mathbb{Z}G$-resolution

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$$R_*^G \text{ constructed from } R_*^Q \text{ and } R_*^A$$

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Free resolutions for

- crystallographic groups
- arbitrary Coxeter groups
- various arithmetic groups
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Let's give some details on the last example.

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Q **well-rounded** if $\mathrm{Span}(\ \mathrm{Min}(Q)\ ) = n$.

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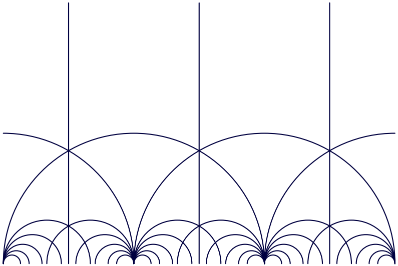

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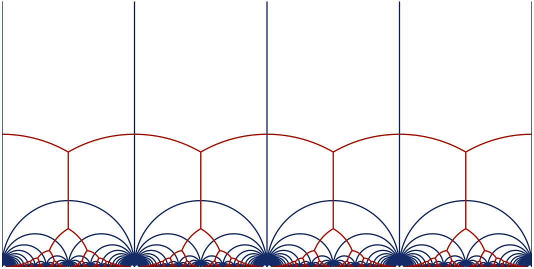
THEOREM (Ash) There is an $SL_n(\mathbb{Z})$-invariant $\binom{n}{2}$-dimensional homotopy retract $S^n_{wr} \subset S^n_{=1}$



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A discrete vector field on S^n_{wr} would yield a contracting homotopy on the resolution R.

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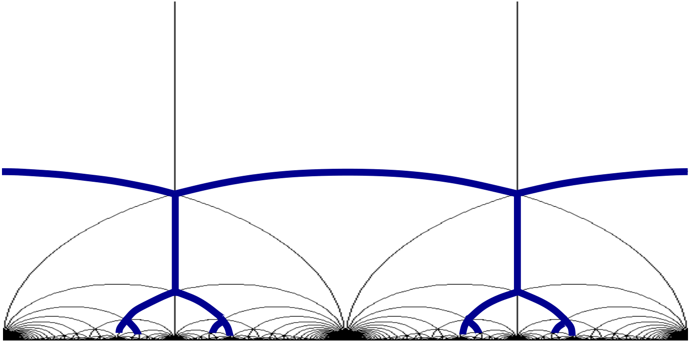

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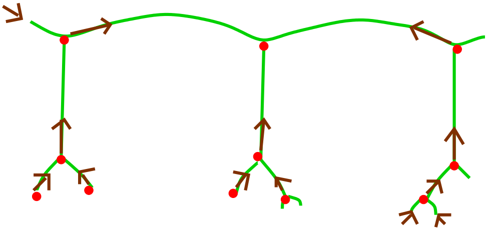
A discrete vector field on $\mathcal{T}$ with one critical cell is an algorithm for expressing an element $A \in SL_2(\mathbb{Z})$ as a word in S and T.



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$$A = \begin{pmatrix} 17 & 29 \\ 7 & 12 \end{pmatrix} \text{ in terms of } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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