p-quotient algorithm

- Go to Presentations
- Go to p-Group Generation

Conclusion Lecture 1

Things we have discussed in the first lecture:

- polycyclic groups, sequences, and series
- polycyclic generating sets (**pcgs**) and relative orders
- polycyclic presentations (**pcp**), power exponents, and consistency
- normal forms and collection
- consistency checks
- weighted polycyclic presentations (**wpcp**)

Conclusion Lecture 1

weighted polycyclic presentation (wpcp):

- all relative orders p
- induced polycyclic series is chief series
- relations are partitioned into definitions and non-definitions

Example

Consider

$$G = \mathrm{Pc}\langle\ x_1, \ldots, x_5 \ \mid\ x_1^2 = x_4,\ x_2^2 = x_3,\ x_3^2 = x_5,\ x_4^2 = x_5,\ x_5^2 = 1 \\ [x_2, x_1] = x_3,\ [x_3, x_1] = x_5\ \rangle.$$

Here $\{x_1, x_2\}$ is a minimal generating set, and we choose $[x_2, x_1] = x_3$ and $x_1^2 = x_4$ and $[x_3, x_1] = x_5$ as definitions for x_3, x_4, and x_5, respectively.

Lecture 2: how to compute a wpcp?

Lower exponent-p series

Lower exponent p-series

The **lower exponent-p series** of a p-group G is

$$G = P_0(G) > P_1(G) > \ldots > P_c(G) = 1$$

where each $P_{i+1}(G) = [G, P_i(G)]P_i(G)^p$; the p-**class** of G is c.

Important properties

- each $P_i(G)$ is characteristic in G;
- $P_1(G) = [G, G]G^p = \Phi(G)$, and $G/P_1(G) \cong C_p^d$ with $d = \text{rank}(G)$;
- each section $P_i(G)/P_{i+1}(G)$ is G-central and elementary abelian;
- if G has p-class c, then its nilpotency class is at most c;
- if θ is a homomorphism, then $\theta(P_i(G)) = P_i(\theta(G))$;
- G/N has p-class c if and only if $P_c(G) \leq N$;
- **weights:** any wpcp on $\{a_1, \ldots, a_n\}$ satisfies $a_i \in P_{\omega(a_i)}(G) \setminus P_{\omega(a_i)+1}(G)$.

Lower exponent-p series

Example 11

Consider

$$G = D_{16} = \mathrm{Pc}\langle a_1, a_2, a_3, a_4 \mid a_1^2 = 1,\ a_2^2 = a_3 a_4,\ a_3^2 = a_4,\ a_4^2 = 1,\ [a_2, a_1] = a_3,\ [a_3, a_1] = a_4\rangle.$$

Here we can read off:

- $P_0(G) = G$
- $P_1(G) = [G, G]G^2 = \langle a_3, a_4\rangle$
- $P_2(G) = [G, P_1(G)]P_1(G)^2 = \langle a_4\rangle$
- $P_3(G) = [G, P_2(G)]P_2(G)^2 = 1$

So G has 2-class 3.

Computing a wpcp of a p-group

p-quotient algorithm[3]

Input: a p-group $G = F/R = \langle x_1, \ldots, x_n \mid \mathcal{R} \rangle$

Output: a wpcp of G

Top-level outline:

1. compute wpcp of $G/P_1(G)$ and epimorphism $G \to G/P_1(G)$, then iterate:
2. given wpcp of $G/P_k(G)$ and epimorphism $G \to G/P_k(G)$, compute wpcp of $G/P_{k+1}(G)$ and epimorphism $G \to G/P_{k+1}(G)$;

For the second step, we use the so-called *p-cover* of $G/P_k(G)$.

More general: a "p-quotient algorithm" computes a consistent wpcp of the largest p-class k quotient (if it exists) of any finitely presented group.

[3] **Historically:** MacDonald (1974), Havas & Newman (1980), Newman & O'Brien (1996)

Computing a wpcp of $G/P_1(G)$

Note that $G/P_1(G)$ is elementary abelian.

Computing wpcp of $G/P_1(G)$

Input: a p-group $G = F/R = \langle x_1, \ldots, x_n \mid \mathcal{R} \rangle$
Output: a wpcp of $G/P_1(G)$ and epimorphism $\theta \colon G \to G/P_1(G)$

Approach:

1. abelianise relations, take exponents modulo p, write these in matrix M
2. compute solution space of M over GF(p)

Then:

- dimension d of solution space is rank of G, that is, $G/P_1(G) \cong C_p^d$
- generating set of $G/P_1(G)$ lifts to subset of given generators;

  set $G/P_1(G) = \mathrm{Pc}\langle a_1, \ldots, a_d \mid a_1^p = \ldots = a_d^p \rangle$ and define θ by

$$\theta(x_i) = a_i \quad \text{for} \quad i = 1, \ldots, d;$$

  images of $\theta(x_j)$ with $j > d$ are determined accordingly.

Computing a wpcp of $G/P_1(G)$

Example 12

$G = \langle x_1, \ldots, x_6 \mid x_6^{10},\ x_1x_2x_3,\ x_2x_3x_4, \ldots, x_4x_5x_6,\ x_5x_6x_1,\ x_1x_6x_2 \rangle$ and $p = 2$

Write coefficients of abelianised and mod-2 reduced equations as rows of matrix, use row-echelonisation, and determine that solution space has dimension 2:

$$\begin{pmatrix} 1&1&1&0&0&0\\ 0&1&1&1&0&0\\ 0&0&1&1&1&0\\ 0&0&0&1&1&1\\ 1&0&0&0&1&1\\ 1&1&0&0&0&1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1&0&0&0&1&1\\ 0&1&0&0&1&0\\ 0&0&1&0&0&1\\ 0&0&0&1&1&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix};$$

Modulo $P_1(G)$, this shows that $x_1 = x_5x_6$, $x_2 = x_5$, $x_3 = x_6$, $x_4 = x_5x_6$, and **Burnside's Basis Theorem** implies that $G = \langle x_5, x_6 \rangle$. Lastly, set

$$G/P_1(G) = \mathrm{Pc}\langle a_1, a_2 \mid a_1^2 = a_2^2 = 1 \rangle,$$

and define $\theta\colon G \to G/P_1(G)$ via $x_5 \mapsto a_1$ and $x_6 \mapsto a_2$.
This determines $\theta(x_1) = a_1a_2$, $\theta(x_2) = a_1$, $\theta(x_3) = a_2$, and $\theta(x_4) = a_1a_2$.

Compute wpcp for $G/P_{k+1}(G)$ from that of $G/P_k(G)$

Given:

- wpcp of d-generator p-group $G/P_k(G)$ and epimorphism $\theta\colon G \to G/P_k(G)$

Want:

- wpcp of $G/P_{k+1}(G)$ and epimorphism $G \to G/P_{k+1}(G)$

In the following:

- $H = G/P_k(G)$ and $K = G/P_{k+1}(G)$ and $Z = P_k(G)/P_{k+1}(G)$
- note that Z is elementary abelian, K-central, and $K/Z \cong H$

Approach: Construct a *covering* H^* of H such that *every* d-generator p-group L with $L/M \cong H$ and $M \leq L$ central elementary abelian, is a quotient of H^*.

Thus, the next steps are:

1. define p-cover H^* and determine a pcp of H^*;
2. make this presentation consistent;
3. construct K as quotient of H^* by enforcing defining relations of G.

p-covering group: definition

Theorem 13: p-covering group

Let H be a d-generator p-group; there is a d-generator p-group H^* with:

- $H^*/M \cong H$ for some central elementary abelian $M \trianglelefteq H^*$;
- if L is a d-generator p-group with $L/Y \cong H$ for some central elementary abelian $Y \leq L$, then L is a quotient of H^*.

The group H^* is unique up to isomorphism.

Proof.

Let $H = F/S$ with F free of rank d. Define $H^* = F/S^*$ with $S^* = [S, F]S^p$.

Now S/S^* is elementary abelian p-group, so H^* is (finite) d-generator p-group.

Let L be as in the theorem, and let $\psi\colon L \to H$ with kernel Y.
Let $\theta\colon F \to H$ with kernel S. Since F is free, θ factors through L, that is,
$\theta\colon F \stackrel{\varphi}{\to} L \stackrel{\psi}{\to} H$, and so $\varphi(S) \leq \ker\psi = Y$. This implies that $\varphi(S^*) = 1$.
In conclusion, φ induces surjective map from $H^* = F/S^*$ onto L.

If H^* and $\tilde{H}^*$ are two such covers, then each is an image of the other.

p-covering group: presentation

Given: a wpcp $\mathrm{Pc}\langle a_1, \ldots, a_m \mid \mathcal{S}\rangle$ for $H = G/P_k(G) \cong F/S$
and epimorphism $\theta\colon G \to H$ with $\theta(x_i) = a_i$ for $i = 1, \ldots, d$

Want: a wpcp for $H^* \cong F/S^*$ where $S^* = [S, F]S^p$

Recall: each of $a_{d+1}, \ldots, a_m$ occurs as right hand side of one relation in $\mathcal{S}$;
write $\mathcal{S} = \mathcal{S}_{\text{def}} \cup \mathcal{S}_{\text{nondef}}$ with $\mathcal{S}_{\text{nondef}} = \{s_1, \ldots, s_q\}$.

Theorem 14

Using the previous notation, $H^* = \mathrm{Pc}\langle a_1, \ldots, a_m, b_1, \ldots, b_q \mid \mathcal{S}^*\rangle$, where

$$\mathcal{S}^* = \mathcal{S}_{\text{def}} \cup \{s_1 b_1, \ldots, s_q b_q\} \cup \{b_1^p, \ldots, b_q^p\}.$$

Note: $M = \langle b_1, \ldots, b_q\rangle \trianglelefteq H^*$ is elementary abelian, central, and $H^*/M \cong H$.

(see Newman, Nickel, Niemeyer: "Descriptions of groups of prime-power order", 1998)

In practice: fewer new generators are introduced.

p-covering group: example

Example 15

If $H = \mathrm{Pc}\langle a_1, a_2 \mid a_1^2 = a_2^2 = 1 \rangle \cong C_2 \times C_2$, then

$$H^* = \mathrm{Pc}\langle a_1, a_2, b_1, b_2, b_3 \mid a_1^2 = b_1,\ a_2^2 = b_2,\ [a_1, a_2] = b_3,\ b_1^2 = b_2^2 = b_3^2 = 1 \rangle;$$

indeed, $H^* \cong (C_4 \times C_2) \colon C_4$, thus we have found a consistent wpcp!

Example 16

If $H = \mathrm{Pc}\langle a_1, a_2, a_3 \mid a_1^2 = a_3^2 = 1, a_2^2 = a_3, [a_2, a_1] = a_3 \rangle \cong D_8$, then

$$H^* = \mathrm{Pc}\langle a_1, a_2, a_3, b_1, \ldots, b_5 \mid \mathcal{T} \cup \{b_1^2, \ldots, b_5^2\} \rangle \quad \text{with}$$

$$\mathcal{T} = \{a_1^2 = b_1, a_2^2 = a_3 b_2, a_3^2 = b_3, [a_2, a_1] = a_3, [a_3, a_1] = b_4, [a_3, a_2] = b_5\};$$

this pcp has power exponents $[2, 2, 2, 2, 2, 2, 2, 2]$.

However, $H^* \cong (C_8 \times C_2) \colon C_4$, so presentation is **not consistent**!

Next step: make the presentation of H^* consistent.

p-covering group: consistency algorithm

By Theorem 8, the presentation $H^* = \mathrm{Pc}\langle u_1, \ldots, u_{m+q} \mid \mathcal{S}^* \rangle$ with $(u_1, \ldots, u_{m+q}) = (a_1, \ldots, a_m, b_1, \ldots, b_q)$ is consistent if and only if

$$u_k(u_j u_i) = (u_k u_j) u_i \qquad (1 \le i < j < k \le m+q)$$

$$(u_j^p) u_i = u_j^{p-1}(u_j u_i) \text{ and } u_j(u_i^p) = (u_j u_i) u_i^{p-1} \qquad (1 \le i < j \le m+q)$$

$$u_j(u_j^p) = (u_j^p) u_j \qquad (1 \le j \le m+q).$$

Consistency Algorithm[4]: find consistent presentation for H^*

- If each pair of words in the above "consistency checks" collects to the same normal word, then the presentation is consistent.
- Otherwise, the quotient of the two different words obtained from one of these conditions is formed and equated to the identity word: this gives a new relation which holds in the group.
- The pcp for H is consistent, so any new relation is an equation in the elementary abelian subgroup M generated by the new generators $\{b_1, \ldots, b_q\}$, which implies that one of these generators is redundant.

[4] **Historically:** Wamsley (1974), Vaughan-Lee (1984)

p-covering group: consistency algorithm

By Theorem 8, the presentation $H^* = \mathrm{Pc}\langle u_1, \ldots, u_{m+q} \mid \mathcal{S}^* \rangle$ with $(u_1, \ldots, u_{m+q}) = (a_1, \ldots, a_m, b_1, \ldots, b_q)$ is consistent if and only if

$$u_k(u_j u_i) = (u_k u_j) u_i \qquad (1 \leq i < j < k \leq m+q)$$

$$(u_j^p) u_i = u_j^{p-1}(u_j u_i) \text{ and } u_j(u_i^p) = (u_j u_i) u_i^{p-1} \qquad (1 \leq i < j \leq m+q)$$

$$u_j(u_j^p) = (u_j^p) u_j \qquad (1 \leq j \leq m+q).$$

Example 17

Consider $G = \mathrm{Pc}\langle u_1, u_2, u_3 \mid u_1^2 = u_2,\ u_2^2 = u_3,\ u_3^2 = 1,\ [u_2, u_1] = u_3 \rangle$.
The last test applied to u_1 yields

$$u_1^3 = (u_1^2) u_1 = u_2 u_1 = u_1 u_2 u_3 \quad \text{and} \quad u_1^3 = u_1(u_1^2) = u_1 u_2,$$

so $u_3 = 1$ in G, hence $G = \mathrm{Pc}\langle u_1, u_2 \mid u_1^2 = u_2,\ u_2^2 = 1 \rangle \cong C_4$.

Construct K from cover H^* of H

So what have we got so far...

- p-group $G = F/R = \langle x_1, \ldots, x_n \mid \mathcal{R} \rangle$
- consistent wpcp of $H = G/P_k(G) = \mathrm{Pc}\langle a_1, \ldots, a_m \mid \mathcal{S} \rangle$
- epimorphism $\theta \colon G \to H$ with $\theta(x_i) = a_i$ for $i = 1, \ldots, d$
- consistent wpcp of cover $H^* = \mathrm{Pc}\langle a_1, \ldots, a_m, b_1, \ldots, b_q \mid \mathcal{S}^* \rangle$; note that $H^*/M \cong H$ where $M = \langle b_1, \ldots, b_q \rangle$

Want:

- consistent wpcp of $K = G/P_{k+1}(G)$ and epimorphism $G \to G/P_{k+1}(G)$

Know:

- $K/Z \cong H$ where $Z = P_k(G)/P_{k+1}(G)$ is elementary abelian, central
- K is quotient of H^*

Idea:

- construct K as quotient of H^*: add relations enforced by G to $\mathcal{S}^*$

Construct K from cover H^* of H

So what have we got so far...

- p-group $G = F/R = \langle x_1, \ldots, x_n \mid \mathcal{R} \rangle$
- consistent wpcp of $H = G/P_k(G) = \mathrm{Pc}\langle a_1, \ldots, a_m \mid \mathcal{S} \rangle$
- epimorphism $\theta\colon G \to H$ with $\theta(x_i) = a_i$ for $i = 1, \ldots, d$
- consistent pcp of cover $H^* = \mathrm{Pc}\langle a_1, \ldots, a_m, b_1, \ldots, b_q \mid \mathcal{S}^* \rangle$; note that $H^*/M \cong H$ where $M = \langle b_1, \ldots, b_q \rangle$

Enforcing relations of G:

- know that $K = G/P_{k+1}(G)$ is quotient of H^*
- lift $\theta\colon G \to H$ to $\hat{\theta}\colon F \to H^*$ such that $\hat{\theta}(x_i) = a_i$ for $i = 1, \ldots, d$
- for every relator $r \in \mathcal{R}$ compute $n_r = \hat{\theta}(r) \in M$; let L be the subgroup of M generated by all these n_r
- by von Dyck's Theorem $H^*/L \to K$ and $G \to H^*/L$ are surjective; since K is the largest p-class $k+1$ quotient of G, we deduce $K = H^*/L$

Finally: find consistent wpcp of $K = H^*/L$ and get epimorphism $G \to K$

Big example: p-quotient algorithm in action

Let $G = \langle x, y \mid [[y,x],x] = x^2,\ (xyx)^4,\ x^4,\ y^4,\ (yx)^3y = x\rangle$ and $p = 2$.

First round:

- compute $G/P_1(G)$ using abelianisation and row-echelonisation:

  obtain $H = G/P_1(G) \cong \mathrm{Pc}\langle a_1, a_2 \mid a_1^2 = a_2^2 = 1\rangle$
  and epimorphism $\theta\colon G \to H$, which is defined by $(x,y) \to (a_1, a_2)$.
- construct covering of H by adding new generators and tails:

  $H^* = \mathrm{Pc}\langle a_1, \ldots, a_5 \mid a_1^2 = a_3, a_2^2 = a_4, [a_2, a_1] = a_5,\ a_3^2 = a_4^2 = a_5^2 = 1\rangle$
- the consistency algorithm shows that this presentation is consistent
- evaluate relations of G in H^*:
  - $1 = [[a_2,a_1],a_1] = \hat\theta([[y,x],x]) = \hat\theta(x^2) = a_1^2 = a_3$ forces $a_3 = 1$
  - $(xyx)^4, x^4, y^4$ impose no conditions
  - $a_1 a_3 = \hat\theta((yx)^3y) = \hat\theta(x) = a_1$ also forces $a_3 = 1$
- construct $G/P_2(G)$ as $H^*/\langle a_3\rangle$; after renaming a_4, a_5:

  $$G/P_2(G) \cong \mathrm{Pc}\langle a_1, \ldots, a_4 \mid a_1^2 = 1, a_2^2 = a_4, [a_2, a_1] = a_3,\ a_3^2 = a_4^2 = 1\rangle$$

  and epimorphism $G \to G/P_2(G)$ defined by $(x,y) \to (a_1, a_2)$.

Big example: p-quotient algorithm in action

$G/P_2(G) = \mathrm{Pc}\langle a_1, \ldots, a_4 \mid a_1^2 = 1, a_2^2 = a_4, [a_2, a_1] = a_3,\ a_3^2 = a_4^2 = 1\rangle$

Second round:

- construct covering of $H = G/P_2(G)$ by adding new generators and tails:
$$H^* = \mathrm{Pc}\langle a_1, \ldots, a_{12} \mid a_1^2 = a_{12}, a_2^2 = a_4, a_3^2 = a_{11}, a_4^2 = a_{10}, [a_2, a_1] = a_3, [a_3, a_1] = a_5, [a_3, a_2] = a_6, [a_4, a_1] = a_7, [a_4, a_2] = a_8, [a_4, a_3] = a_9,\ a_5^2 = \ldots = a_{12}^2 = 1\rangle$$
- the consistency algorithm shows only the following inconsistencies:
  - $a_2(a_2a_2) = a_2a_4$ and $(a_2a_2)a_2 = a_4a_2 = a_2a_4a_8$ $\implies \boldsymbol{a_8 = 1}$
  - $a_2(a_1a_1) = a_2a_{12}$ and $(a_2a_1)a_1 = a_1a_2a_3a_1 = \ldots = a_2a_5a_{11}a_{12}$ $\implies \boldsymbol{a_5a_{11} = 1}$
  - $a_2(a_2a_1) = a_1a_2^2a_3^2a_6 = a_1a_4a_6a_{11}$ and $(a_2a_2)a_1 = a_1a_4a_7$ $\implies \boldsymbol{a_6a_7a_{11} = 1}$
  - $a_3(a_2a_2) = a_3a_4$ and $(a_3a_2)a_2 = a_2a_3a_6a_2 = a_2^2a_3a_6^2 = a_3a_4a_9$ $\implies \boldsymbol{a_9 = 1}$
- removing redundant gens (and renaming), we obtain the consistent wpcp
$$H^* = \mathrm{Pc}\langle a_1, \ldots, a_8 \mid a_1^2 = a_8, a_2^2 = a_4, a_3^2 = a_7, a_4^2 = a_6,\ a_5^2 = \ldots = a_8^2 = 1, [a_2, a_1] = a_3, [a_3, a_1] = a_7, [a_3, a_2] = a_5a_7, [a_4, a_1] = a_5\rangle$$

Big example: p-quotient algorithm in action

Still second round:

- $G = \langle x, y \mid [[y,x],x] = x^2, (xyx)^4, x^4, y^4, (yx)^3 y = x\rangle$ and $p = 2$;
- epimorphism $\theta\colon G \to H$ onto $H = G/P_2(H)$ defined by $(x,y) \to (a_1, a_2)$
- $H^* = \mathrm{Pc}\langle a_1, \ldots, a_8 \mid a_1^2 = a_8, a_2^2 = a_4, a_3^2 = a_7, a_4^2 = a_6,\ a_5^2 = \ldots = a_8^2 = 1$
  $[a_2,a_1] = a_3, [a_3,a_1] = a_7, [a_3,a_2] = a_5a_7, [a_4,a_1] = a_5\rangle$

Evaluate relations of G in H^*:

- $a_7 = [[a_2,a_1],a_1] = \hat\theta([[y,x],x]) = \hat\theta(x^2) = a_1^2 = a_8$ forces $a_7 = a_8$
- $(xyx)^4$ forces $a_6 = 1$; x^4 and y^4 impose no condition
- $\hat\theta((yx)^3y) = \hat\theta(x)$ forces $a_7a_8 = 1$

Now construct $G/P_3(G)$ as $H^*/\langle a_7a_8, a_6\rangle$; after renaming:

$$G/P_3(G) = \mathrm{Pc}\langle a_1, \ldots, a_6 \mid a_1^2 = a_6, a_2^2 = a_4, a_3^2 = a_6, a_4^2 = 1,\ a_5^2 = a_6^2 = 1,$$
$$[a_2,a_1] = a_3, [a_3,a_1] = a_6, [a_3,a_2] = a_5a_6, [a_4,a_1] = a_5\rangle$$

and the epimorphism $G \to G/P_3(G)$ is defined by $(x,y) \to (a_1,a_2)$.

Big example: p-quotient algorithm in action

In conclusion:

We started with

$$G = \langle x, y \mid [[y,x],x] = x^2, (xyx)^4, x^4, y^4, (yx)^3 y = x \rangle$$

and computed $G/P_3(G)$ as

$$\begin{aligned} \mathrm{Pc}\langle a_1, \ldots, a_6 \mid\ & a_1^2 = a_6,\ a_2^2 = a_4,\ a_3^2 = a_6,\ a_4^2 = a_5^2 = a_6^2 = 1, \\ & [a_2,a_1] = a_3, [a_3,a_1] = a_6, [a_3,a_2] = a_5 a_6, [a_4,a_1] = a_5 \rangle \end{aligned}$$

with epimorphism $G \to G/P_3(G)$ defined by $(x,y) \to (a_1,a_2)$.

One can check that $|G| = |G/P_3(G)| = 2^6$, hence $G \cong G/P_3(G)$.

In particular, we have found a consistent wpcp for G.

In general: if our input group is a finite p-group, then the p-quotient algorithm constructs a consistent wpcp of that group.

Motivation and Application: Burnside problem

Burnside Problems

- **Generalised Burnside Problem** (GBP), 1902:
  Is every finitely generated torsion group finite?
- **Burnside Problem** (BP), 1902:
  Let $B(d,n)$ be the largest d-generator group with $g^n = 1$ for all $g \in G$.
  Is this group finite? If so, what is its order?
- **Restricted Burnside Problem** (RBP), $\sim$1940:
  What is order of largest finite quotient $R(d,n)$ of $B(d,n)$, if it exists?

- Golod-Šafarevič (1964): answer to GBP is "no";
  (cf. Ol'shanskii's Tarski monster)
- Various authors: $B(d,n)$ is finite for $n = 2, 3, 4, 6$, but in no other cases with $d > 1$ is it known to be finite; is $B(2,5)$ finite?
- Higman-Hall (1956): reduced (RBP) to prime-power n.
- Zel'manov (1990-91): $R(d,n)$ always exists! (**Fields medal 1994**)

Motivation and Application: Burnside problem

Burnside groups:

- $B(d,n) = \langle x_1, \ldots, x_d \mid g^n = 1$ for all words g in $x_1, \ldots, x_n \rangle$
- $R(d,n)$ largest finite quotient of $B(d,n)$; exists by Zel'manov

Recall: the p-quotient algorithm computes a consistent wpcp of the largest p-class k quotient (if it exists) of any finitely presented group.

Implementations of the p-quotient algorithm have been used to determine the order and compute pcp's for various of these groups.

Group	Order	Authors
$B(3,4)$	2^{69}	Bayes, Kautsky & Wamsley (1974)
$R(2,5)$	5^{34}	Havas, Wall & Wamsley (1974)
$B(4,4)$	2^{422}	Alford, Havas & Newman (1975)
$R(3,5)$	5^{2282}	Vaughan-Lee (1988); Newman & O'Brien (1996)
$B(5,4)$	2^{2728}	Newman & O'Brien (1996)
$R(2,7)$	7^{20416}	O'Brien & Vaughan-Lee (2002)