Computational Group Cohomology

Bangalore, November 2016

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Slides available at **http://hamilton.nuigalway.ie/Bangalore**

Password: **Galway**

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- Hecke operators, Bredon homology, Tate cohomology ...

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- uses information about an explicit contracting homotopy

$$X \simeq *$$

when computing cohomology groups, Steenrod algebras etc.

Outline

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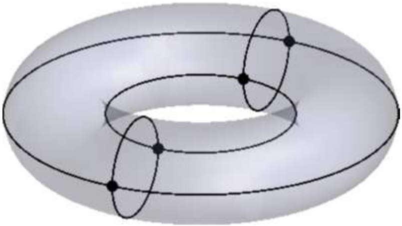
- Lecture 4: Steenrod algebras of finite 2-groups
- Lecture 5: Curvature and classifying spaces of groups

JHC Whitehead

CW spaces
Simple homotopy theory
Crossed modules
...

CW space

Hausdorff space X with open subsets $e_\lambda \subset X$ for $\lambda \in \Lambda$



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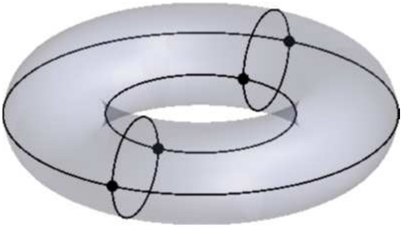

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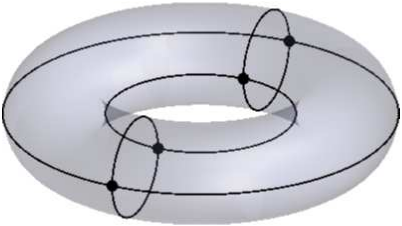

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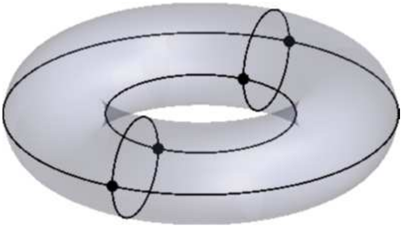

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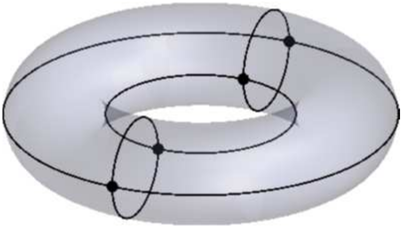

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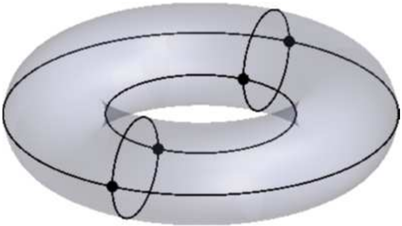

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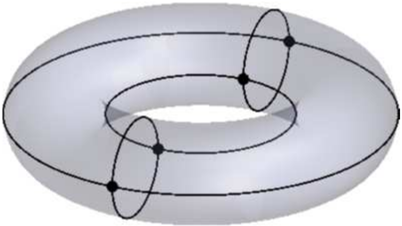

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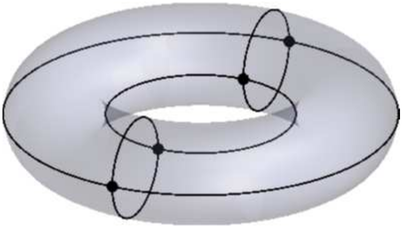
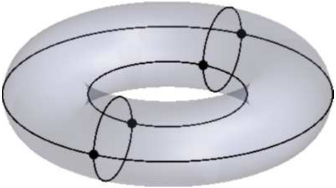
- some extra topological conditions if Λ is infinite



Regular CW space

Attaching maps $\phi_\lambda \colon \mathbb{D}^n \to e_\lambda$ are homeomorphisms in this case.



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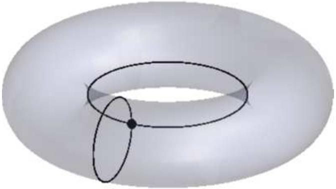

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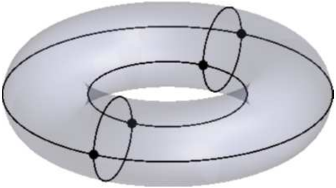

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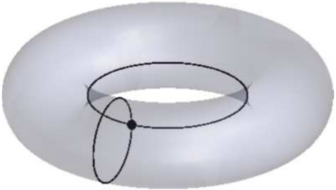

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Represented in GAP as a component object X:

- X!.boundaries[n+1][k] is a list of integers $[t, a_1, ..., a_t]$ recording that the a_ith cell of dimension $n-1$ lies in the boundary of the kth cell of dimension n.

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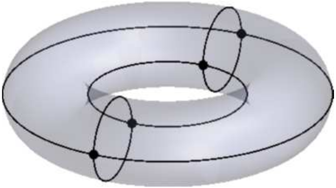

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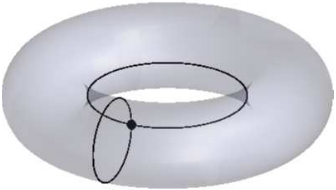

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Given a set S of points randomly sampled from an unknown manifold M, what can we infer about the topology of M?

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For instance, $S \subset M \subset \mathbb{E}^2$.

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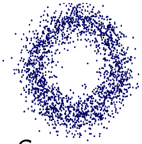

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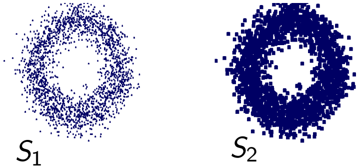

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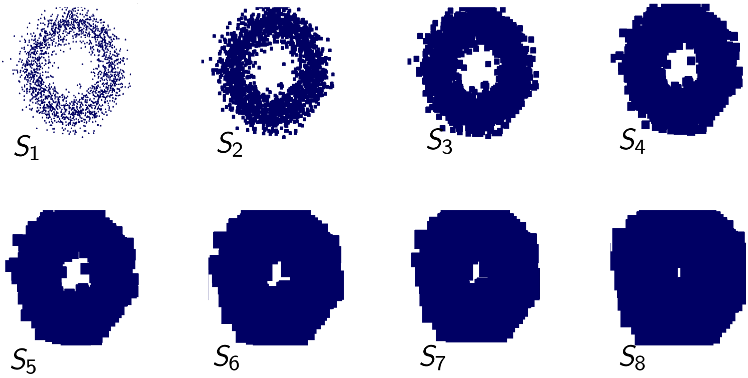

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These numbers are consistent with the sample coming from some region with the homotopy type of a circle.

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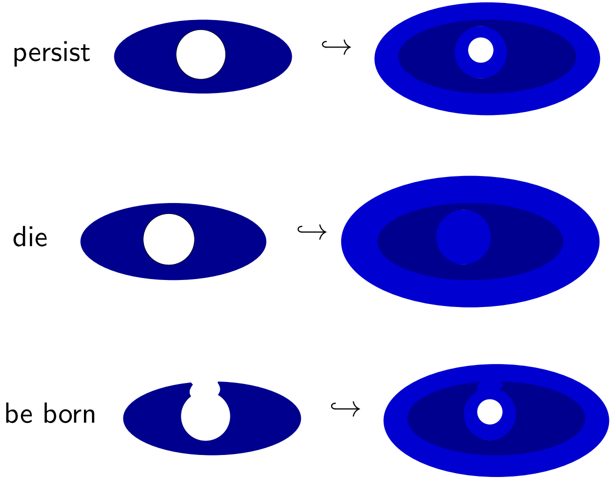

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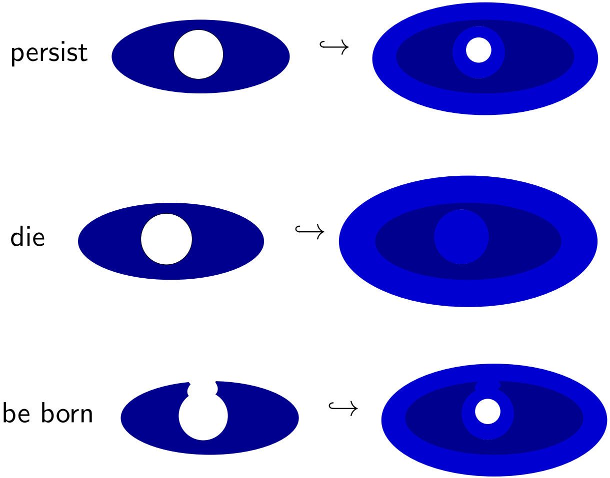

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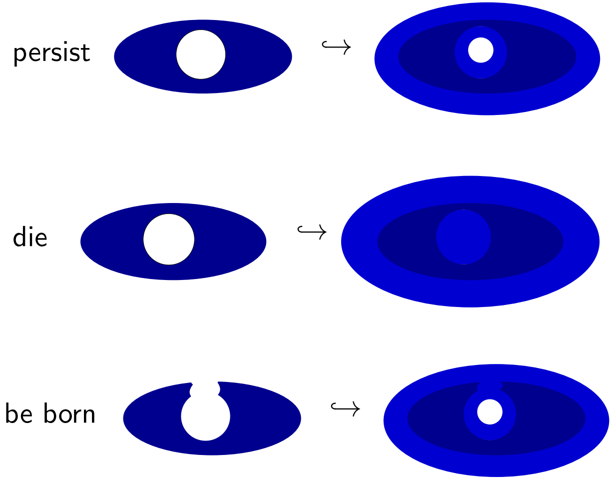

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$$(\beta_n^{s\,t}) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

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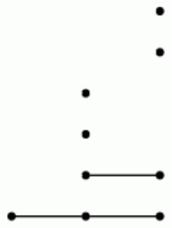
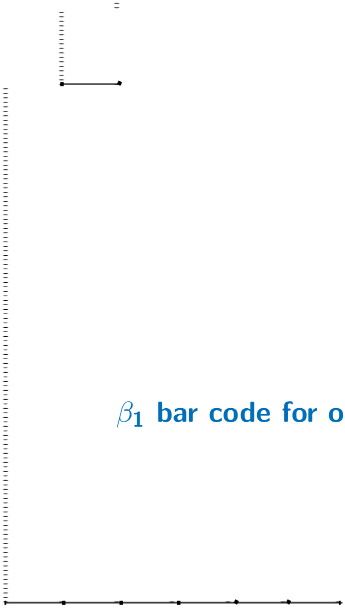
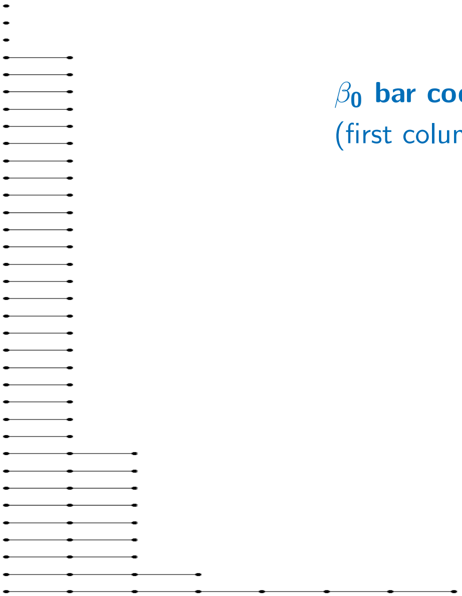


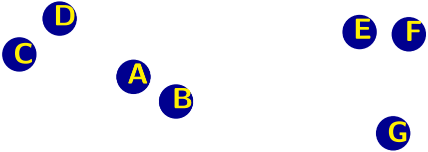
β_1 bar code for our example



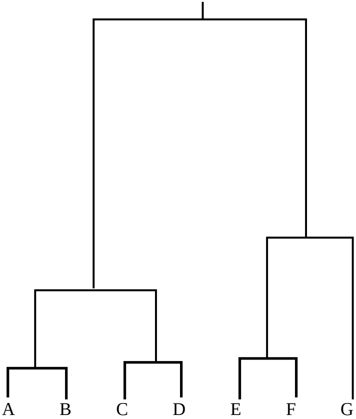


β_0 bar code for our example

(first column cropped)



β_0 bar codes could be enhanced to dendrograms



Second applied topology toy example

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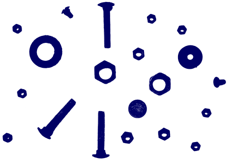

$$F : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \mapsto (r, g, b)$$

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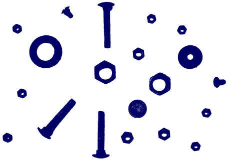

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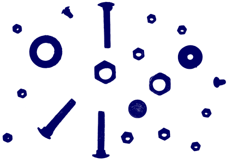

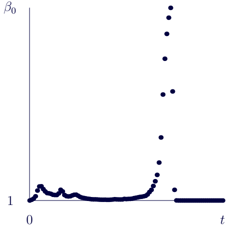

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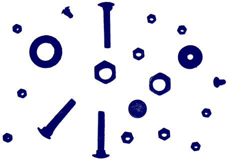

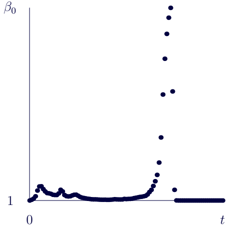

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$$\begin{aligned} \beta_0(X_t) &= \text{number of connected components of } X_t \\ &= \text{rank } H_0(X_t, \mathbb{Z}) \end{aligned}$$

Homology functors

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for $s \leq t$.

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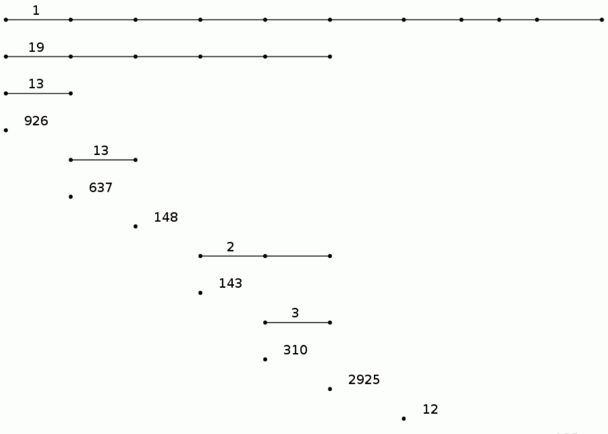

Homology functors (degree 1)

$X_s \hookrightarrow X_t$ induces

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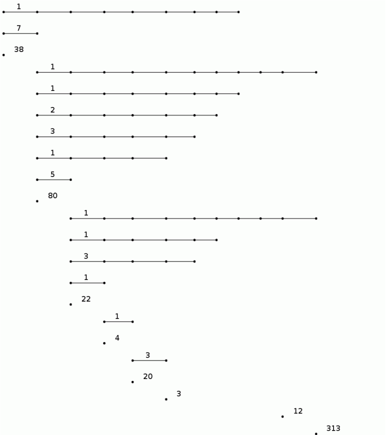

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$$\partial_n \partial_{n+1} = 0$$

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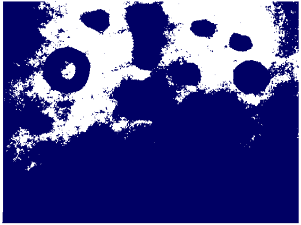
Sign of $\epsilon_{\lambda,\mu}$ chosen (non-uniquely) to ensure

$$\partial_n \partial_{n+1} = 0$$

Homology of a regular CW space X

$$H_n(X,\mathbb{Z}) = H_n(C_*X) = \ker \partial_n / \text{image}\, \partial_{n+1}$$

Example

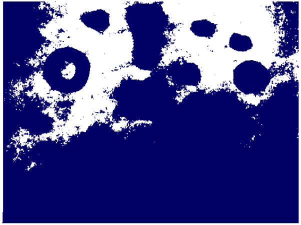
$X =$ 

$$C_3X \xrightarrow{\partial_3} C_2X \xrightarrow{\partial_2} C_1X \xrightarrow{\partial_1} C_0X$$

$$0 \longrightarrow \mathbb{Z}^{249702} \longrightarrow \mathbb{Z}^{509417} \longrightarrow \mathbb{Z}^{259601}$$

Example

$X =$



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$$0 \longrightarrow \mathbb{Z}^{249702} \longrightarrow \mathbb{Z}^{509417} \longrightarrow \mathbb{Z}^{259601}$$

The matrix of ∂_1 has 132245162617 entries.

Direct application of Smith Normal Form is not so practical in such a situation.

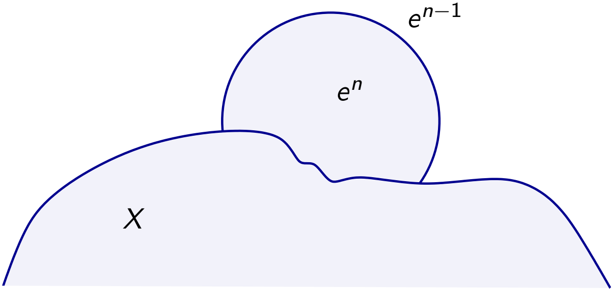
A simple homotopy collapse

Write

$$Y \searrow X$$

if

$$Y = X \cup e^n \cup e^{n-1}$$



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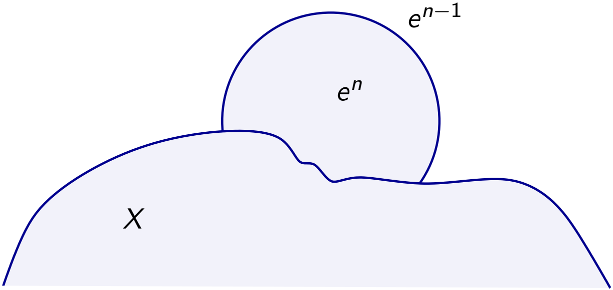
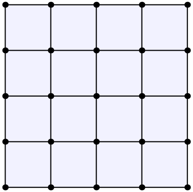


Such a collapse is recorded as an arrow $e^{n-1} \longrightarrow e^n$.

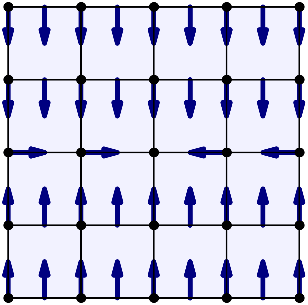
Example

The homotopy equivalence $[0,4] \times [0,4] \simeq *$



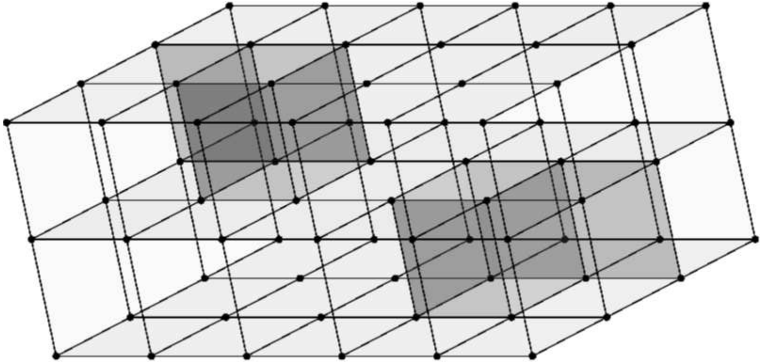
$\simeq *$

can be realized as a collection of simple homotopy collapses

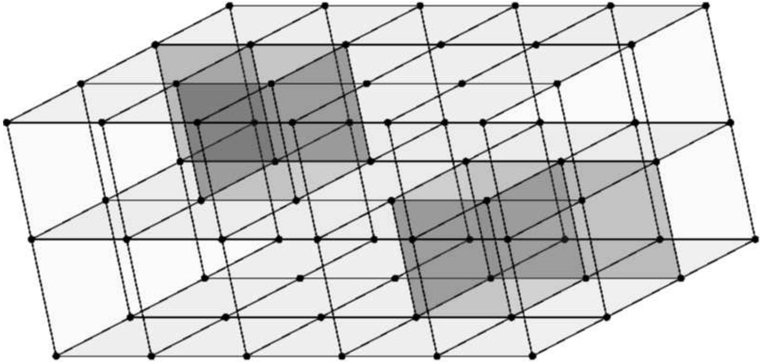


Bing's house

Bing's house



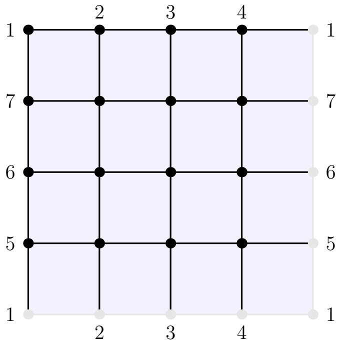
Bing's house

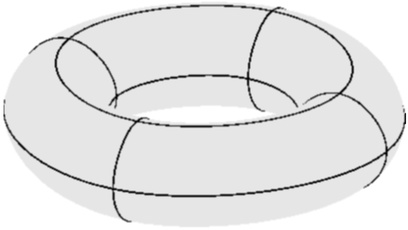


The homotopy equivalence

$$\text{Bing's house} \simeq *$$

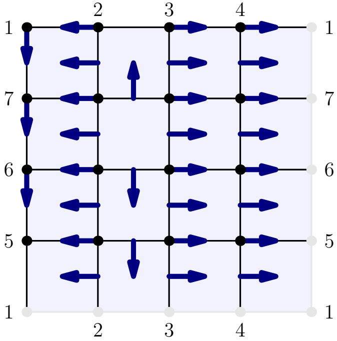
is not representable as a collection of simple homotopy collapses.

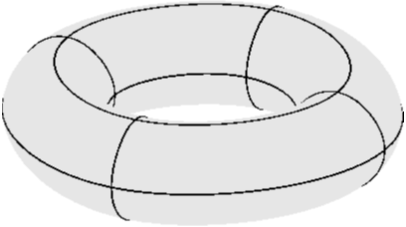

2
3
4
1
1
7
7
6
6
5
5
1
1
2
3
4



A discrete vector field

is a collection of arrows $e^{n-1} \longrightarrow e^n$ with e^{n-1} in the boundary of e^n and with any cell involved in at most one arrow.



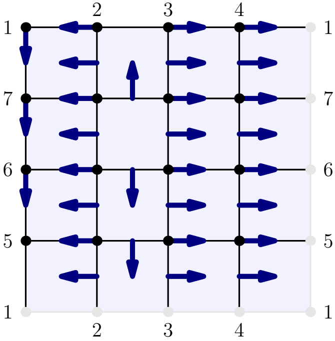


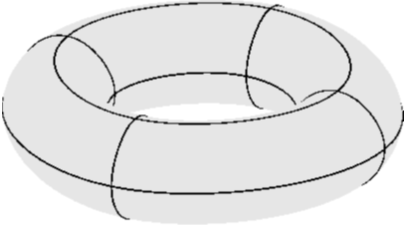
A discrete vector field

is a collection of arrows $e^{n-1} \longrightarrow e^n$ with e^{n-1} in the boundary of e^n and with any cell involved in at most one arrow. It is **admissible** if there is no chain

$$\cdots (e_1^{n-1} \to e_1^n), (e_2^{n-1} \to e_2^n), (e_3^{n-1} \to e_3^n), \cdots$$

with each e_{i+1}^{n-1} in the boundary of e_i^n and with infinitely many (not necessarily distinct) terms to the right.



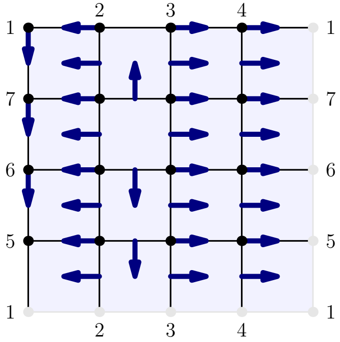


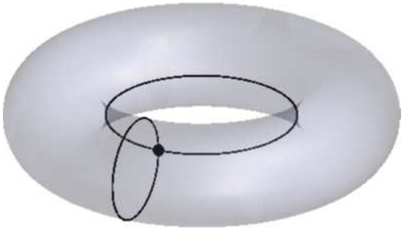
Theorem

A regular CW complex X with admissible discrete vector field represents a homotopy equivalence

$$X \xrightarrow{\simeq} Y$$

where Y is a CW complex whose cells correspond to those of X that are neither source nor target of any arrow.





Theorem

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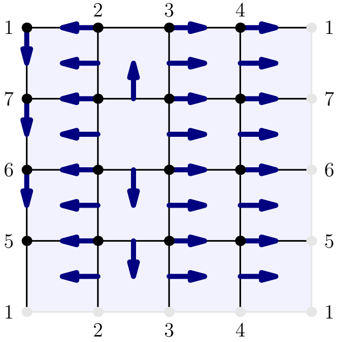
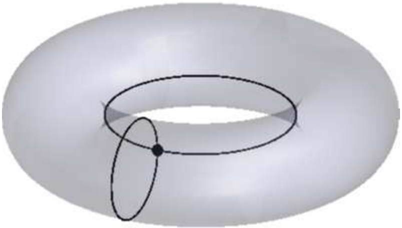
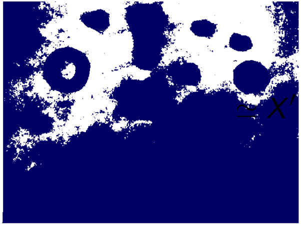
where Y is a CW complex whose cells correspond to those of X that are neither source nor target of any arrow. Such cells of X are **critical**.





Second toy example revisited

$X =$  $\simeq X'$

$$C_3X' \xrightarrow{\partial_3} C_2X' \xrightarrow{\partial_2} C_1X' \xrightarrow{\partial_1} C_0X'$$

$$0 \longrightarrow \mathbb{Z}^0 \longrightarrow \mathbb{Z}^{476} \longrightarrow \mathbb{Z}^{362}$$

```
gap> M:=ReadImageAsPureCubicalComplex("file.png",300);
Pure cubical complex of dimension 2.
```

```
gap> M:=ReadImageAsPureCubicalComplex("file.png",300);
Pure cubical complex of dimension 2.

gap> Y:=RegularCWComplex(M);
Regular CW-complex of dimension 2
```

```
gap> M:=ReadImageAsPureCubicalComplex("file.png",300);
Pure cubical complex of dimension 2.

gap> Y:=RegularCWComplex(M);
Regular CW-complex of dimension 2

gap> C:=ChainComplexOfRegularCWComplex(Y);
Chain complex of length 2 in characteristic 0 .

gap> List([0..2],C!.dimension);
[ 259601, 509417, 249702 ]
```

```
gap> M:=ReadImageAsPureCubicalComplex("file.png",300);
Pure cubical complex of dimension 2.

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gap> C:=ChainComplexOfRegularCWComplex(Y);
Chain complex of length 2 in characteristic 0 .

gap> List([0..2],C!.dimension);
[ 259601, 509417, 249702 ]

gap> MM:=ContractedComplex(M);;
gap> YY:=RegularCWComplex(MM);;
gap> YY:=ContractedComplex(YY);;
gap> CC:=ChainComplex(YY);;
gap> List([0..2],CC!.dimension);
[ 362, 476, 0 ]
```

Chain equivalence

Let X be a regular CW-space with discrete vector field. Let D_n denote the subgroup of C_nX freely generated by critical n-cells.

There are commuting homomorphisms

$$\begin{array}{ccc} C_nX & \xrightarrow{\phi_n} & D_n \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \delta_n} \\ C_{n-1}X & \xrightarrow{\phi_{n-1}} & D_n \end{array} \qquad \begin{array}{ccc} C_nX & \xleftarrow{\psi_n} & D_n \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \delta_n} \\ C_{n-1}X & \xleftarrow{\psi_{n-1}} & D_n \end{array}$$

satisfying

$$\partial_n\partial_{n+1} = 0, \qquad \delta_n\delta_{n+1} = 0$$

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satisfying

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and inducing isomorphisms

$$H_n(\phi_*)\colon H_n(C_*X) \xrightarrow{\cong} H_n(D_*)$$

$$H_n(\psi_*) = H_n(\phi_*)^{-1}\colon H_n(D_*) \xrightarrow{\cong} H_n(C_*X)$$

$$\mathbf{H^n(G,\mathbb{Z}) = H^n(Hom_{\mathbb{Z}G}(C_*X,\mathbb{Z}))}$$

G	group
X	contractible CW-space with free G-action

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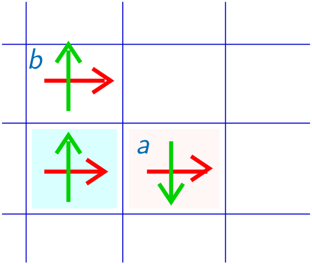
Example

$G = \langle a, b | bab = a \rangle, \quad X = \mathbb{R}^2,$



$$\mathbf{H^n(G, \mathbb{Z}) = H^n(Hom_{\mathbb{Z}G}(C_*X, \mathbb{Z}))}$$

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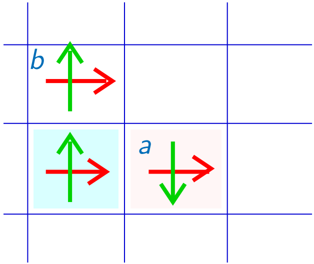
X	contractible CW-space with free G-action

Example

$$G = \langle a, b | bab = a \rangle, \quad X = \mathbb{R}^2, \qquad G \setminus X = \text{Klein bottle}$$



$$\mathbf{H^n(G,\mathbb{Z}) = H^n(Hom_{\mathbb{Z}G}(C_*X,\mathbb{Z}))}$$

G	group
X	contractible CW-space with free G-action
$H^n(-,\mathbb{Z})$	contravariant functor *Groups* $\longrightarrow$ *Abelian groups*

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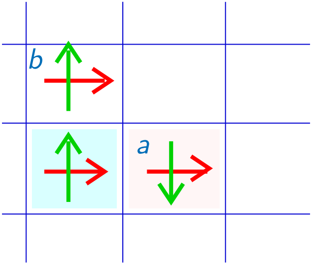

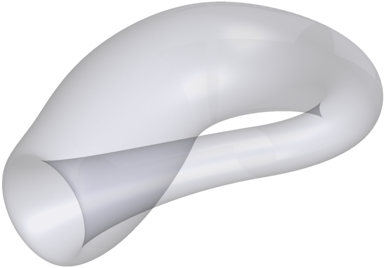

Contracting homotopy

Exactness of the free $\mathbb{Z}G$-resolution

$$R_* = C_*X : \quad \cdots \longrightarrow R_n \longrightarrow R_{n-1} \longrightarrow \cdots \longrightarrow R_0$$

is encoded as $\mathbb{Z}$-linear homomorphisms $h_n \colon R_n \longrightarrow R_{n+1}$ satisfying

$$h_{n-1}\partial_n + \partial_{n+1}h_n \qquad (n \geq 1).$$

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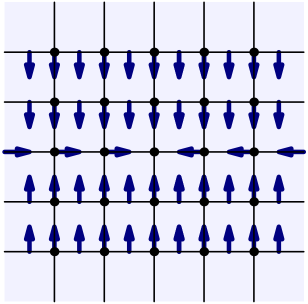
Example continued

h_n can encode a contracting distrete vector field on X



Classical homological algebra

Given

$$x \in \ker(R_n \xrightarrow{\partial_n} R_{n-1})$$

we can, by exactness, choose some

$$\tilde{x} \in R_{n+1}$$

such that

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such that

$$\partial_{n+1}\tilde{x} = x$$

Constructive homological algebra

We can simply set

$$\tilde{\mathbf{x}} = \mathbf{h_n}(\mathbf{x})$$

A group theoretic example of vector fields

$\mathcal{A}_p(G)$ = poset of non-trivial elementary abelian subgroups in G.
$\Delta\mathcal{A}_p(G)$ = order (simplicial) complex of $\mathcal{A}_p(G)$

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Proposition (Ksontini) $\Delta\mathcal{A}_2(S_7) = \vee_{i=1}^{160}\mathbb{S}^2$

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Proposition (Ksontini) $\Delta\mathcal{A}_2(S_7) = \vee_{i=1}^{160}\mathbb{S}^2$

```
gap> K:=QuillenComplex(SymmetricGroup(7),2);
Simplicial complex of dimension 2.
gap> Size(K);
11291
gap> Y:=RegularCWComplex(K);;
gap> C:=CriticalCells(Y);;
gap> n:=0;;Length(Filtered(C,c->c[1]=n));
1
gap> n:=1;;Length(Filtered(C,c->c[1]=n));
0
gap> n:=2;;Length(Filtered(C,c->c[1]=n));
160
```