Bounds on Joule dissipation for dynamos

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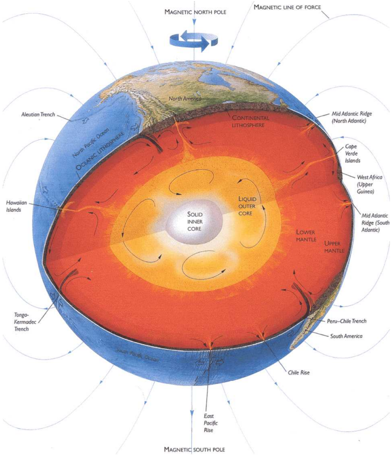
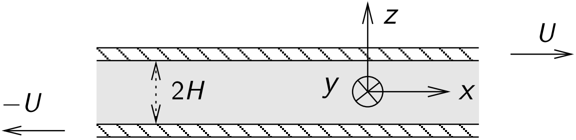
Earth, numerical and experimental dynamos



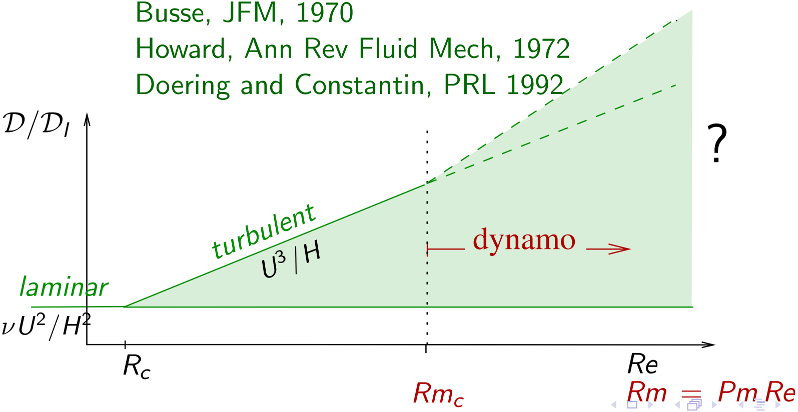
What is the upper bound for dissipation?

What is the Joule fration of dissipation?

A simple shear flow



Busse, JFM, 1970
Howard, Ann Rev Fluid Mech, 1972
Doering and Constantin, PRL 1992



The Hopf-Doering-Constantin method

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + Re^{-1} \nabla^2 \mathbf{u}$$

Background flow decomposition $\mathbf{u}(x, y, z, t) = \phi(z)\mathbf{e}_x + \mathbf{v}(x, y, z, t)$ with $\phi(1) = 1$ and $\phi(-1) = -1$. Substitute into dimensionless NS

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \phi \partial_x \mathbf{v} + v_z \phi' \mathbf{e}_x = -\nabla p + Re^{-1} \left[\phi'' \mathbf{e}_x + \nabla^2 \mathbf{v} \right]$$

Dot product with $\mathbf{v}$, integrate over volume and time

$$\langle \phi' v_x v_z \rangle = -Re^{-1} \langle \phi' \partial_z v_x \rangle - Re^{-1} \langle \nabla \mathbf{v} : \nabla \mathbf{v} \rangle$$

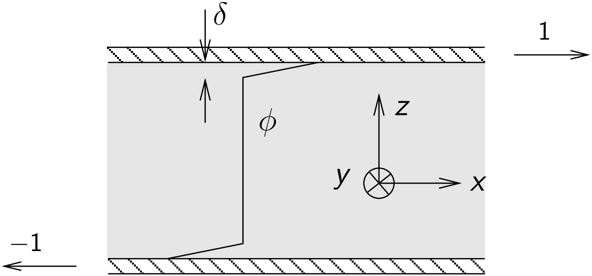
Express viscous dissipation

$$\mathcal{D} \hat{=} \langle \nabla \mathbf{u} : \nabla \mathbf{u} \rangle = \langle \nabla \mathbf{v} : \nabla \mathbf{v} \rangle + \left\langle \phi'^2 \right\rangle + 2 \langle \phi' \partial_z v_x \rangle$$

and combine with the fluctuations energy balance above

$$\mathcal{D} = \left\langle \phi'^2 \right\rangle - \left[\langle \nabla \mathbf{v} : \nabla \mathbf{v} \rangle + 2Re \langle \phi' v_x v_z \rangle \right]$$

A possible choice of background flow



The spectral constraint

The bracket $[\langle \nabla \mathbf{v} : \nabla \mathbf{v} \rangle + 2Re \langle \phi' v_x v_z \rangle]$ must be made positive for every possible $\mathbf{v}$, by choosing an appropriate ϕ. As $\mathbf{v}$ must vanish on the walls, we have by virtue of Schwartz inequality

$$v_x = \int_{-1}^{z} \partial_z v_x \, dz' \le \sqrt{z+1} \sqrt{\int_{-1}^{1} [\partial_z v_x]^2 \, dz'}$$

and then

$$v_x v_z \le \frac{1}{2}(z+1) \int_{-1}^{1} \nabla \mathbf{v} : \nabla \mathbf{v} \, dz'$$

So, when choosing ϕ piecewise linear with 2 boundary layers of thickness δ and a stagnant core, one gets

$$|\langle \phi' v_x v_z \rangle| \le \frac{\delta}{2} \langle \nabla \mathbf{v} : \nabla \mathbf{v} \rangle \quad \longrightarrow \quad \delta \le Re^{-1} \quad \longrightarrow \quad \mathcal{D} \le 2\, Re$$

What is changed under dynamo action?

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathbf{j} \times \mathbf{B} + Re^{-1} \nabla^2 \mathbf{u}$$

$$\partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + R_m^{-1} \nabla^2 \mathbf{B}$$

Substituting Hopf decomposition, dot product equations with **v** and **B** respectively, then space-time average

$$\langle \phi' v_x v_z \rangle = \langle (\mathbf{j} \times \mathbf{B}) \cdot \mathbf{v} \rangle - Re^{-1} \langle \phi' \partial_z v_x \rangle - Re^{-1} \langle \nabla \mathbf{v} : \nabla \mathbf{v} \rangle$$

$$0 = \langle \phi' B_x B_z \rangle + \langle \mathbf{j} \cdot (\mathbf{v} \times \mathbf{B}) \rangle - R_m^{-1} \langle \mathbf{j}^2 \rangle$$

One must add up these equations to get rid of intractable cubic terms. Then the sum of Joule and viscous dissipations can be written

$$\mathcal{D} = \left\langle \phi'^2 \right\rangle - \left[\langle \nabla \mathbf{v} : \nabla \mathbf{v} \rangle + 2Re \langle \phi' v_x v_z \rangle \right] - P_m^{-1} \left[\left\langle \mathbf{j}^2 \right\rangle - 2R_m \langle \phi' B_x B_z \rangle \right]$$

Big change in magnetic spectral condition

$$\left[\left\langle \mathbf{j}^2 \right\rangle - 2R_m \left\langle \phi' B_x B_z \right\rangle\right]$$

The magnetic field does not vanish at the boundary, but matches a harmonic external field. However, it can be shown that, at $z = \pm 1$

$$\overline{B_x B_z} = 0$$

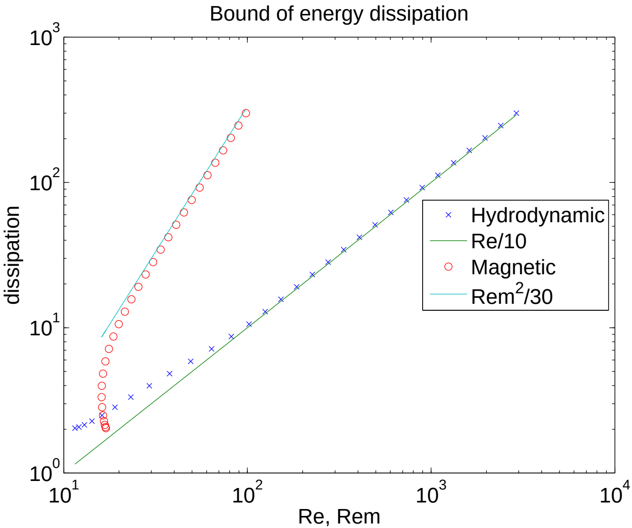
corresponding physically to a zero mean Maxwell stress. Then, using Schwartz inequality

$$\left| \left\langle B_x B_z \phi' \right\rangle \right| \leq \frac{8}{3\pi} \delta^{1/2} \left\langle \mathbf{j}^2 \right\rangle$$

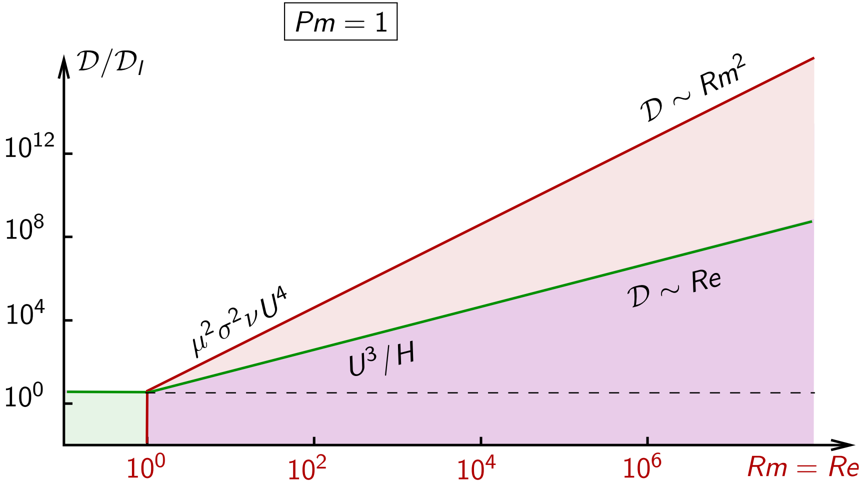
Hence, δ must be of order Rm^{-2} and dissipation then scales as Rm^2

$$\mathcal{D} \leq \frac{512\, Rm^2}{9\, \pi^2}$$

Spectral results

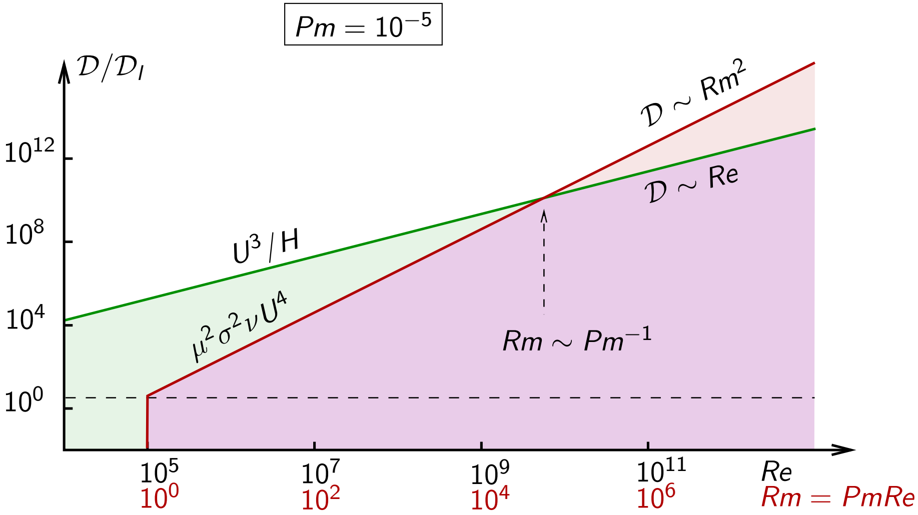


Global bound for dissipation



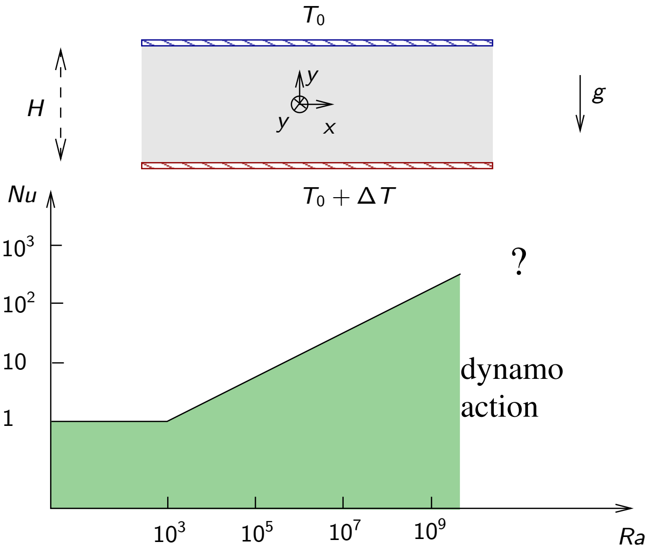
Alboussière, PRE, 2009

Global bound for dissipation



Alboussière, PRE, 2009

Rayleigh-Bénard



The Hopf-Doering-Constantin method with Boussinesq

$$\frac{1}{Pr}\left(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + Ra\, T\mathbf{e}_z + \nabla^2\mathbf{u}$$

$$\partial_t T + \mathbf{u}\cdot\nabla T = \nabla^2 T$$

Background temperature decomposition
$\mathbf{T}(x,y,z,t) = \tau(z) + \theta(x,y,z,t)$ with $\tau(-1/2) = 1/2$ and $\tau(1/2) = -1/2$.
Dot product NS with $\mathbf{u}$, integrate over volume and time

$$0 = \langle \theta u_z \rangle - Ra^{-1}\langle \nabla\mathbf{u} : \nabla\mathbf{u}\rangle$$

Product of thermal equation by θ

$$\langle \tau' u_z \theta\rangle = -\langle \partial_z\theta\ \tau'\rangle - \langle \nabla\theta\cdot\nabla\theta\rangle$$

Express the heat flow

Integration of the product of T with the thermal equation (without Hopf decomposition) leads to

$$Nu = \langle \nabla T \cdot \nabla T \rangle$$

Now substituting the decomposition

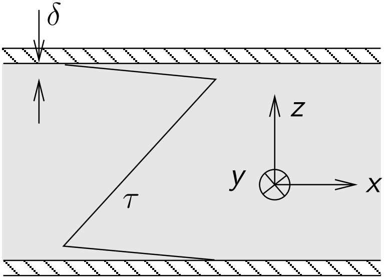
$$Nu = \left\langle \tau'^2 \right\rangle + 2 \langle \tau' \partial_z \theta \rangle + \langle \nabla \theta \cdot \nabla \theta \rangle$$

Using the equations on previous slide

$$Nu = \left\langle \tau'^2 \right\rangle - \left[\langle u_z (\tau' - 1) \theta \rangle + \langle \nabla \theta \cdot \nabla \theta \rangle + Ra^{-1} \langle \nabla \mathbf{u} : \nabla \mathbf{u} \rangle \right]$$

Background temperature profile

Make $\tau' = 1$ in the bulk of the flow and satisfy boundary conditions in thin layers so as to have control of $\mathbf{u}$ and θ with $\langle \nabla \mathbf{u} : \nabla \mathbf{u} \rangle$ and $\langle \nabla \theta \cdot \nabla \theta \rangle$.



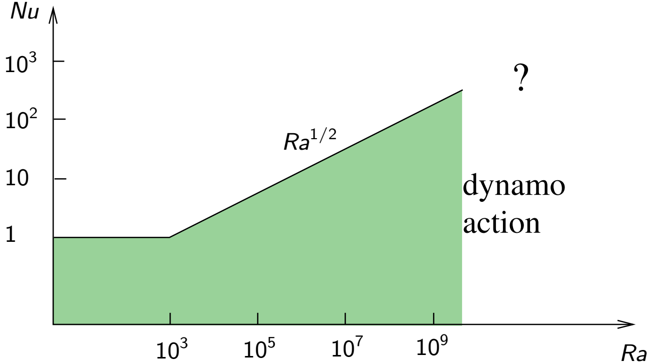
In the boundary layers : $\tau' \sim \delta^{-1}$

Flux bound

Using Schwartz inequality, and choosing $\delta = 4Ra^{-1/2}$, the spectral condition is satisfied (bracket in the last equation is positive), the flux can be bounded as

$$Nu \leq \langle \tau'^2 \rangle = \frac{1}{2} Ra^{1/2} - 3$$

Doering & Gibbon, CUP 1995



The induction equation

Adding the induction equation has not effect on the bound. Adding kinetic and magnetic energy equations cancel the electromotive power and the power of Lorentz forces. The bound is again

$$Nu \leq \frac{1}{2} Ra^{1/2} - 3$$

Dissipation

The dimensional heat flux Φ is $\overline{\rho c_p u_z T} - k\partial_z \overline{T}$, where denotes horizontal and time average. Integrated over the thickness of the layer, we have

$$\Phi = \left\langle \frac{\rho c_p u_z T}{H} \right\rangle - k\frac{\delta T}{H}$$

Dot product of NS with $\mathbf{u}$ leads to

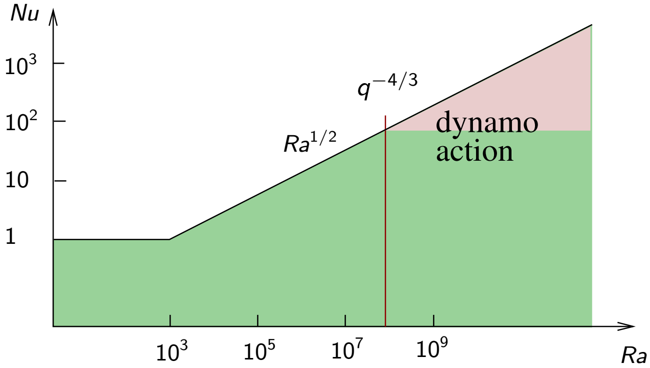
$$\langle \alpha \rho g T u_z \rangle - \rho\nu \langle \nabla \mathbf{u} : \nabla \mathbf{u} \rangle - \frac{1}{\sigma} \langle \mathbf{j}^2 \rangle = 0$$

Combining both equations, we get

$$\mathcal{D}_\nu + \mathcal{D}_J = \frac{\alpha g H}{c_p} (\Phi - \Phi_{cond})$$

Dissipation

When that dissipation is translated into a magnetic Reynolds number, it is found that Rm cannot be of order unity if $Ra < q^{-4/3}$.



Conclusions, Perspectives

Some bounds are obtained for dynamo action, but it is hard to see what they mean.

- Effect of rotation not well captured
- Effect of compressibility difficult to handle

References

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Infinite Prandtl number:

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