Energy Transfers, Spectrum,and Flux in Quasi-Static MHD Turbulence

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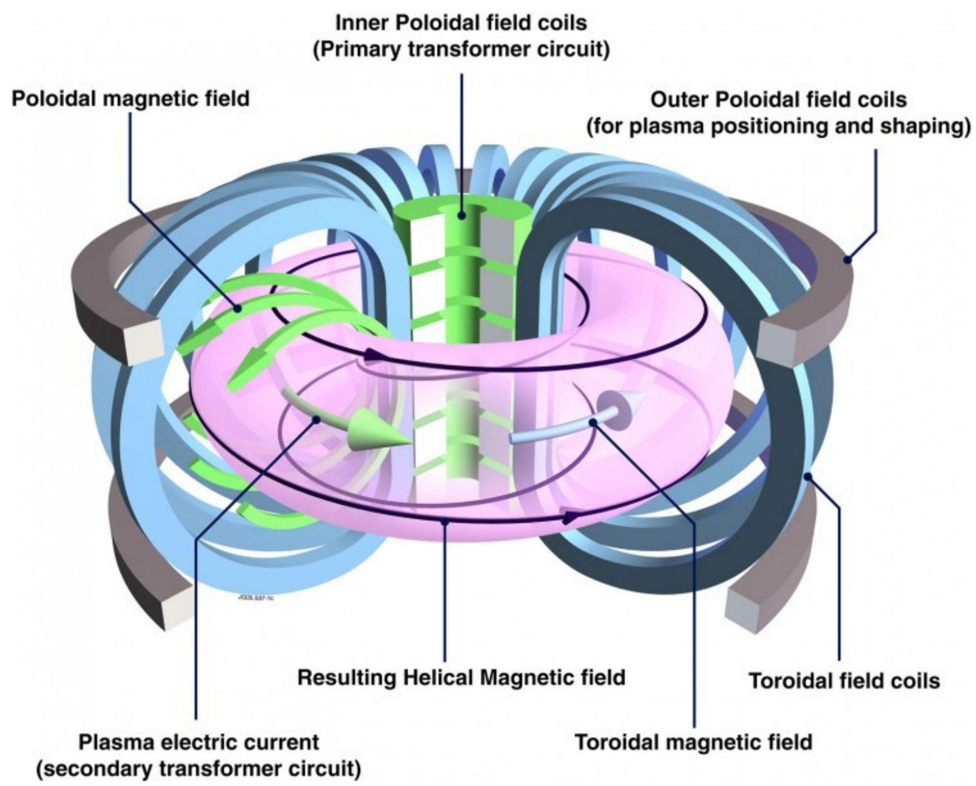
Outline of the talk

- Introduction
- Quasi-static MHD approximation
- Energy spectrum
- Anisotropic energy transfers
- Summary

Introduction

- Liquid metals are used as heat exchangers in fusion reactors (International Thermonuclear Experimental Reactor).
- High external magnetic fields for plasma confinement.
- Strong external magnetic field affects the flow properties.



http://magfuzio.hu/tanulmanyok/magneses-osszetartas/

MHD Equations

Governing equations for MHD:

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla(p/\rho) + \frac{1}{\mu\rho}(\mathbf{b}\cdot\nabla)\mathbf{b} + \frac{1}{\mu\rho}(\mathbf{B}\cdot\nabla)\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{f} \\
\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} &= (\mathbf{b}\cdot\nabla)\mathbf{u} + (\mathbf{B}\cdot\nabla)\mathbf{u} + \eta\nabla^2\mathbf{b} \\
\nabla\cdot\mathbf{u} &= 0 \\
\nabla\cdot\mathbf{b} &= 0
\end{aligned}
$$

u is the velocity field, **B** is the external magnetic field, **b** is the induced magnetic field, ν is the kinematic viscosity, μ is the magnetic permeability, and η is the magnetic diffusivity.

Non-dimensional parameters

- Kinetic Reynolds number Re = UL/ν — For Liquid metals **b«B**
- Magnetic Reynolds number Rm = UL/η — Rm → 0
- Magnetic Prandtl number Pm = ν/η — Pm → 0

Roberts, 1967

Quasi-static MHD Turbulence

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla(p/\rho) + \frac{1}{\mu\rho}(\mathbf{b}\cdot\nabla)\mathbf{b} + \frac{1}{\mu\rho}(\mathbf{B}\cdot\nabla)\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{f} \\
\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} &= (\mathbf{b}\cdot\nabla)\mathbf{u} + (\mathbf{B}\cdot\nabla)\mathbf{u} + \eta\nabla^2\mathbf{b} \\
\nabla\cdot\mathbf{u} &= 0 \\
\nabla\cdot\mathbf{b} &= 0
\end{aligned}
$$

Order of magnitude analysis

$$
\mathcal{O}((\mathbf{u}\cdot\nabla)\mathbf{b}) = \frac{ub}{L}, \mathcal{O}((\mathbf{b}\cdot\nabla)\mathbf{u}) = \frac{ub}{L}, \mathcal{O}(\eta\nabla^2\mathbf{b}) = \frac{\eta b}{L^2}
$$

$$
\mathrm{Rm} = \frac{uL}{\eta} = \frac{\mathcal{O}((\mathbf{u}\cdot\nabla)\mathbf{b})}{\mathcal{O}(\eta\nabla^2\mathbf{b})} = \frac{\mathcal{O}((\mathbf{b}\cdot\nabla)\mathbf{u})}{\mathcal{O}(\eta\nabla^2\mathbf{b})} \to 0
$$

If $\mathbf{B} = B_0\hat{\mathbf{z}}$

$$
\frac{\partial \mathbf{b}}{\partial t} + B_0\frac{\partial \mathbf{u}}{\partial z} = -\eta\nabla^2\mathbf{b}
$$

$$
\frac{\partial}{\partial t}\mathbf{b} \approx \mathbf{0}
$$

$$
\mathbf{b} = \frac{1}{\eta\nabla^2}B_0\frac{\partial \mathbf{u}}{\partial z}
$$

Roberts, 1967

Quasi-static MHD Turbulence

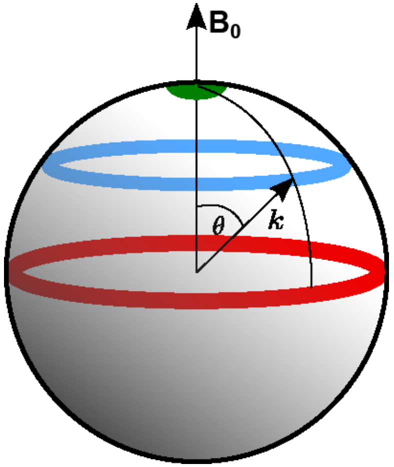
- Governing Equations of liquid metal MHD under quasi-static approximations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla(p/\rho) - \frac{\sigma {B_0}^2}{\rho}\frac{1}{\nabla^2}\frac{\partial^2 \mathbf{u}}{\partial z^2} + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

- Continuity equation $\nabla\cdot\mathbf{u} = 0$

- Interaction parameter = Lorentz term / Nonlinear advective term

$$N = \frac{\sigma {B_0}^2 L}{\rho u'}$$

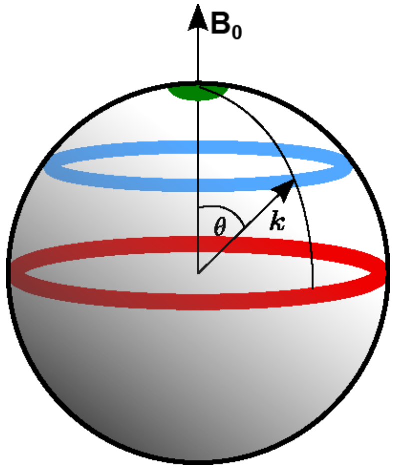


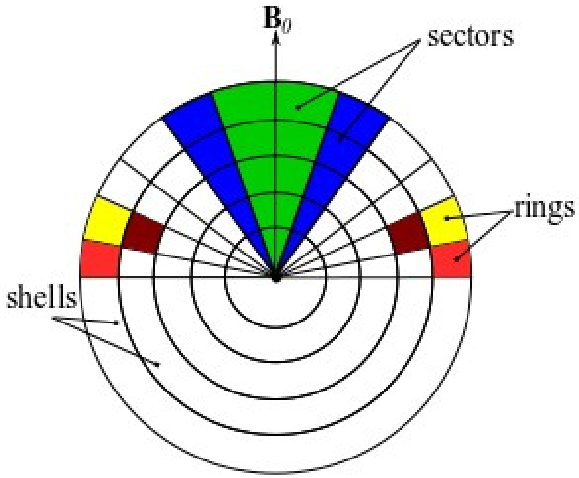
The quasi-static MHD equation in Fourier space

$$\begin{aligned}\frac{\partial \hat{u}_i(\mathbf{k})}{\partial t} + ik_j \sum_{\mathbf{k}=\mathbf{p}+\mathbf{q}} \hat{u}_j(\mathbf{q})\hat{u}_i(\mathbf{k}-\mathbf{q}) &= -\frac{ik_i\hat{p}(\mathbf{k})}{\rho} - \frac{\sigma {B_0}^2}{\rho}\cos^2(\theta)\hat{u}_i(\mathbf{k}) \\ &- \nu k^2\hat{u}_i(\mathbf{k}) + \hat{f}_i(\mathbf{k})\end{aligned}$$

Simulation Method

- TARANG : A pseudo-spectral code
- Dealiased
- Time Stepping: Runge-Kutta fourth order
- Forced simulations
- Grid Resolution: 256^3
- N = 0-220
- Periodic boundary condition
- Fourier space is divided into shells and rings (Teaca *et al*. PRE (2009).

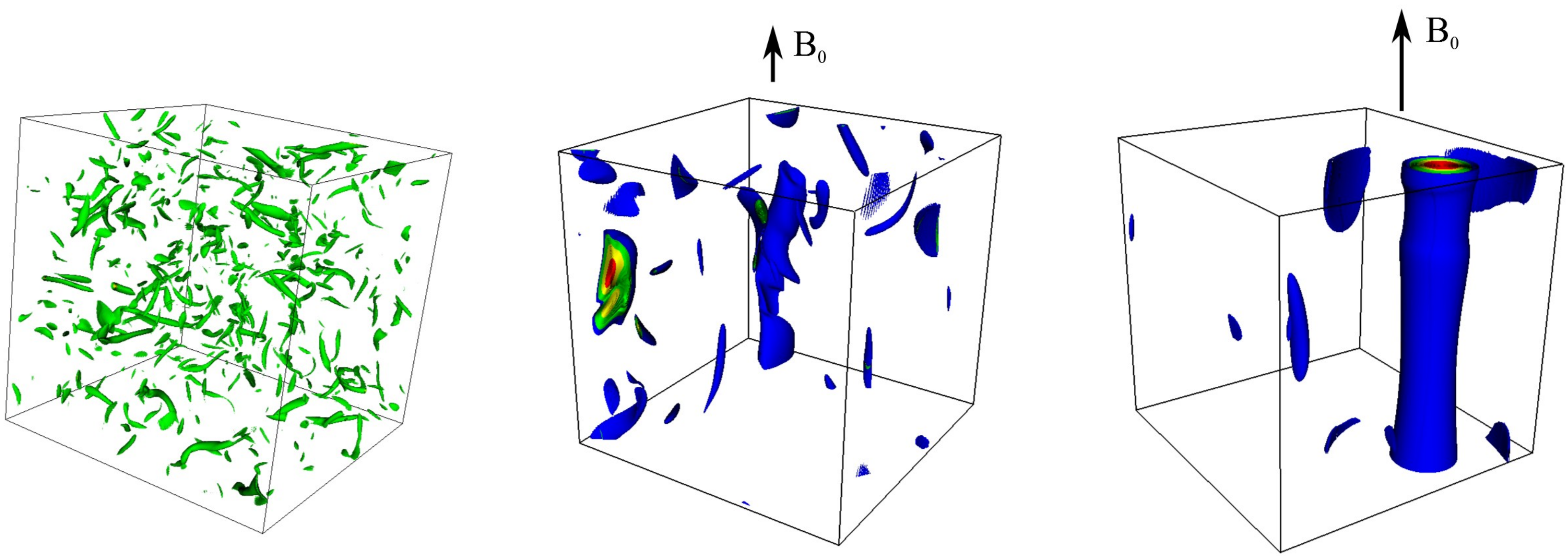




Results

Flow visualization

- Iso-surfaces of vorticity fields.
- Flow field is anisotropic for non-zero interaction parameters.



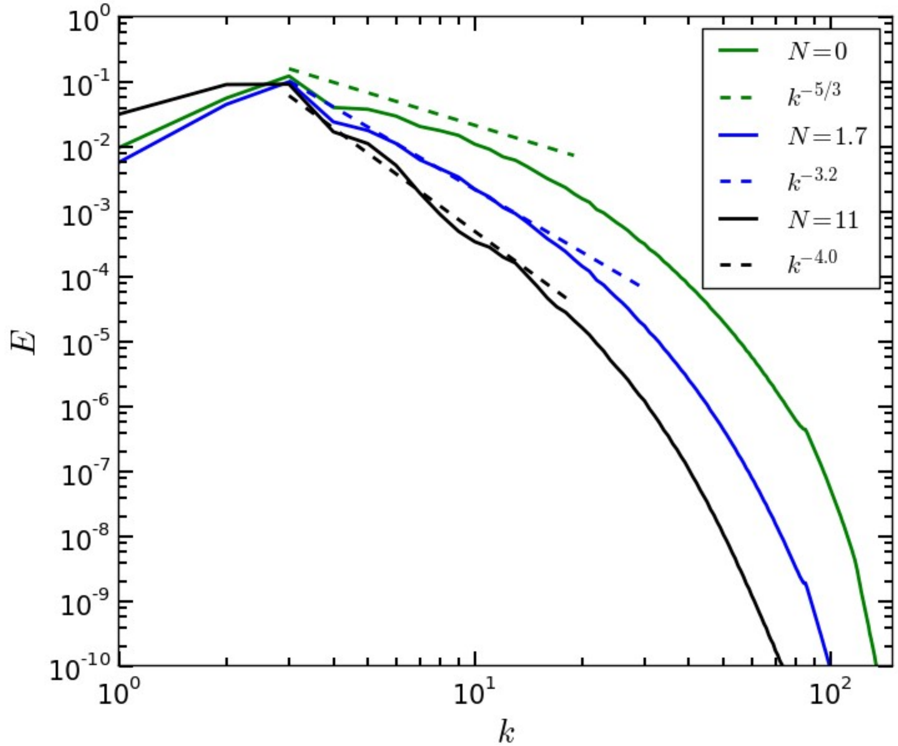
$N = 0$ Isotropic

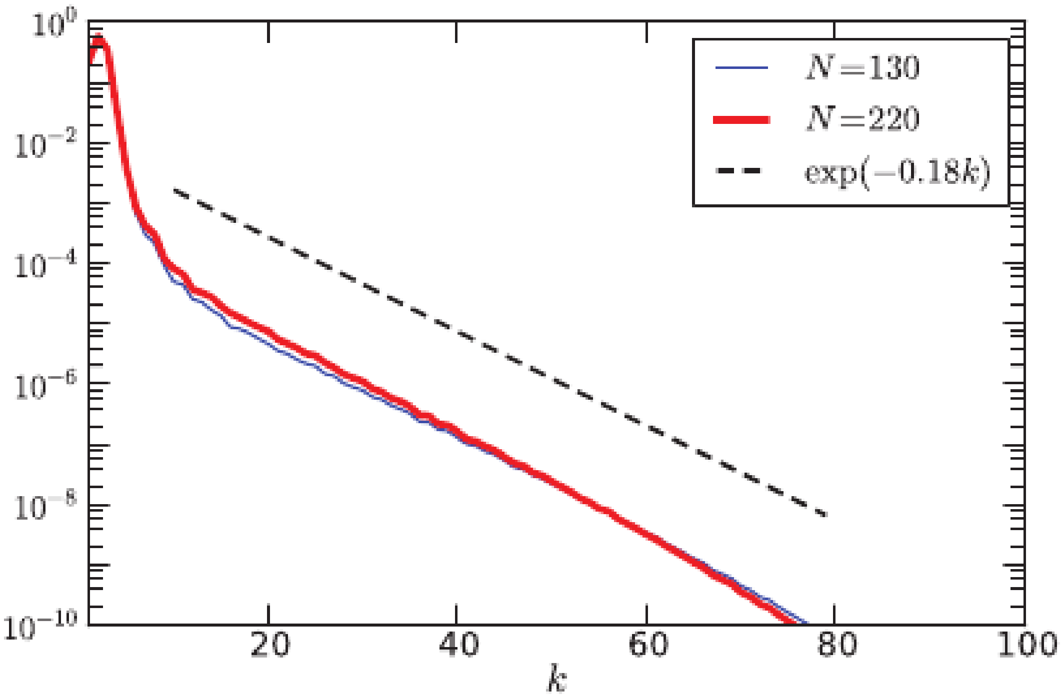
$N = 5.5$ Weakly Anisotropic

$N = 18$ Strongly Anisotropic

Reddy and Verma, *Phys. of Fluids* (2014)

Shell spectrum

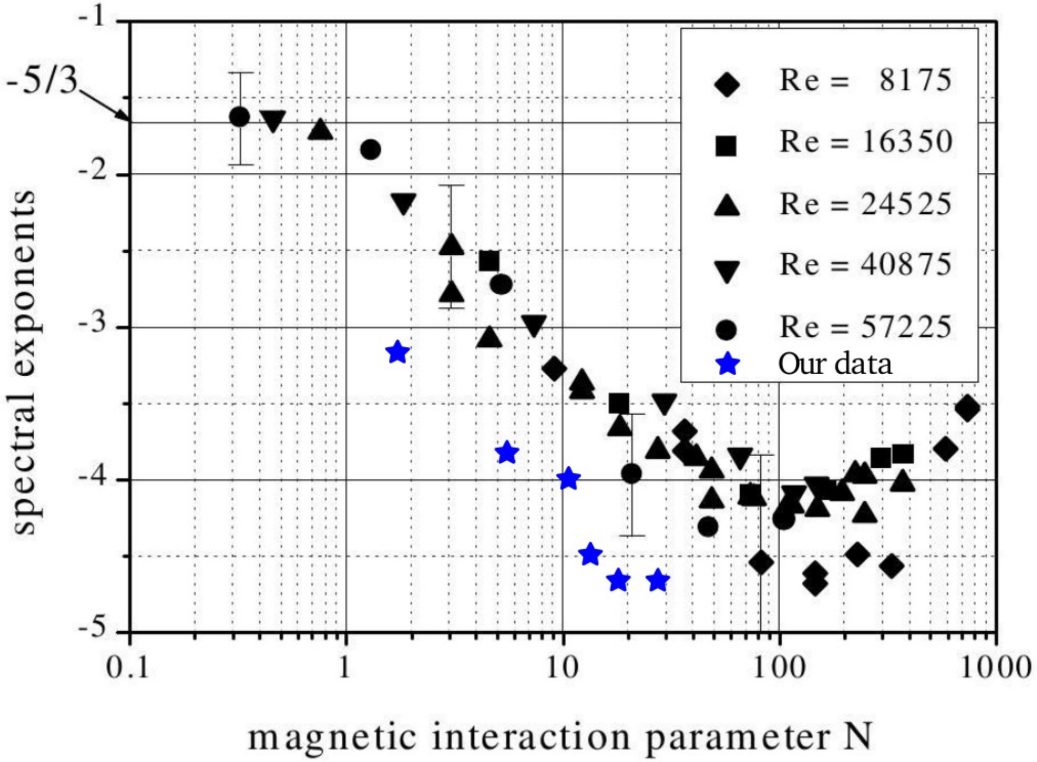




- $N = 0$, Kolmogorov's $k^{-5/3}$
- $N \neq 0$, exponent of the energy spectrum decreases.
- For large N, energy spectrum is exponential.

Reddy and Verma, *Phys. of Fluids* (2014)

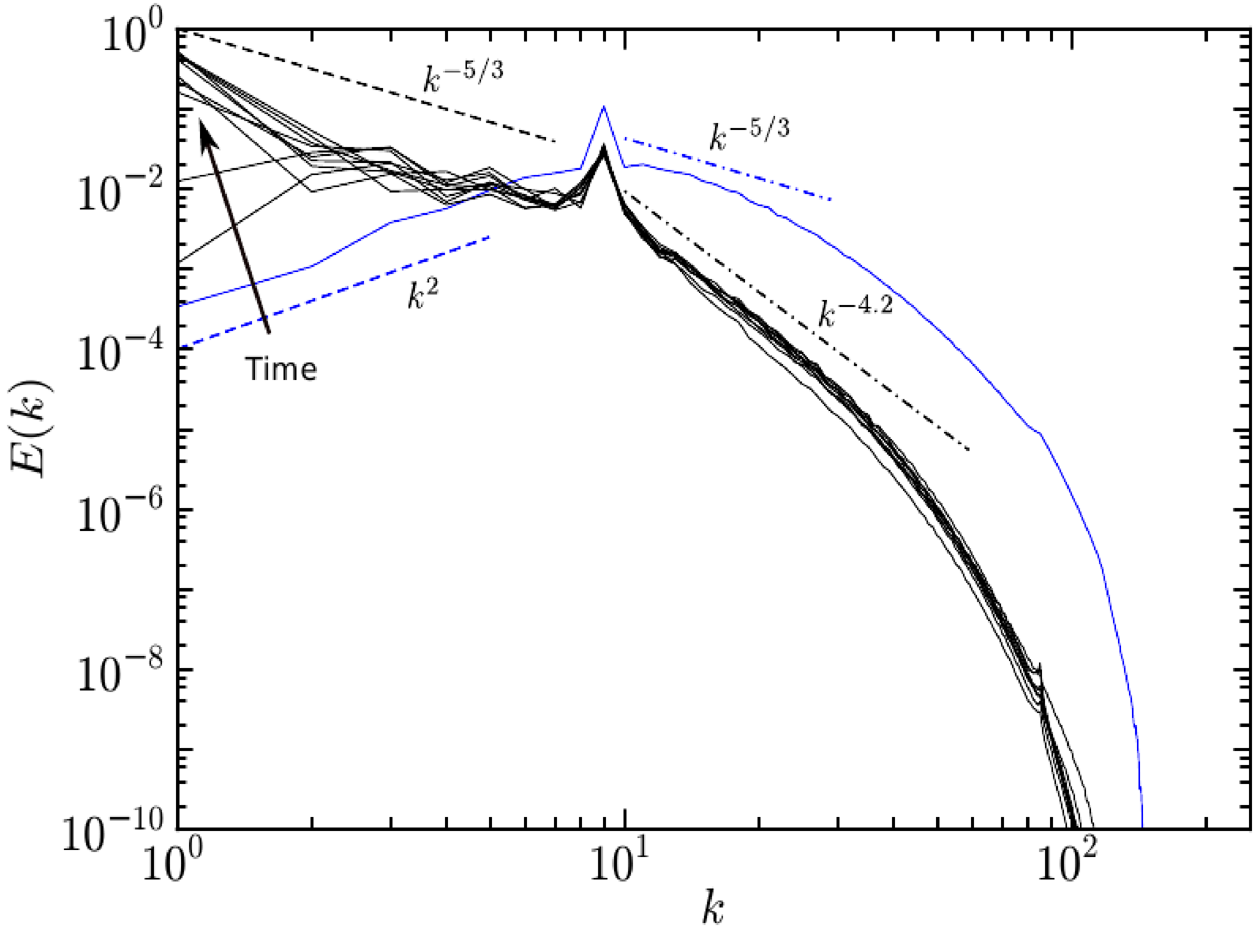
Exponents of shell spectrum



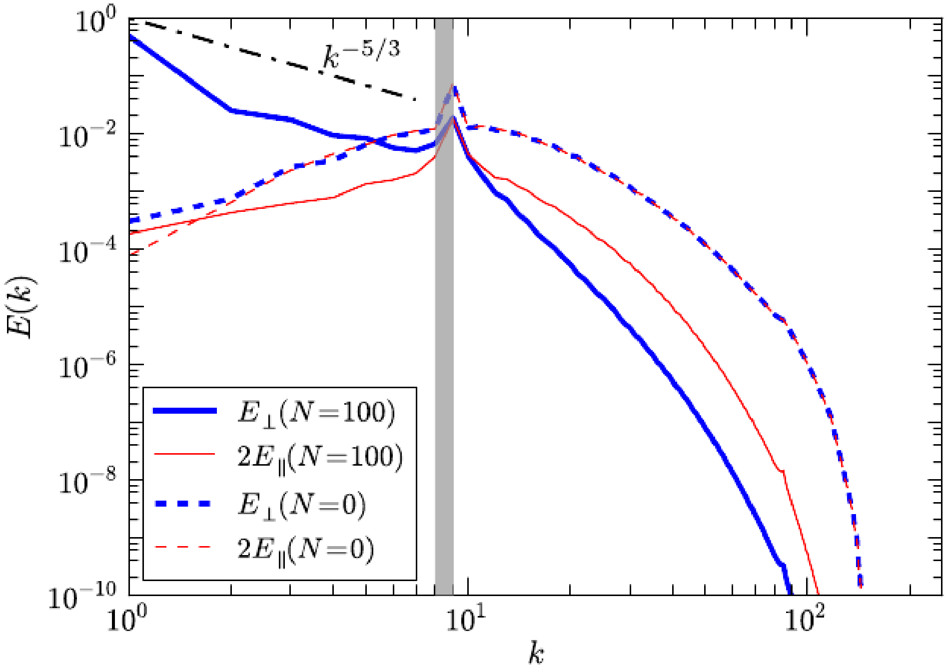
Experimental results of Eckert et al. IJHFF (2001).

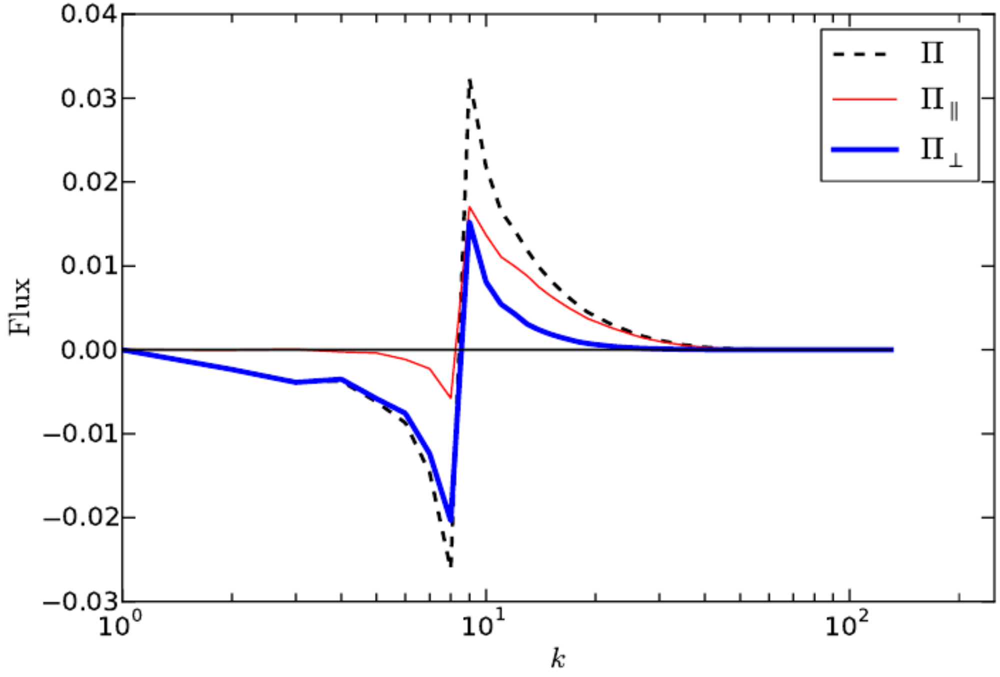
Inverse Cascade in QS MHD

- Forcing: $|\mathbf{k}|$ = 8-9
- Inverse cascade of energy.
- Flow two-dimensionalization.



Inverse Cascade in QS MHD





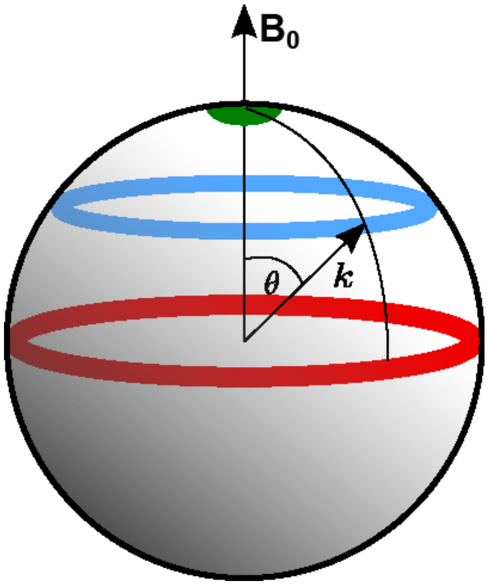
$$E_\perp = \frac{1}{2}(u_x^2 + u_y^2)$$

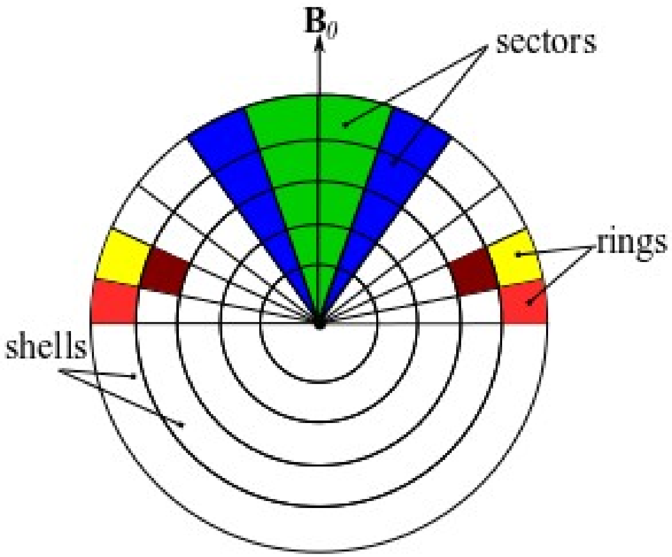
$$E_\parallel = \frac{1}{2}u_z^2$$

$$\Pi_\perp(k_0) = \sum_{|\mathbf{k}|\geq k_0} \sum_{|\mathbf{p}|<k_0} S_\perp(\mathbf{k}|\mathbf{p}|\mathbf{q})$$

$$\Pi_\parallel(k_0) = \sum_{|\mathbf{k}|\geq k_0} \sum_{|\mathbf{p}|<k_0} S_\parallel(\mathbf{k}|\mathbf{p}|\mathbf{q})$$

Energy Transfers

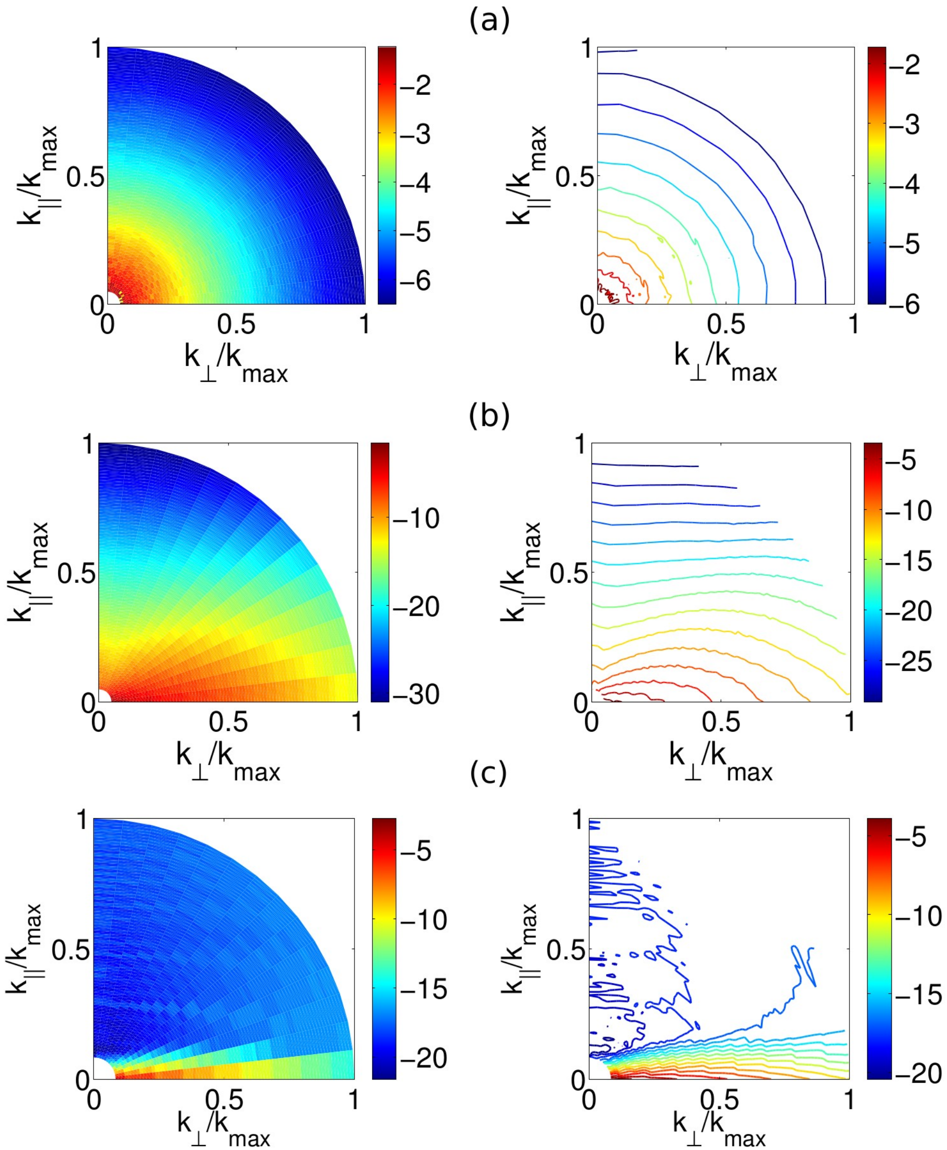




Division of spectral space into 20 shells and 15 sectors.

Angular distribution of energy

Ring spectrum of kinetic energy



N = 0, energy is uniformly distributed.

N = 18,

Accumulation of energy is increasing near the equator with N.

N = 130

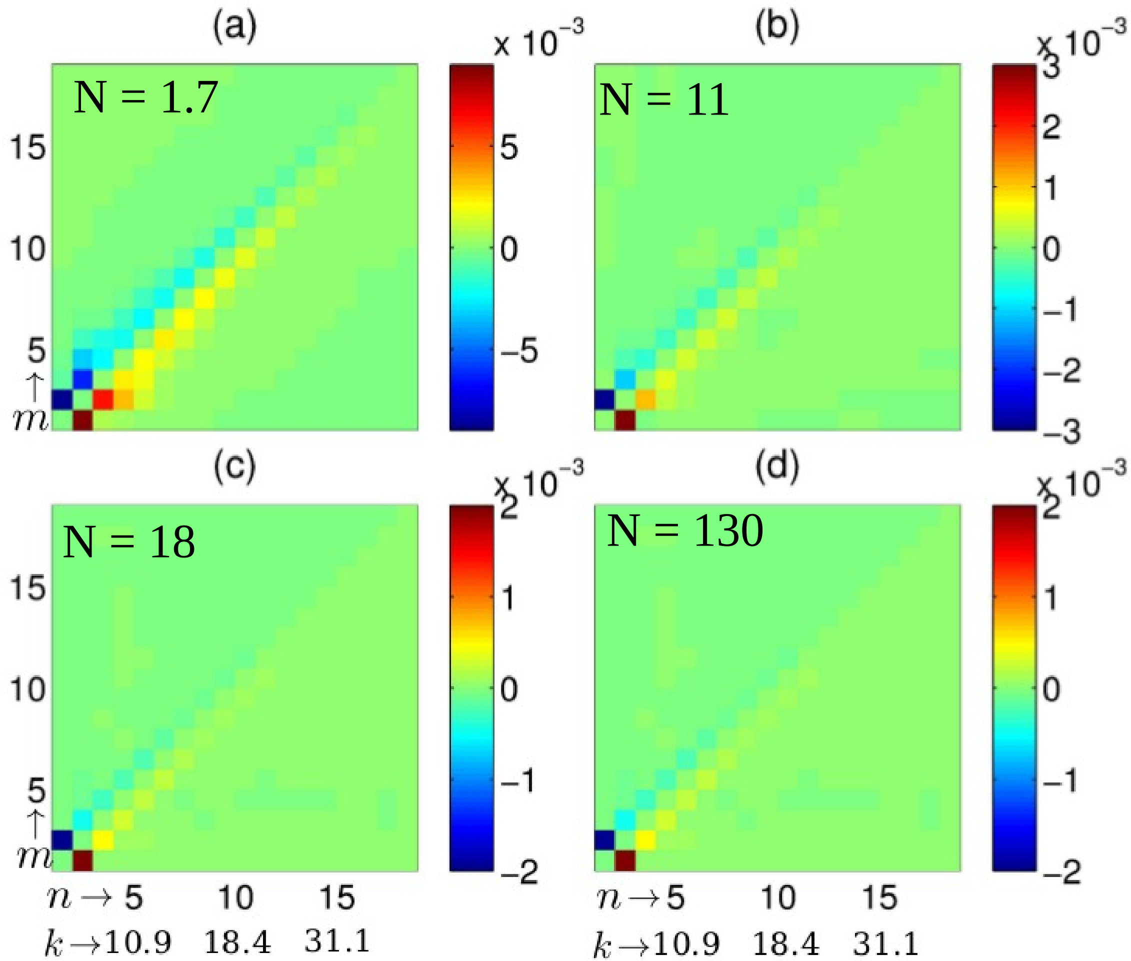
Shell-to-shell Transfers

$$T_n^m = \sum_{\mathbf{k}\in n}\sum_{\mathbf{p}\in m} S(\mathbf{k}|\mathbf{p}|\mathbf{q})$$

Mode-to-mode energy transfer

Dar, Verma, and Eswaran, Physica D (2001)

- i^{th} shell gives energy to $(i+1)^{th}$ shell.
- i^{th} shell receives energy from $(i-1)^{th}$ shell.
- Maximum +ve transfer from i^{th} to $(i+1)^{th}$.
- Shell-to-shell energy transfers are forward and local.



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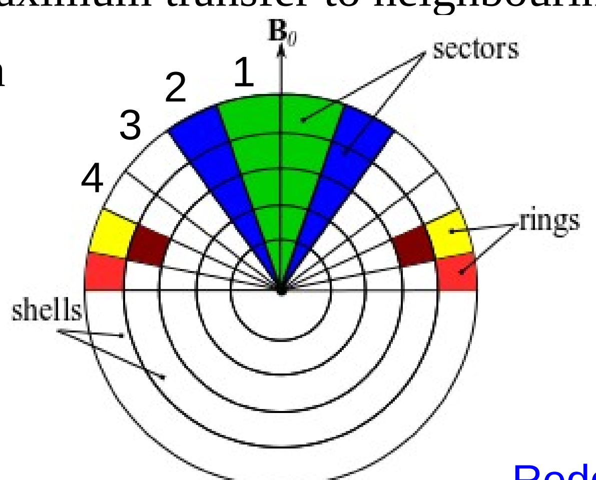
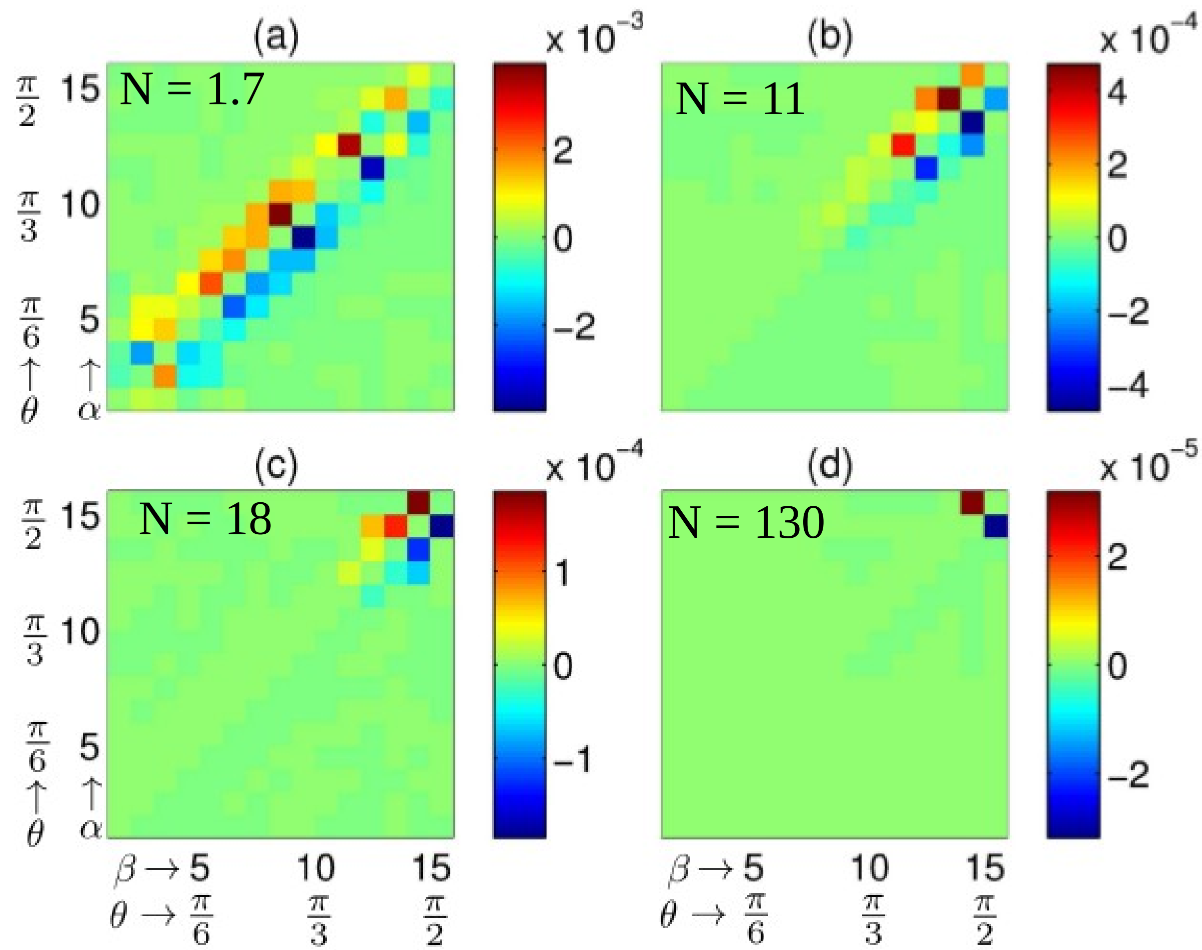
Ring-to-ring Transfers

$$T_{(n,\beta)}^{(m,\alpha)} = \sum_{\mathbf{k}\in(n,\beta)} \sum_{\mathbf{p}\in(m,\alpha)} S(\mathbf{k}|\mathbf{p}|\mathbf{q})$$

$$T_{(9,\beta)}^{(9,\alpha)}$$

- Transfers within rings of 9th shell.
- (m,α) → (n,β)
- Transfers from α → α-1 are positive.
- Maximum transfer to neighbouring rin





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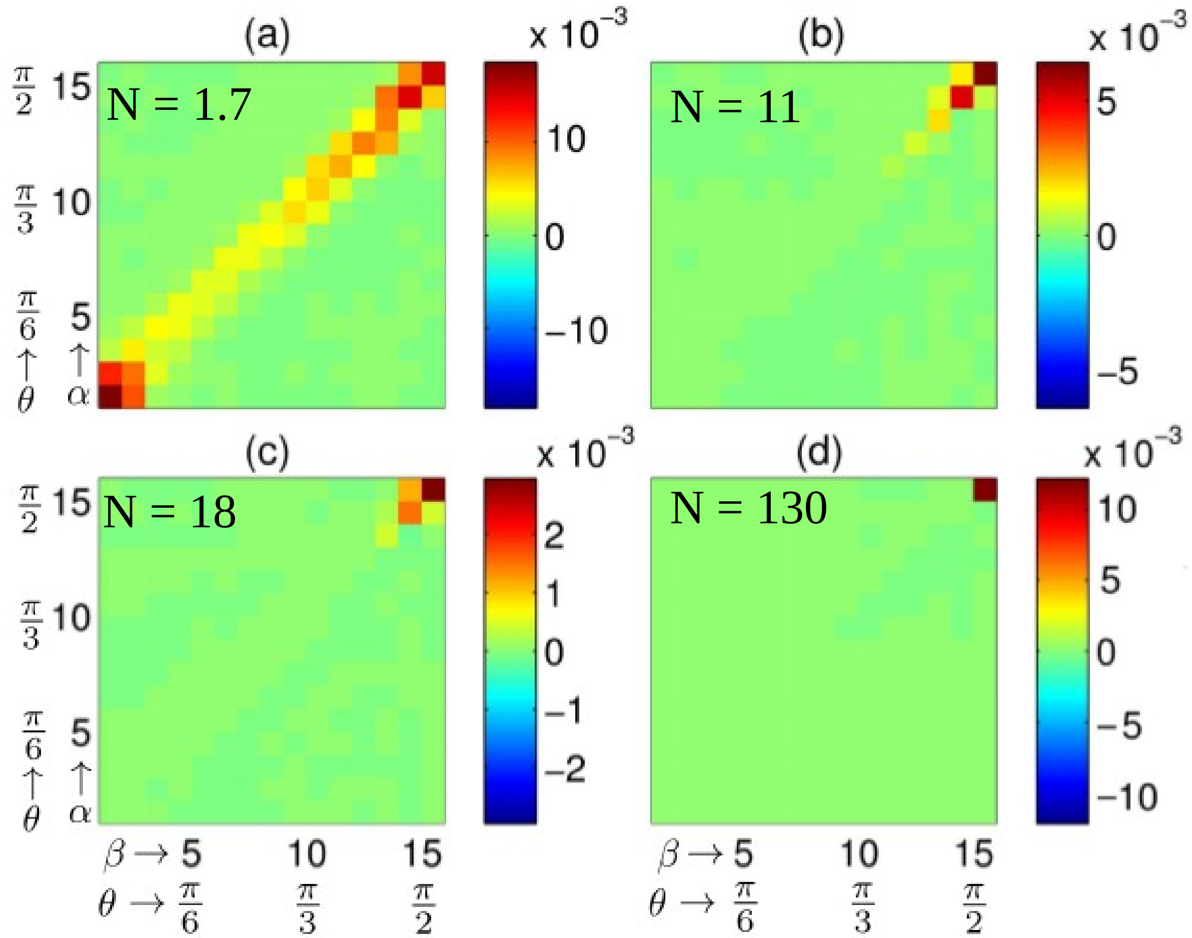
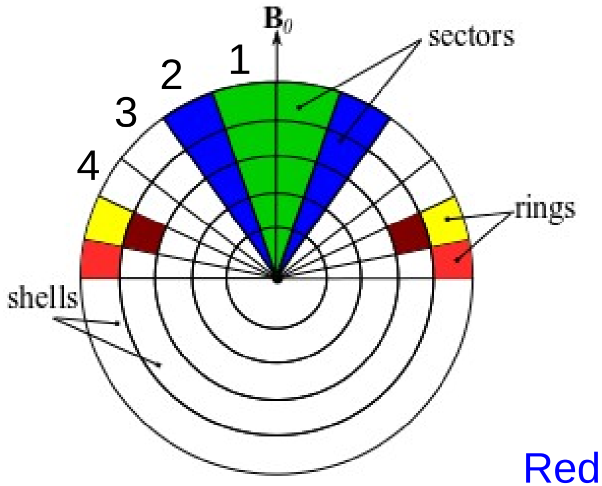
Ring-to-ring Transfers

$$T^{(9,\alpha)}_{(10,\beta)}$$

- Transfers from rings of 9^{th} shell to rings of 10^{th} shell.
- Transfers from $\alpha \rightarrow \alpha$-1 are positive.
- Maximum transfer to neighbouring rings.





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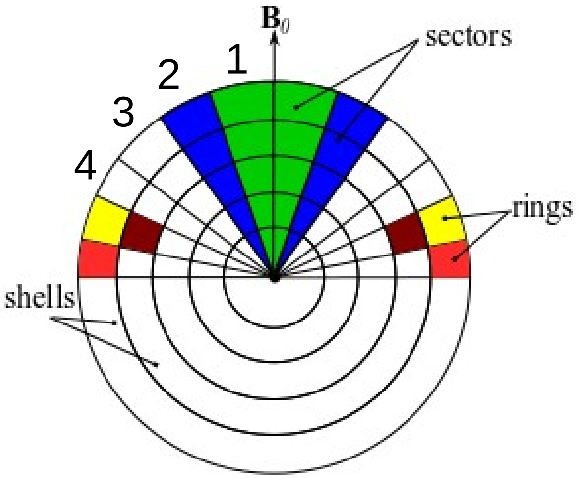
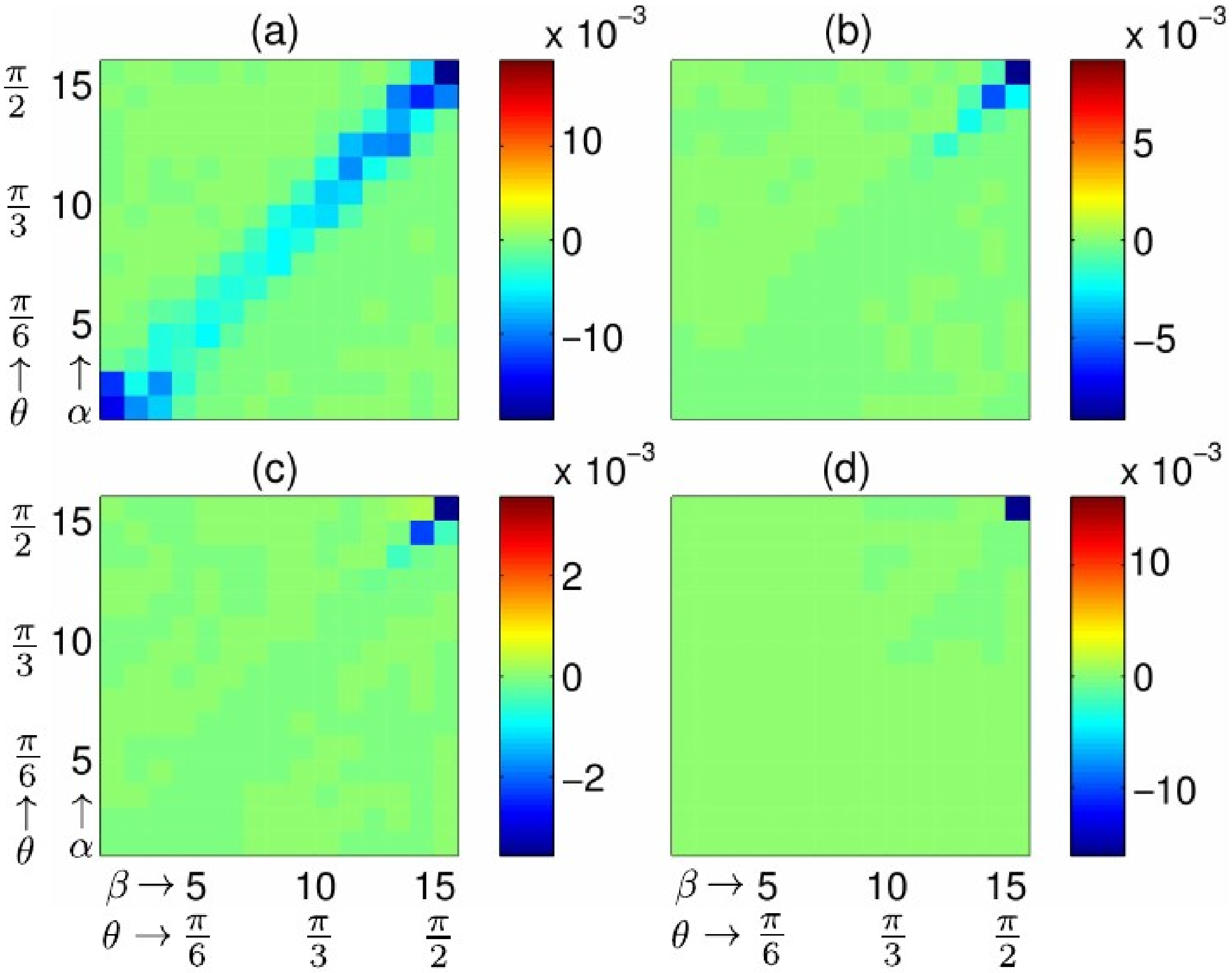
Ring-to-ring Transfers

- Transfers from rings of 9th shell to rings of 8th shell.
- Transfers from α→α-1 are negative.
- Maximum transfer to neighbouring rings.

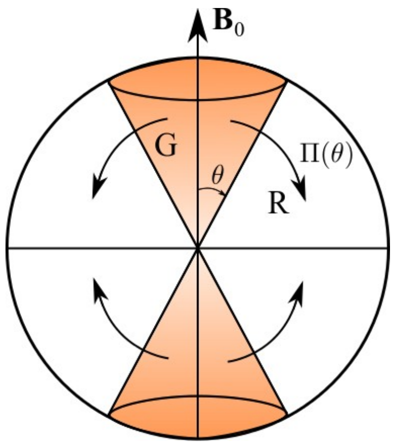


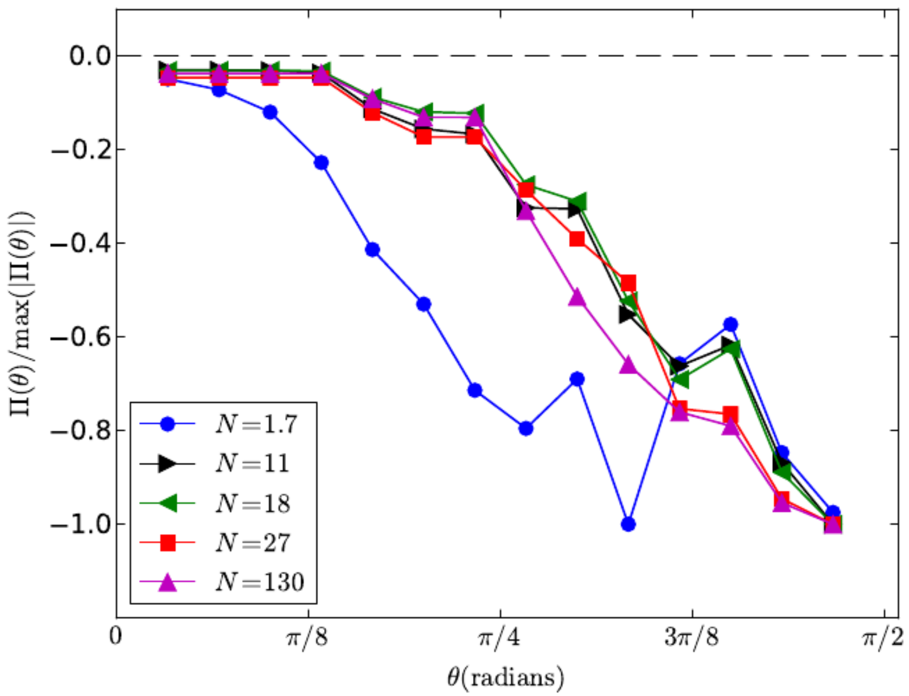
$$T^{(9,\alpha)}_{(8,\beta)}$$



Conical Energy Flux

$$\Pi(\theta) = \sum_{\mathbf{k}\in R} \sum_{\mathbf{p}\in G} S(\mathbf{k}|\mathbf{p}|\mathbf{q})$$





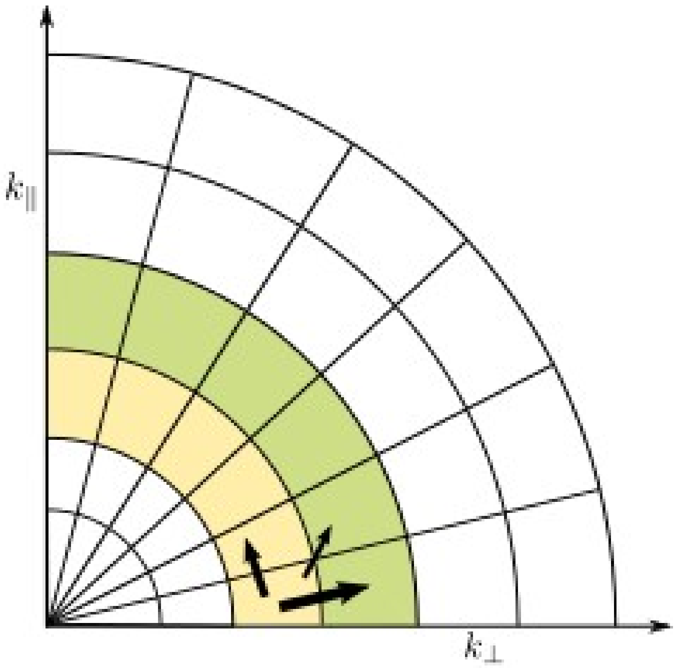
Conical transfers are from equator to polar region.

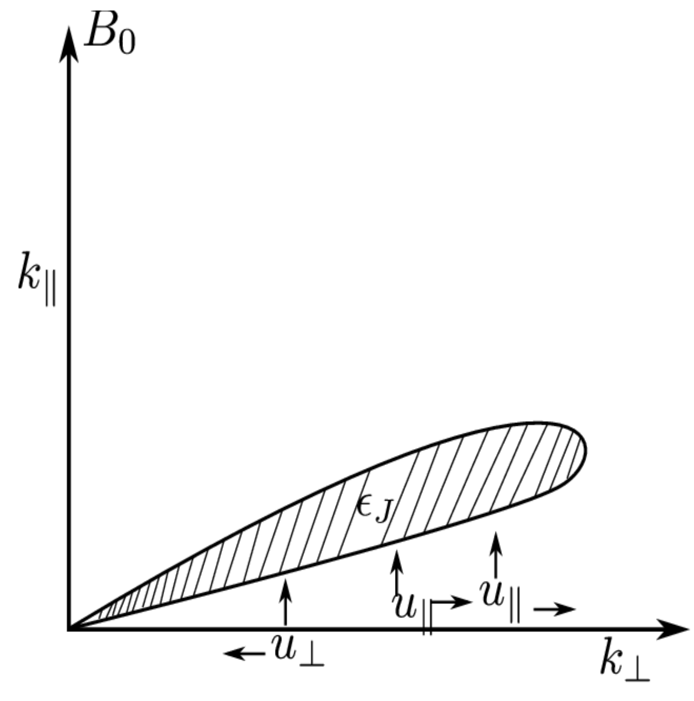
Reddy, Kumar, and Verma, *Phys. of Plasmas* (2014)

Inverse Cascade in QS MHD

$E_\perp$ shows inverse cascade

$E_{||}$ shows forward cascade





Summary

- DNS of quasi-static MHD turbulence.
- Shell spectrum and ring spectrum.
- Power law and exponential behaviour.
- Shell-to-shell, ring-to-ring transfers are local.
- Energy is transferred from equator to poles.
- 2-D behaviour.
- Inverse cascade of energy for perpendicular component.
- Forward cascade of energy for parallel component

References

Knaepen, Kassinos, and Carati, *JFM* (2004)

Teaca et. al. PRE (2009)

Dar, Verma, and Eswaran, *Physica D* (2001)

Reddy and Verma, *Phys. of Fluids* (2014)

Reddy, Kumar, and Verma, *Phys. of Plasmas* (2014)

Eckert et. al. IJHFF (2001)

Thank You