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Effects of flow inhomogeneity in turbulent dynamo

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Present IIS

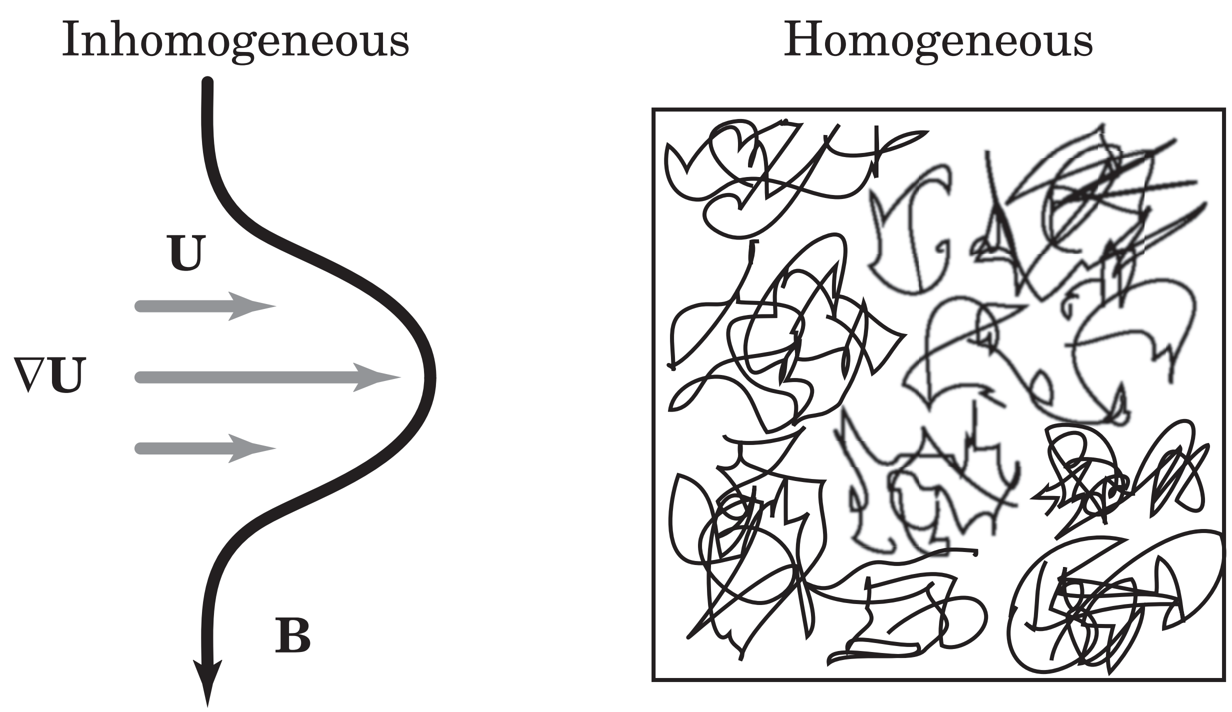
II. Flow inhomogeneities in dynamos

Mean field

$$\mathbf{b} = \mathbf{B} + \mathbf{b}', \quad \mathbf{B} = \langle \mathbf{b} \rangle$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times \langle \mathbf{u}' \times \mathbf{b}' \rangle + \eta \nabla^2 \mathbf{B}$$

$(\mathbf{B} \cdot \nabla)\mathbf{U}$ ⟶ differential rotation, "Ω effect"



Turbulence

$$\mathbf{U} = \mathbf{U}_0(\text{constant}) \ \text{ or } \ \mathbf{0}$$

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{u}' = (\mathbf{B} \cdot \nabla)\mathbf{b}' + (\mathbf{b}' \cdot \nabla)\mathbf{B} \cancel{- (\mathbf{u}' \cdot \nabla)\mathbf{U}} + \cdots$$

$$\frac{\partial \mathbf{b}'}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{b}' = (\mathbf{B} \cdot \nabla)\mathbf{u}' - (\mathbf{u}' \cdot \nabla)\mathbf{B} \cancel{+ (\mathbf{b}' \cdot \nabla)\mathbf{U}} + \cdots$$

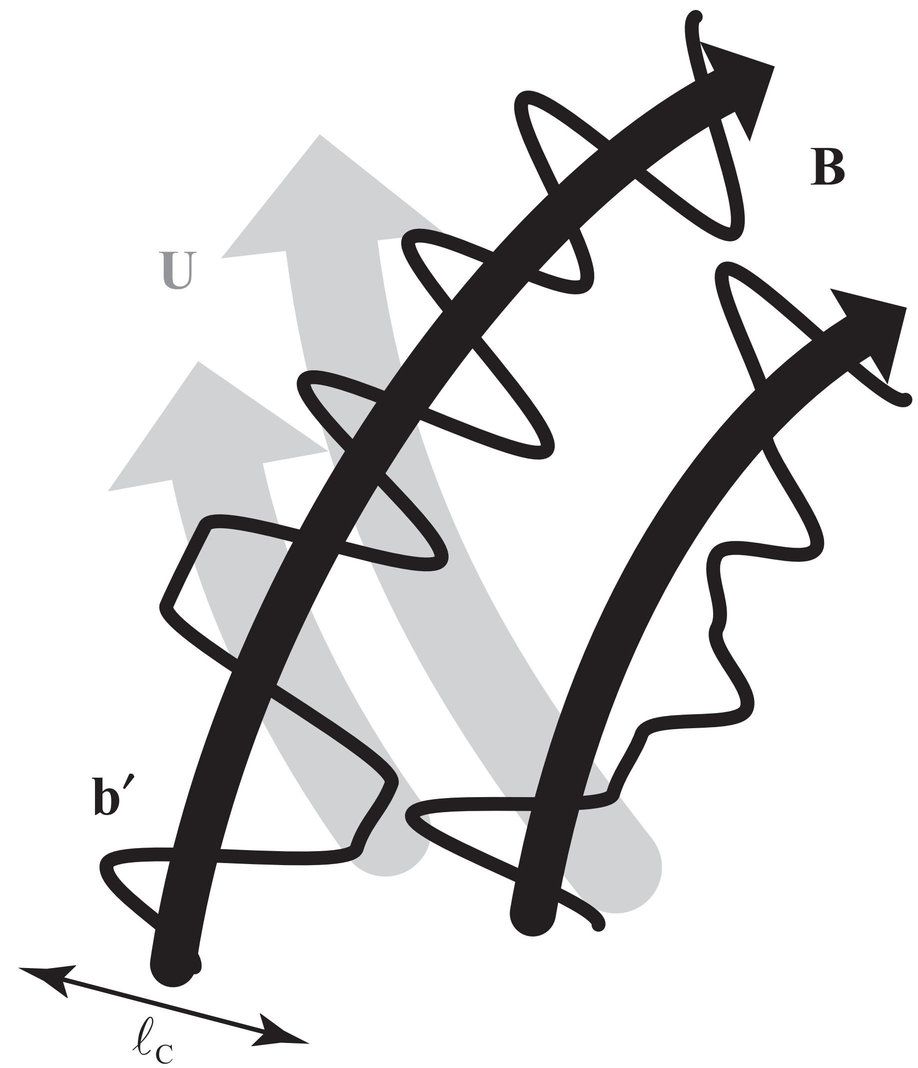
⟶ $$\langle \mathbf{u}' \times \mathbf{b}' \rangle^{\alpha} = \alpha^{\alpha a} B^{a} + \beta^{\alpha a b} \frac{\partial B^{a}}{\partial x^{b}} + \cdots$$

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{u}' = (\mathbf{B}\cdot\nabla)\mathbf{b}' + (\mathbf{b}'\cdot\nabla)\mathbf{B} - (\mathbf{u}'\cdot\nabla)\mathbf{U} + \cdots$$

$$\frac{\partial \mathbf{b}'}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{b}' = (\mathbf{B}\cdot\nabla)\mathbf{u}' - (\mathbf{u}'\cdot\nabla)\mathbf{B} + (\mathbf{b}'\cdot\nabla)\mathbf{U} + \cdots$$

$$\left\langle \frac{\partial \mathbf{u}'}{\partial t} \times \mathbf{b}' \right\rangle + \left\langle \mathbf{u}' \times \frac{\partial \mathbf{b}'}{\partial t} \right\rangle = \cdots$$

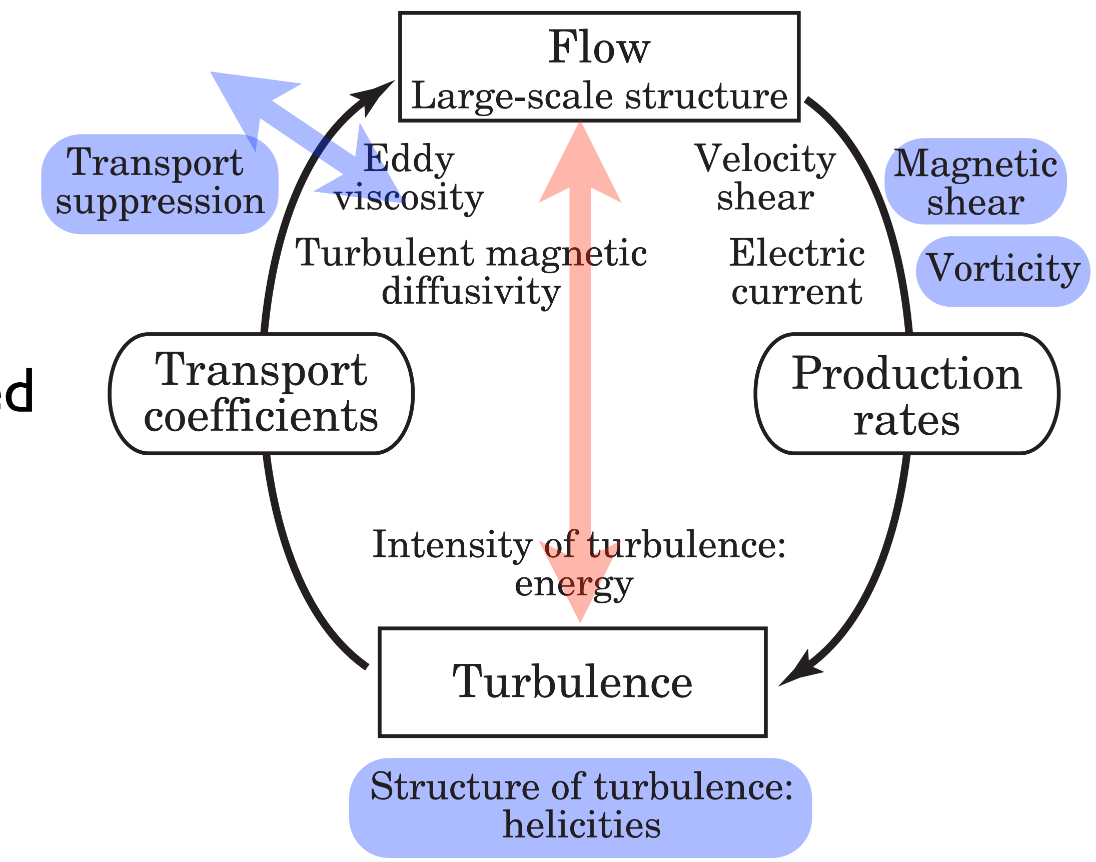
$$\begin{aligned}
&\tau\langle \mathbf{u}' \times [(\mathbf{b}'\cdot\nabla)\mathbf{U}] + [(\mathbf{u}'\cdot\nabla)\mathbf{U}] \times \mathbf{b}' \rangle^{\alpha} \\
&\qquad = \epsilon^{\alpha a b}\tau\langle u'^{a} b'^{c}\rangle \frac{\partial U^{b}}{\partial x^{c}} - \epsilon^{\alpha b a}\tau\langle b'^{a} u'^{c}\rangle \frac{\partial U^{b}}{\partial x^{c}} \\
&\qquad = \tau\left(\langle u'^{a} b'^{c}\rangle + \langle u'^{c} b'^{a}\rangle\right)\epsilon^{\alpha a b}\frac{\partial U^{b}}{\partial x^{c}}
\end{aligned}$$



$\longrightarrow$ $$\langle \mathbf{u}' \times \mathbf{b}' \rangle = \cdots + \tau\langle \mathbf{u}' \cdot \mathbf{b}' \rangle \nabla \times \mathbf{U} + \cdots$$

cross helicity

Inhomogeneous turbulence



- Homogeneous turbulence

  Turbulence energy externally injected

- Large-scale inhomogeneities

Turbulent energy

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle / 2 = - \langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}' \rangle \cdot \nabla \mathbf{U} - \langle \mathbf{u}' \times \mathbf{b}' \rangle \cdot \nabla \times \mathbf{B} - \varepsilon_K + \cdots$$

Turbulent cross helicity

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \langle \mathbf{u}' \cdot \mathbf{b}' \rangle = - \langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}' \rangle \cdot \nabla \mathbf{B} - \langle \mathbf{u}' \times \mathbf{b}' \rangle \cdot \nabla \times \mathbf{U} - \varepsilon_W + \cdots$$

Turbulent residual helicity

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \langle \mathbf{b}' \cdot \mathbf{j}' - \mathbf{u}' \cdot \boldsymbol{\omega} \rangle = - \langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}' \rangle \cdot \nabla \boldsymbol{\Omega} - \frac{1}{\tau \beta} \langle \mathbf{u}' \times \mathbf{b}' \rangle \cdot \mathbf{B} - \varepsilon_H + \cdots$$

$$\mathbf{j}' = \nabla \times \mathbf{b}', \;\; \boldsymbol{\omega}' = \nabla \times \mathbf{u}', \;\; \boldsymbol{\Omega} = \nabla \times \mathbf{U}$$

II. Cross helicity and related dynamo

Turbulence effects

Reynolds (+ turbulent Maxwell) stress

$$\mathcal{R}_{\rm D}^{\alpha\beta} \equiv \left\langle u'^{\alpha} u'^{\beta} - b'^{\alpha} b'^{\beta} \right\rangle_{\rm D}$$
$$= -\nu_{\rm K} \mathcal{S}_{\rm D}^{\alpha\beta} + \nu_{\rm M} \mathcal{M}_{\rm D}^{\alpha\beta} + [\boldsymbol{\Gamma\Omega}]_{\rm D}^{\alpha\beta}$$

Enhancement Suppression

$$\mathcal{S}^{\alpha\beta} = \frac{\partial U^\beta}{\partial x^\alpha} + \frac{\partial U^\alpha}{\partial x^\beta}$$
$$\mathcal{M}^{\alpha\beta} = \frac{\partial B^\beta}{\partial x^\alpha} + \frac{\partial B^\alpha}{\partial x^\beta}$$

Turbulent electromotive force

$$\mathbf{E}_{\rm M} \equiv \langle \mathbf{u}' \times \mathbf{b}' \rangle$$
$$= -\beta \mathbf{J} + \alpha \mathbf{B} + \gamma \boldsymbol{\Omega}$$

Enhancement Suppression

$$\alpha = \frac{1}{3} \int d\mathbf{k} \int_{-\infty}^{\tau} d\tau_1 G(k, \mathbf{x}; \tau, \tau_1, t) \left[-H_{uu}(k, \mathbf{x}; \tau, \tau_1, t) + H_{bb}(k, \mathbf{x}; \tau, \tau_1, t) \right] \qquad \alpha \propto \tau H$$

$$\beta = \frac{1}{3} \int d\mathbf{k} \int_{-\infty}^{\tau} d\tau_1 G(k, \mathbf{x}; \tau, \tau_1, t) \left[Q_{uu}(k, \mathbf{x}; \tau, \tau_1, t) + Q_{bb}(k, \mathbf{x}; \tau, \tau_1, t) \right] = \frac{5}{7} \nu_{\rm K} \qquad \beta \propto \tau K$$

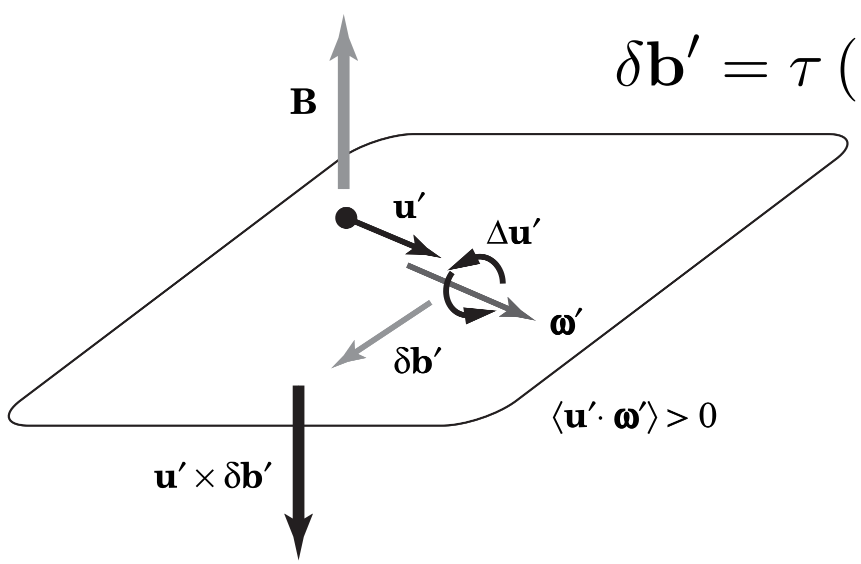
$$\gamma = \frac{1}{3} \int d\mathbf{k} \int_{-\infty}^{\tau} d\tau_1 G(k, \mathbf{x}; \tau, \tau_1, t) \left[Q_{ub}(k, \mathbf{x}; \tau, \tau_1, t) + Q_{bu}(k, \mathbf{x}; \tau, \tau_1, t) \right] = \frac{5}{7} \nu_{\rm M} \qquad \gamma \propto \tau W$$

$$\boldsymbol{\Gamma} = \frac{1}{15} \int d\mathbf{k}\ k^{-2} \int_{-\infty}^{\tau} d\tau_1 G(k, \mathbf{x}; \tau, \tau_1, t) \nabla H_{uu}(k, \mathbf{x}; \tau, \tau_1, t) \qquad \boldsymbol{\Gamma} \propto \tau \ell^2 \nabla H$$

α and cross-helicity effect

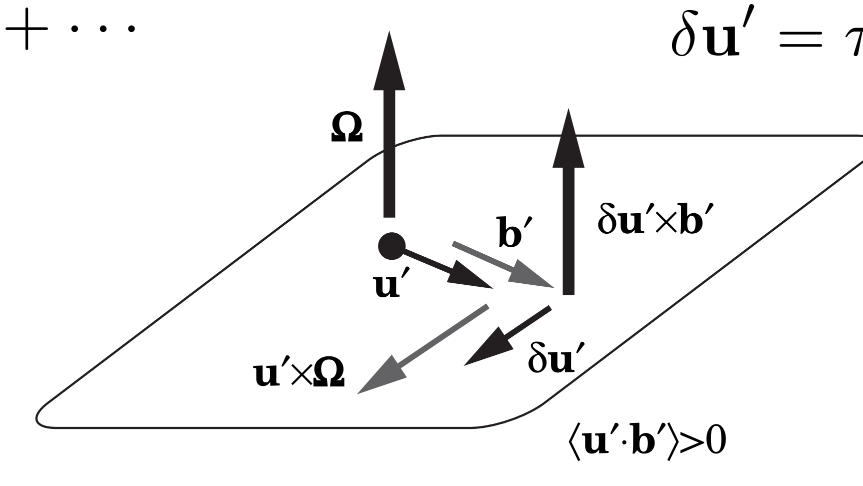
(Yokoi, GAFD **107**, 114, 2013)

$$\delta \mathbf{b}' = \tau \left(\mathbf{B} \cdot \nabla \right) \mathbf{u}' + \cdots$$



kinetic helicity effect

$$\delta \mathbf{u}' = \tau \mathbf{u}' \times \mathbf{\Omega} + \cdots$$



mean vorticity
$\mathbf{\Omega} = \nabla \times \mathbf{U}$

cross-helicity effect

α dynamo

$$\langle \mathbf{u}' \times \mathbf{b}' \rangle = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B} + \gamma \nabla \times \mathbf{U}$$

cross-helicity dynamo

Mean induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B} + \underline{\mathbf{E}_\mathrm{M}}) + \eta \nabla^2 \mathbf{B}$$

Turbulence

$$\mathbf{E}_\mathrm{M} = -\beta \mathbf{J} + \alpha \mathbf{B} + \gamma \boldsymbol{\Omega}$$

$$\mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B}, \quad \mathbf{J} = \mathbf{J}_0 + \delta \mathbf{J}$$

Reference $$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}_0) + \nabla \times (\alpha \mathbf{B}_0 - \beta \nabla \times \mathbf{B}_0)$$

Modulation $$\frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \delta \mathbf{B}) - \nabla \times \left[\beta \nabla \times \left(\delta \mathbf{B} - \frac{\gamma}{\beta} \mathbf{U} \right) \right]$$

→ $$\delta \mathbf{B} = \frac{\gamma}{\beta} \mathbf{U} = C_\mathrm{W} \frac{W}{K} \mathbf{U}$$

$$\frac{|W|}{K} = \frac{|\langle \mathbf{u}' \cdot \mathbf{b}' \rangle|}{\langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle / 2} \leq 1$$

c.f. $$\nabla \times \left(\frac{\gamma}{\beta} \mathbf{U} \right) = \frac{\gamma}{\beta} \nabla \times \mathbf{U} + \nabla \left(\frac{\gamma}{\beta} \right) \times \mathbf{U}$$

Mean momentum equation

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{U} \times \boldsymbol{\Omega} + \mathbf{J} \times \mathbf{B} \underline{- \nabla \cdot \boldsymbol{\mathcal{R}}} + \mathbf{F} - \nabla \left(P + \frac{1}{2}\mathbf{U}^2 + \left\langle \frac{1}{2}\mathbf{b}'^2 \right\rangle \right)$$

Turbulence

$$\mathbf{J} = \sigma \left(\mathbf{E} + \mathbf{U} \times \mathbf{B} + \underline{\mathbf{E}_{\rm M}} \right)$$

Turbulence

$$\begin{cases} \mathcal{R}^{\alpha\beta} = \frac{2}{3} K_{\rm R} \delta^{\alpha\beta} - \nu_{\rm K} \mathcal{S}^{\alpha\beta} + \nu_{\rm M} \mathcal{M}^{\alpha\beta} \\ \mathbf{E}_{\rm M} = -\beta \mathbf{J} + \alpha \mathbf{B} + \gamma \boldsymbol{\Omega} \end{cases}$$

Mean Lorentz force $\mathbf{J} \times \mathbf{B} = \frac{1}{\beta} (\mathbf{U} \times \mathbf{B}) \times \mathbf{B} + \frac{\gamma}{\beta} \boldsymbol{\Omega} \times \mathbf{B} - \frac{1}{\beta} \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right) \times \mathbf{B}$

$$\mathbf{U} = \mathbf{U}_0 + \delta \mathbf{U}, \quad \boldsymbol{\Omega} = \boldsymbol{\Omega}_0 + \delta \boldsymbol{\Omega}$$

Reference $\frac{\partial \boldsymbol{\Omega}_0}{\partial t} = \nabla \times \left[\mathbf{U}_0 \times \boldsymbol{\Omega}_0 + \nu_{\rm K} \nabla^2 \mathbf{U}_0 + \mathbf{F} - \frac{1}{\beta} \left(\frac{\partial \mathbf{A}}{\partial t} + \nabla \varphi \right) \times \mathbf{B} \right]$

Modulation $\frac{\partial \delta \boldsymbol{\Omega}}{\partial t} = \nabla \times \left[\left(\delta \mathbf{U} - \frac{\gamma}{\beta} \mathbf{B} \right) \times \boldsymbol{\Omega}_0 + \nu_{\rm K} \nabla^2 \left(\delta \mathbf{U} - \frac{\gamma}{\beta} \mathbf{B} \right) \right]$

$\longrightarrow$ $\delta \mathbf{U} = \frac{\gamma}{\beta} \mathbf{B} = C_\gamma \frac{W}{K} \mathbf{B}$

$$\frac{|W|}{K} = \frac{|\langle \mathbf{u}' \cdot \mathbf{b}' \rangle|}{\langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle / 2} \le 1$$

Cross-helicity generation mechanism

Evolution equation of the turbulent cross helicity $W \equiv \langle \mathbf{u}' \cdot \mathbf{b}' \rangle$

$$\frac{DW}{Dt} \equiv \left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) W = P_W - \varepsilon_W + \nabla \cdot \mathbf{T}_W$$

where $$P_W = -\mathcal{R}^{ab} \frac{\partial B^a}{\partial x^b} - \mathbf{E}_\mathrm{M} \cdot \mathbf{\Omega}$$ production rate

$$\varepsilon_W = (\nu + \eta) \left\langle \frac{\partial u'^a}{\partial x^b} \frac{\partial b'^a}{\partial x^b} \right\rangle$$ dissipation rate

$$\mathbf{T}_W = K\mathbf{B} - \left\langle (\mathbf{u}' \cdot \mathbf{b}')\, \mathbf{u}' - \left(\frac{\mathbf{u}'^2 + \mathbf{b}'^2}{2} - p'_\mathrm{M} \right) \mathbf{b}' \right\rangle$$ transport rate

$$\nabla \cdot \mathbf{T}_W = \mathbf{B} \cdot \nabla K + \cdots \qquad K \equiv \left\langle \mathbf{u}'^2 + \mathbf{b}'^2 \right\rangle / 2$$

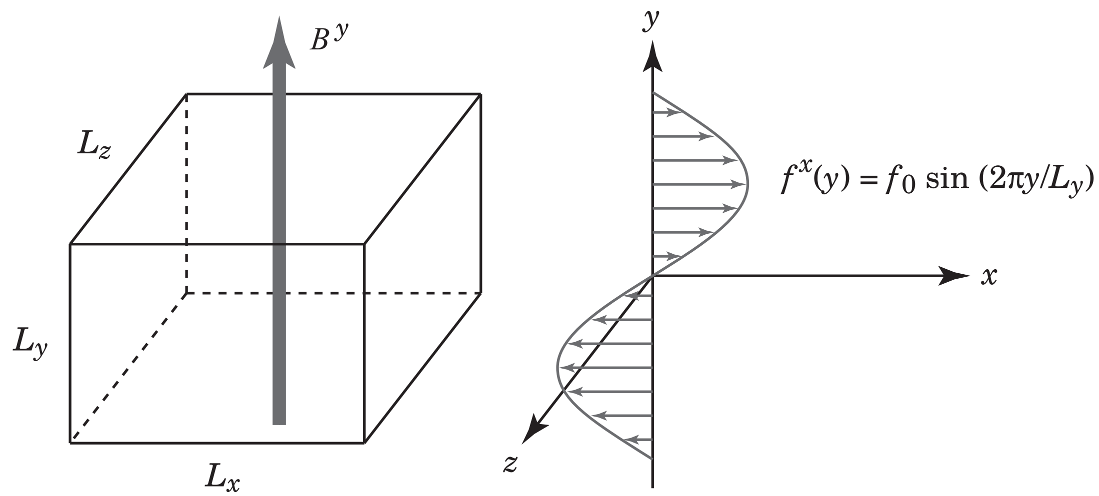
with $\mathcal{R}^{\alpha\beta} = \left\langle u'^\alpha u'^\beta - b'^\alpha b'^\beta \right\rangle$ Reynolds stress

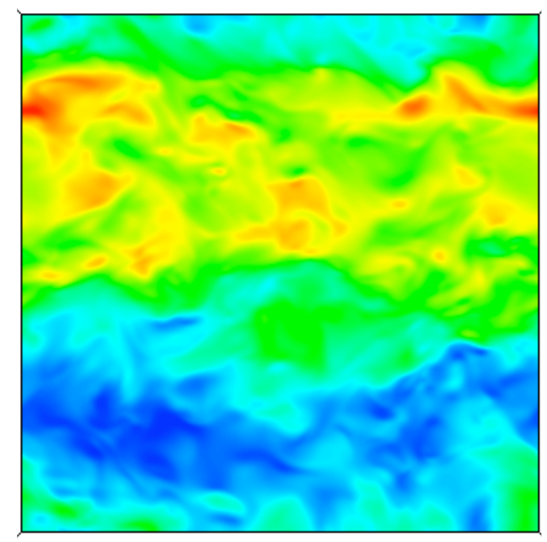
$$\mathbf{E}_\mathrm{M} \equiv \langle \mathbf{u}' \times \mathbf{b}' \rangle$$ Turbulent electromotive force

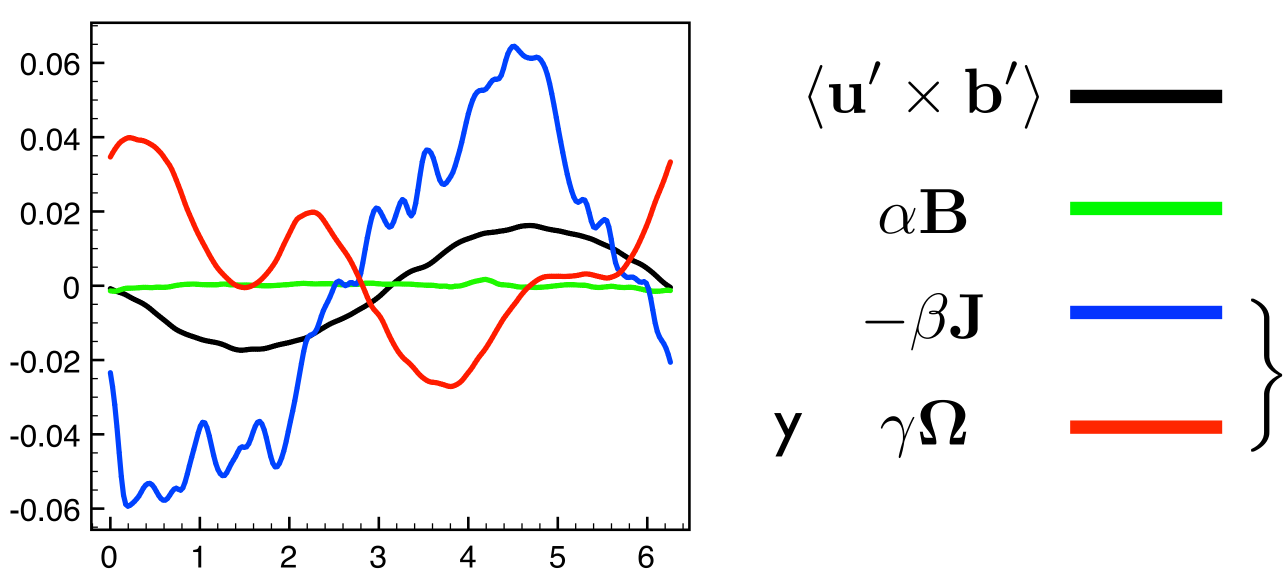
$$= \alpha \mathbf{B} - \beta \mathbf{J} + \gamma \mathbf{\Omega}$$

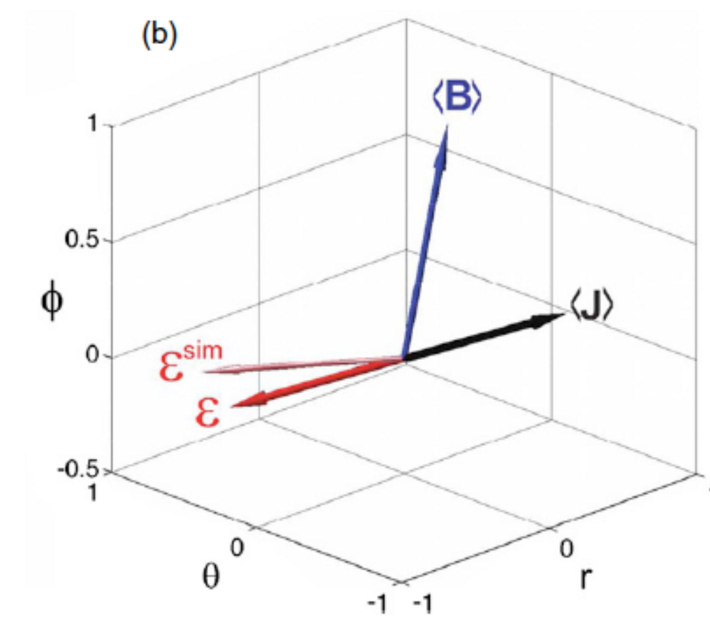
Cross helicity in dynamos

DNS of electromotive force in Kolmogorov flow (Yokoi & Balarac, 2011)









Rahbarnia, et al. ApJ., (2012)

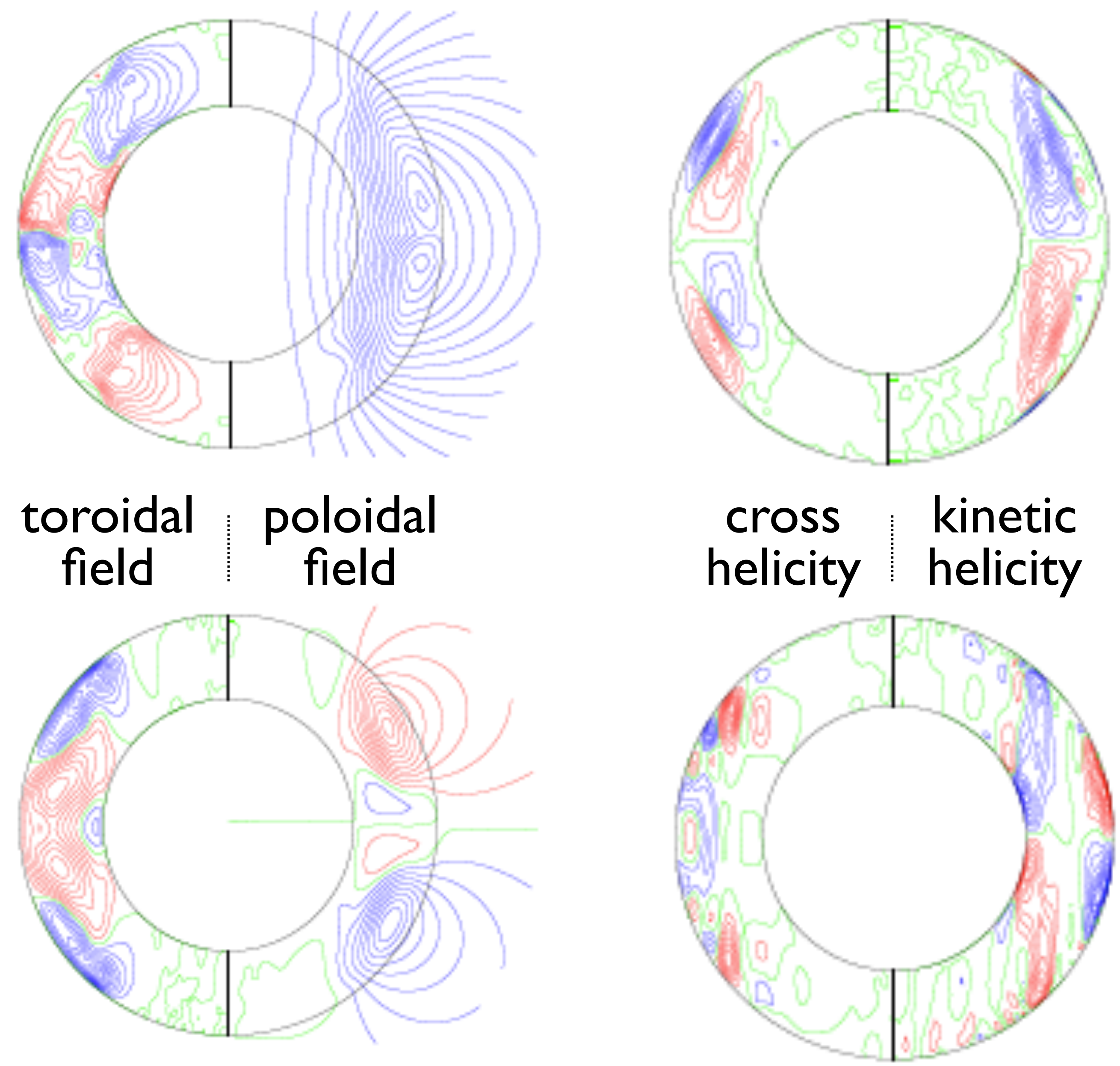
III. Solar-Cycle Dynamo

Pseudoscalar

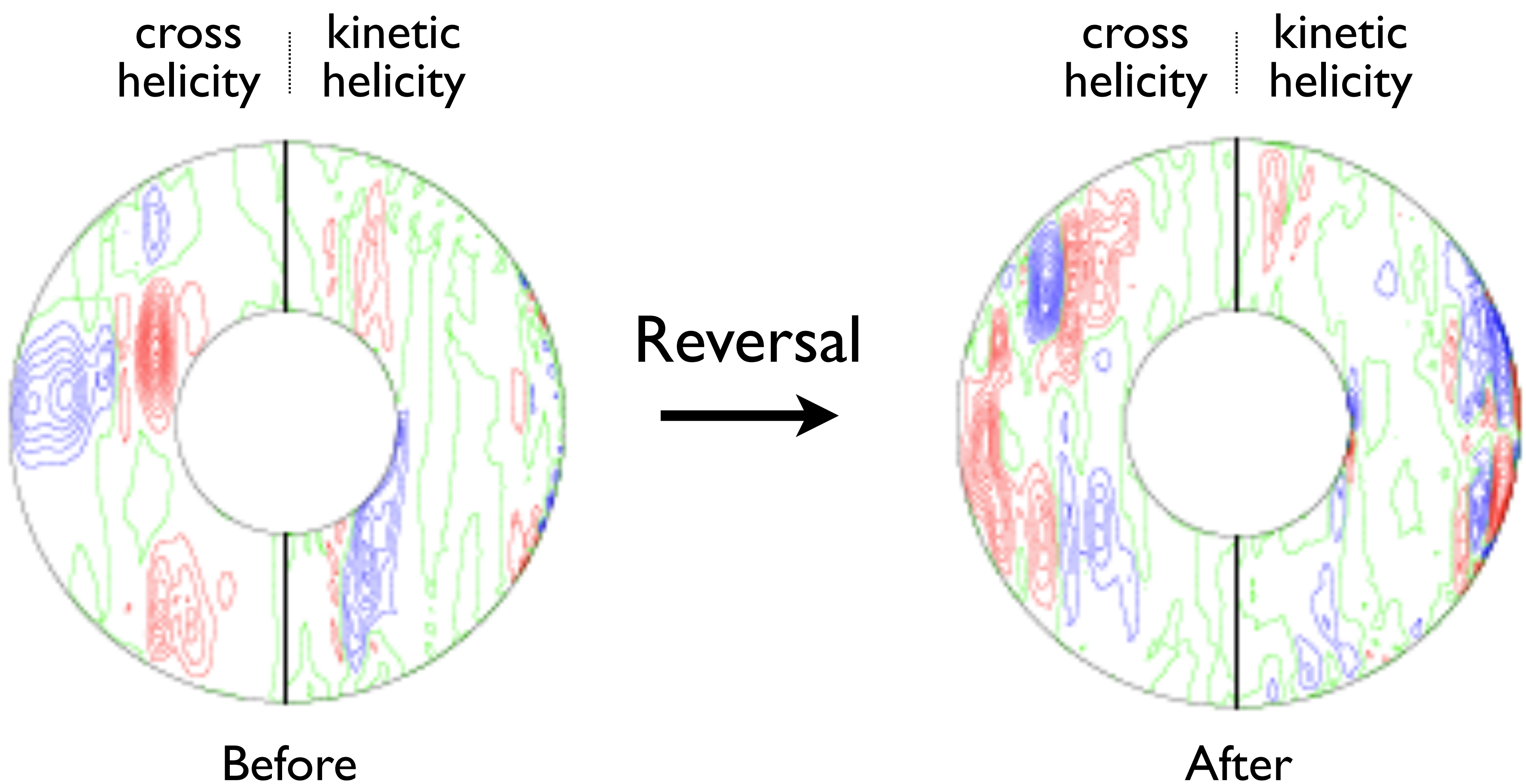
Spatial distribution (with R. Simitev & F. Busse)

Dipole-like case

Quadrupole-like case



Signs of cross helicity and helicity during the polarity reversal



Cross helicity changes its sign

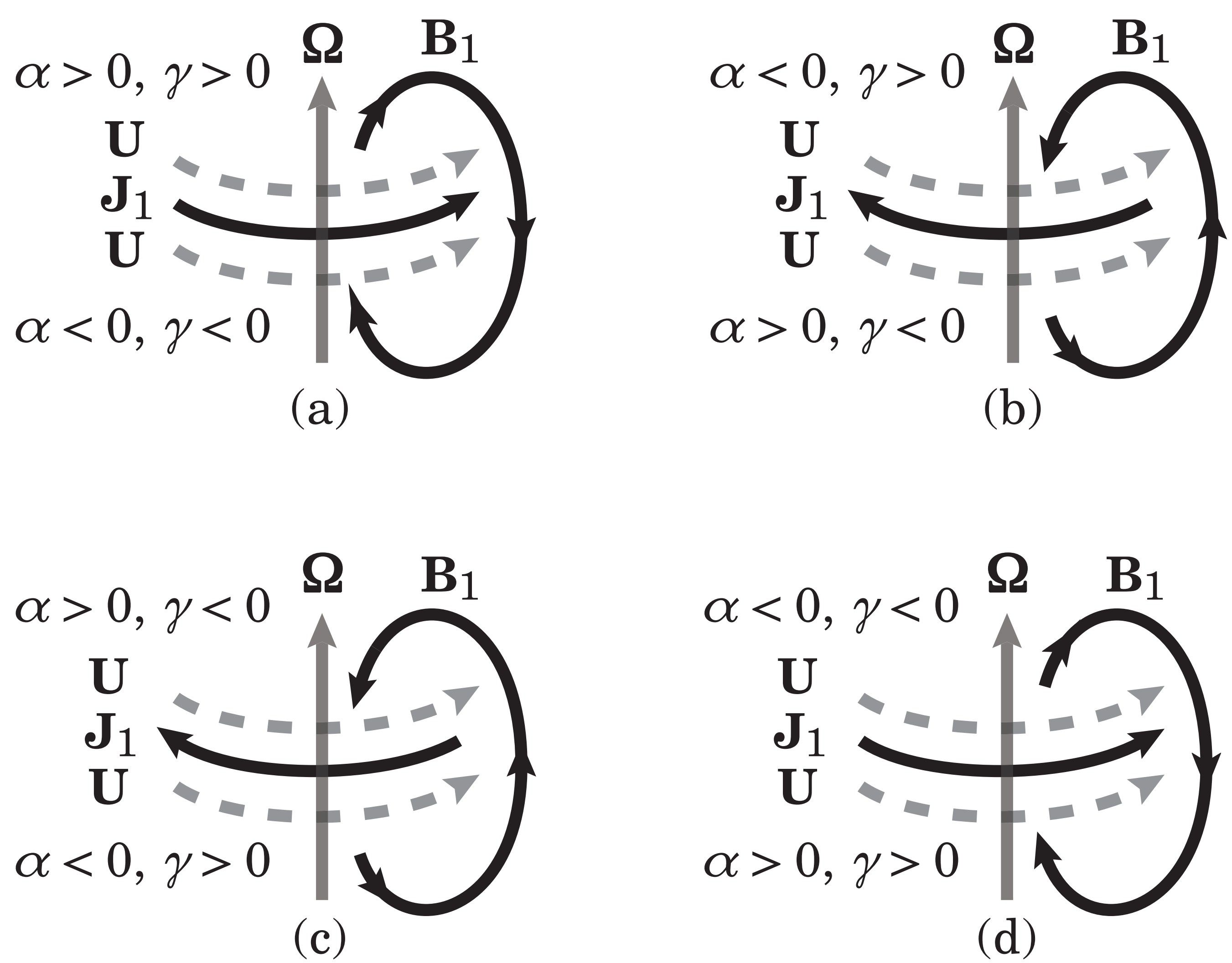
Kinetic helicity does not

Turbulent cross-helicity equation

$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\langle \mathbf{u}'\cdot\mathbf{b}'\rangle = -\langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}'\rangle \cdot \nabla\mathbf{B} \underline{- \langle \mathbf{u}'\times\mathbf{b}'\rangle\cdot\nabla\times\mathbf{U}} - \varepsilon_W + \cdots$$

Negative production of the turb. cross helicity ← $-\alpha\mathbf{B}\cdot\mathbf{\Omega}$

$$P_{W1} = -\alpha\mathbf{B}_1\cdot\mathbf{\Omega} > 0 \qquad \text{for} \qquad \alpha \gtrless 0, \gamma < 0$$



Scenario for periodic reversal

Positive cross helicity

↓ $\mathbf{B}_0 = \frac{\gamma}{\beta}\mathbf{U}$

Generation of the toroidal field due to the cross-helicity (γ) effect

Toroidal magnetic field B_0

↓ $\mathbf{B}_1 = \frac{\alpha}{\beta}\mathbf{B}_0 = \frac{\alpha}{\beta}\frac{\gamma}{\beta}\mathbf{U}$

Generation of the poloidal field due to the helicity (α) effect

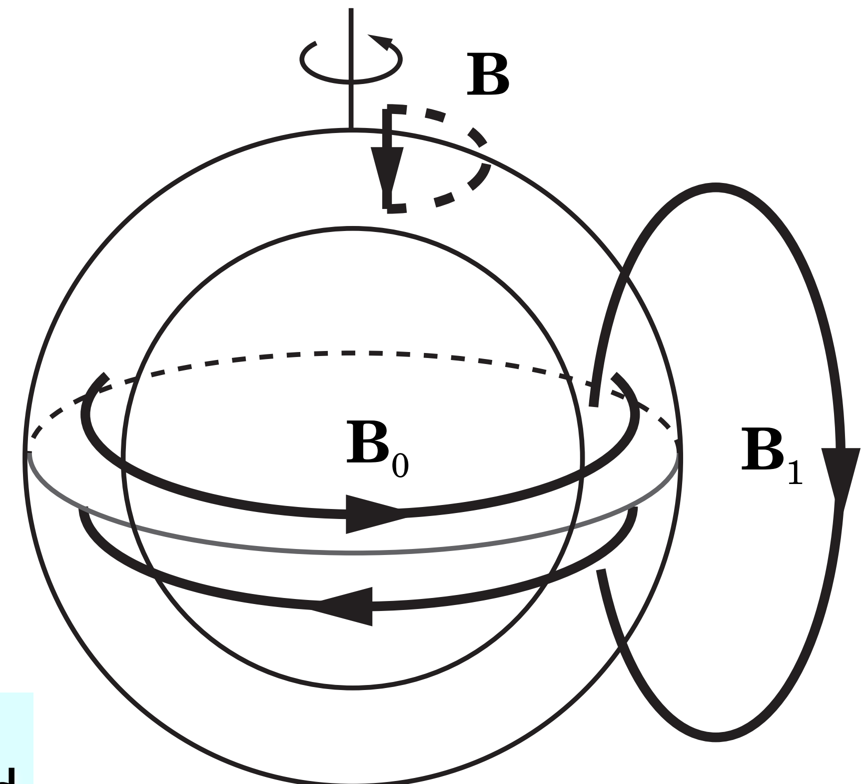
Poloidal magnetic field B_1

↓ $P_{W1} = -\alpha \mathbf{B}_1 \cdot \mathbf{\Omega}$

Negative cross-helicity generation due to the induced poloidal magnetic field B_1

Negative cross helicity

→ Periodic reversal



Turbulent cross-helicity equation

$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\langle \mathbf{u}'\cdot\mathbf{b}'\rangle = -\langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}'\rangle \cdot \nabla\mathbf{B} \underline{- \langle \mathbf{u}'\times\mathbf{b}'\rangle \cdot \nabla\times\mathbf{U}} - \varepsilon_W + \cdots$$

Negative production of the turb. cross helicity ← $-\alpha\mathbf{B}\cdot\mathbf{\Omega}$

Dynamo equations with cross-helicity equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times (-\beta \nabla \times \mathbf{B} + \alpha \mathbf{B} + \gamma \boldsymbol{\Omega})$$

$$\frac{\partial \gamma}{\partial t} = \beta \nabla^2 \gamma - \alpha \tau \mathbf{B} \cdot \boldsymbol{\Omega} + \beta \tau (\nabla \times \mathbf{B}) \cdot \boldsymbol{\Omega} - \gamma \tau \boldsymbol{\Omega}^2$$

with vorticity $\boldsymbol{\Omega} = \nabla \times \mathbf{U}$

cross helicity $\gamma = \tau \langle \mathbf{u}' \cdot \mathbf{b}' \rangle$

Cartesian coordinate $(x, y, z) \simeq (\theta, \phi, r)$

Axisymmetry $\frac{\partial}{\partial y} = 0$

Rotation $\mathbf{U} = (U^x, U^y, U^z) = (0, U^y, 0)$

with shears $\frac{\partial U^y}{\partial x}, \quad \frac{\partial U^y}{\partial z}$ omit meridional circulation

Magnetic field
$$\mathbf{B} = (B^x, B^y, B^z) = \mathbf{B}_{\rm tor} + \mathbf{B}_{\rm pol} = (0, B^y, 0) + \nabla \times (0, A^y, 0)$$
$$= (0, B^y, 0) + \left(-\frac{\partial A^y}{\partial z}, 0, \frac{\partial A^y}{\partial x} \right)$$

$$\begin{cases} \frac{\partial A}{\partial t} = \beta \frac{\partial^2 A}{\partial x^2} + \alpha B \\ \frac{\partial B}{\partial t} = \beta \frac{\partial^2 B}{\partial x^2} - \frac{\partial}{\partial x} \left(\gamma \frac{\partial U}{\partial x} \right) \\ \frac{\partial \gamma}{dt} = \beta \frac{\partial^2 \gamma}{\partial x^2} - \alpha \tau \frac{\partial U}{\partial x} \frac{\partial A}{\partial x} \end{cases}$$

$\longleftarrow$ $\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\cancel{\mathbf{U} \times \mathbf{B}_1} + \alpha \mathbf{B}_0 - \beta \mathbf{J}_1)$

$\longleftarrow$ $\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\cancel{\mathbf{U} \times \mathbf{B}_0} - \beta \mathbf{J}_0 + \gamma \boldsymbol{\Omega})$

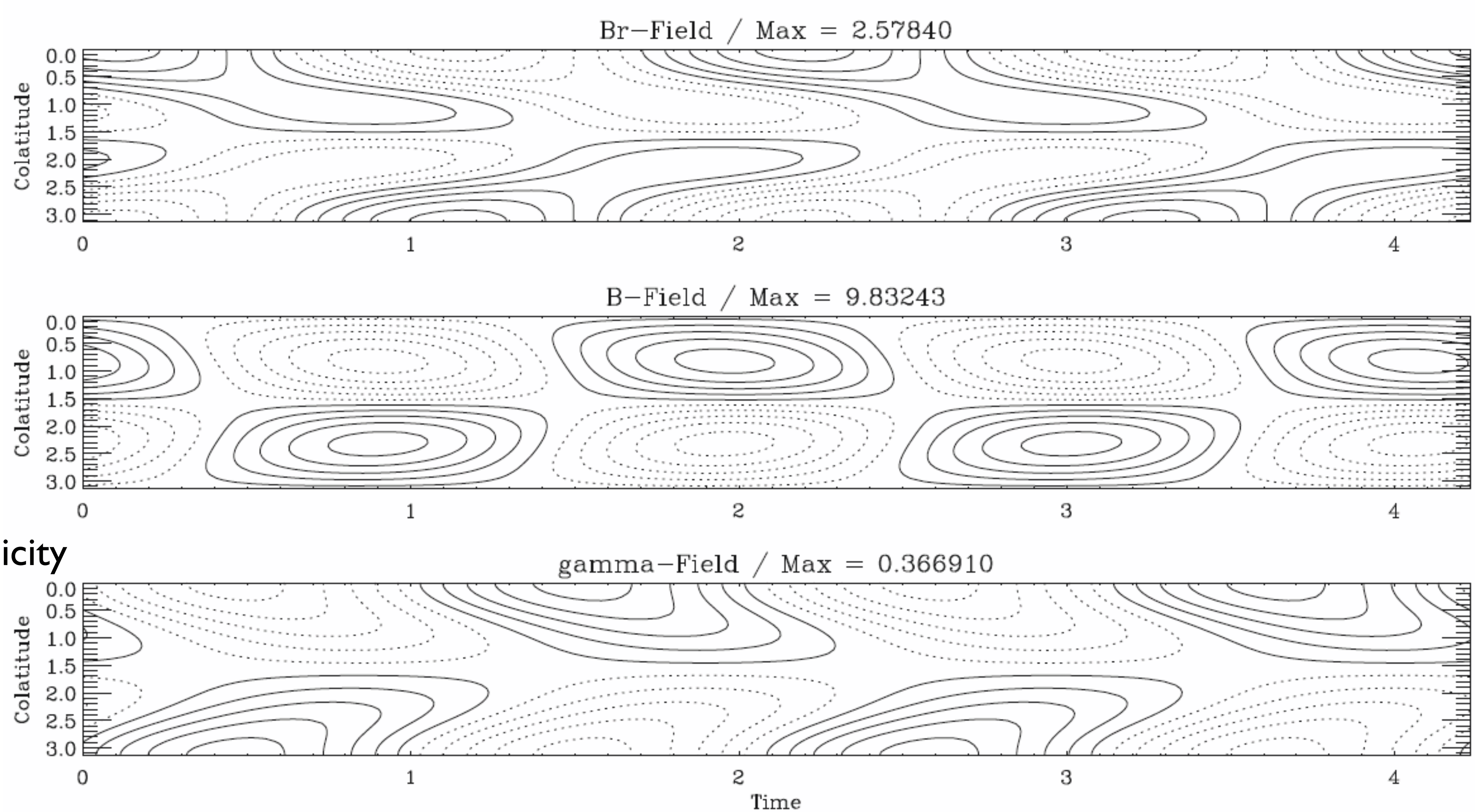
$\frac{\partial W}{\partial t} = -\alpha \mathbf{B}_1 \cdot \boldsymbol{\Omega} + \cdots$

$$P_{\mathrm{crit}} = 17.53 \quad \text{with} \quad \omega_{\mathrm{I}} = 2.97$$

Poloidal

Toroidal

Cross helicity



(with Valery Pipin)

Poloidal

$$\frac{\partial A}{\partial t} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial A\sin\theta}{\partial\theta} + B\cos\theta$$

Toroidal

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial(B\sin\theta)}{\partial\theta} - 2\gamma C_\gamma \mathcal{D}(x\sin\theta+1)f(\theta)$$

Cross helicity

$$\frac{\partial \gamma}{\partial t} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\gamma}{\partial\theta} - \frac{\xi}{\sin\theta}\frac{\partial A\sin\theta}{\partial\theta}$$

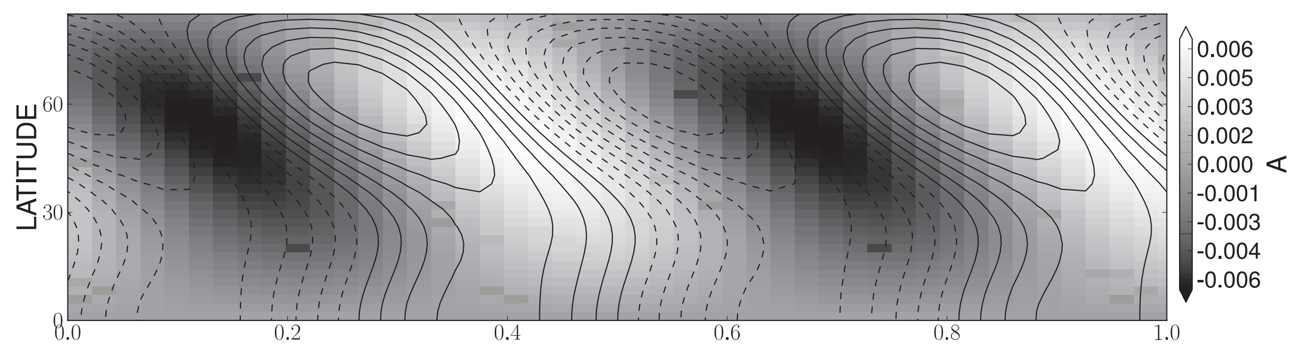
$\mathcal{D}$ dynamo number

$f(\theta) = \frac{\partial\Omega}{\partial x}$ radial derivative of the shear

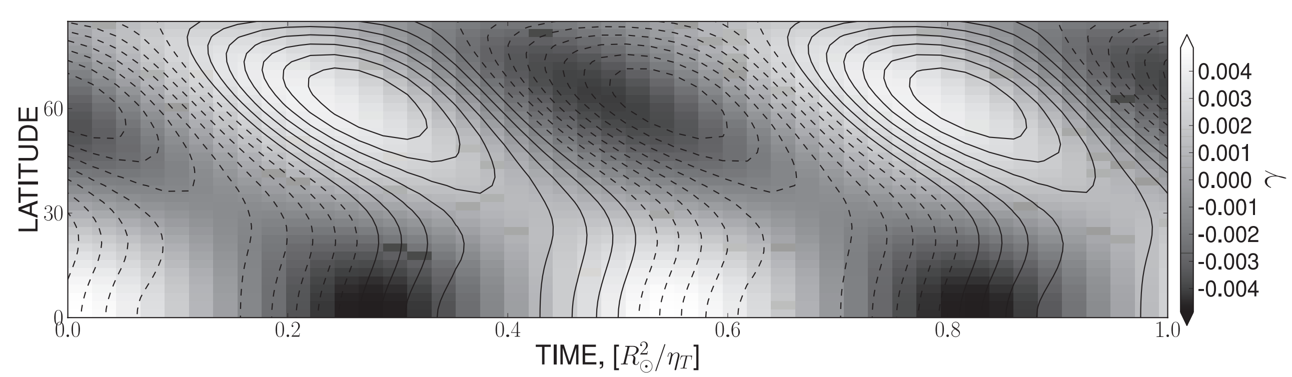
ξ stratification parameter

C_γ model constant related to the cross-helicity generation

Toroidal (in contour) vs poloidal



Cross helicity



"Butterfly diagram" is generated without resorting to the Ω effect

$$
\left\{
\begin{aligned}
&\frac{\partial A}{\partial t} = \beta \frac{\partial^2 A}{\partial x^2} + \alpha B \\
&\frac{\partial B}{\partial t} = \beta \frac{\partial^2 B}{\partial x^2} - \frac{\partial}{\partial x}\left(\gamma \frac{\partial U}{\partial x} \right) \\
&\frac{\partial \gamma}{dt} = \beta \frac{\partial^2 \gamma}{\partial x^2} - \alpha \tau \frac{\partial U}{\partial x} \frac{\partial A}{\partial x}
\end{aligned}
\right.
\quad
\begin{aligned}
&\longleftarrow \\
&\longleftarrow \\
&\longleftarrow
\end{aligned}
\quad
\begin{aligned}
&\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\cancel{\mathbf{U} \times \mathbf{B}_1} + \alpha \mathbf{B}_0 - \beta \mathbf{J}_1) \\
&\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\cancel{\mathbf{U} \times \mathbf{B}_0} - \beta \mathbf{J}_0 + \gamma \mathbf{\Omega}) \\
&\frac{\partial W}{\partial t} = -\alpha \mathbf{B}_1 \cdot \mathbf{\Omega} + \cdots
\end{aligned}
$$

- Cross-helicity and α effects:
  - Toroidal field is generated by the cross-helicity effect
  - Poloidal field is generated by the α effect;
  - Induced poloidal field reduces the cross-helicity generation (produces turbulence cross helicity with the opposite sign);
  - Oscillatory behavior of magnetic field through the cross helicity oscillation;
- Dynamic dynamo equation with cross-helicity evolution equation

Future work

- Red Dwarfs (cool stars)
  - highly turbulent but no differential rotation
- Inclusion of modes with other symmetry
  - dipole, quadruple magnetic fields
  - Maunder-minimum-like behavior
- Meridional circulation
- evolution equation for α
  - fluctuation of α

Magnetic reconnection

Simplest reconnection model

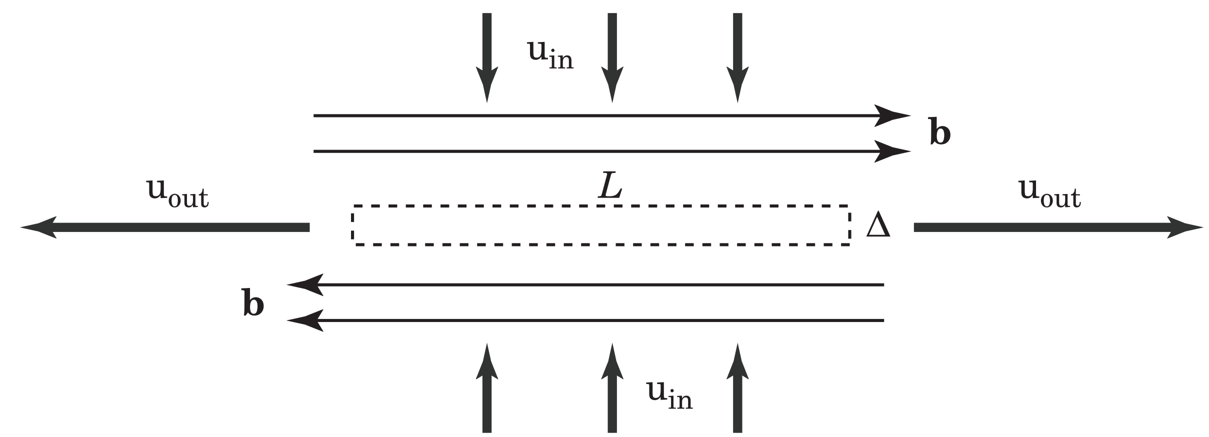
Sweet (1958), Parker (1957)

Mass conservation

$$u_{\rm in} L = u_{\rm out}\Delta$$

Energy conservation

$$\frac{1}{2\mu_0} b_{\rm in}^2 u_{\rm in} L = \frac{1}{2}\rho u_{\rm out}^2 u_{\rm out}\Delta$$



Out-flow speed can be estimated by the Alfvén speed

$$u_{\rm out} = b_{\rm in}/(\mu_0\rho)^{1/2} \equiv V_{\rm A}$$

In-flow Alfvén Mach number

$$M_{\rm in} = \frac{u_{\rm in}}{V_{\rm A}} = \frac{u_{\rm in}}{u_{\rm out}} = \frac{\Delta}{L}$$

$$\Delta = \eta/u_{\rm i}$$

Magnetic field diffused by η in the domain of Δ is supplied by magnetic field convected by inflow

$$\eta\frac{b_{\rm in}}{\Delta} = b_{\rm in}u_{\rm in}$$

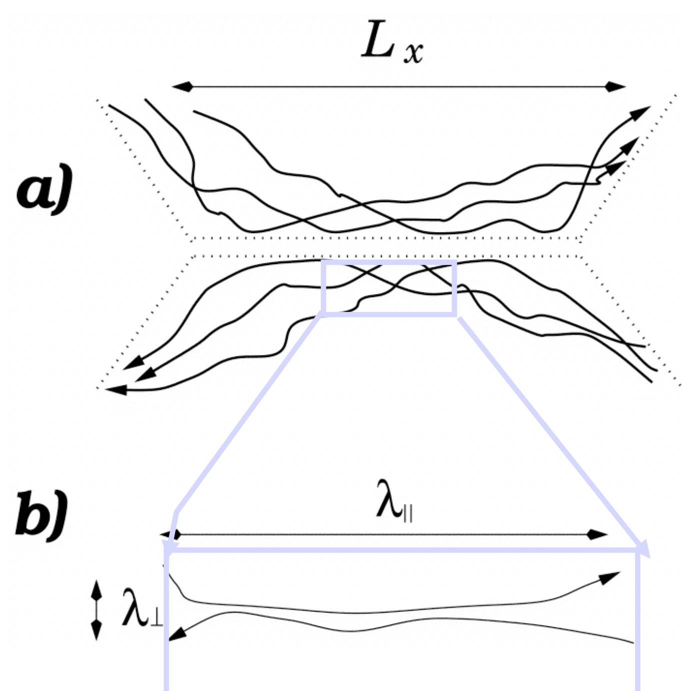
Reconnection rate

$$M_{\rm in} = S^{-1/2}$$

where the Lundquist number S is defined by

$$S = \frac{V_{\rm A}L}{\eta}$$

Turbulent reconnection



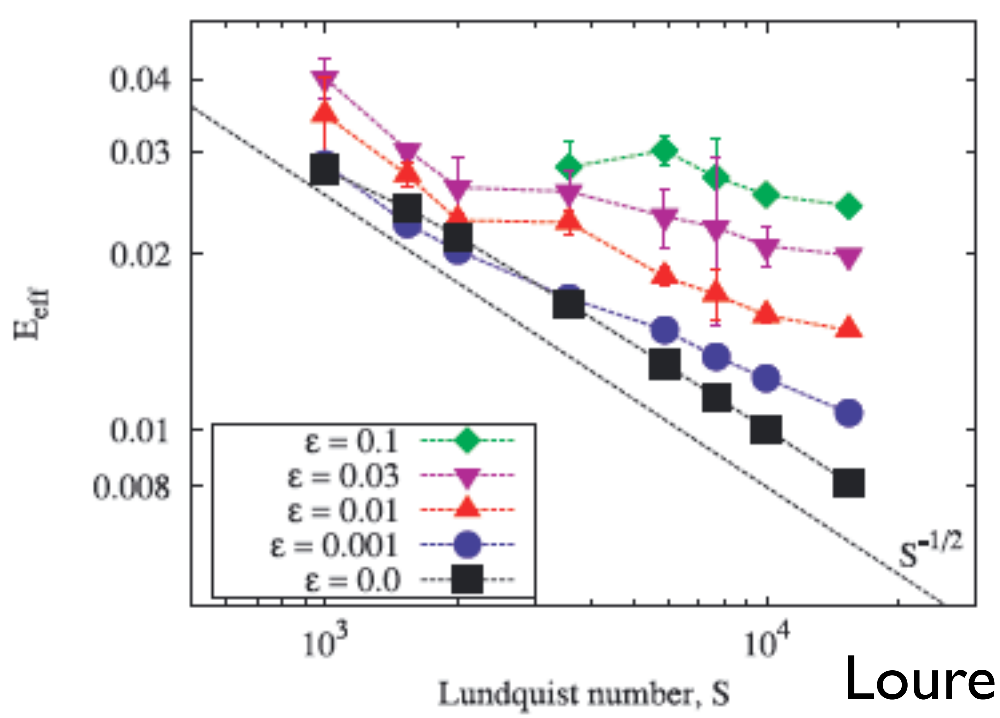
Matthaeus & Lamkin (1985)

Lazarian & Vishniac (1999)

$$M_{\rm in} = \frac{U_{\rm in}}{V_{\rm Ain}} \leq M_{\rm turb}$$

$M_{\rm turb}$: large-scale magnetic Mach number of turbulence

Eyink, et al. (2011) Lagrangian trajectory



Loureiro, et al. (2009)

Numerical simulations

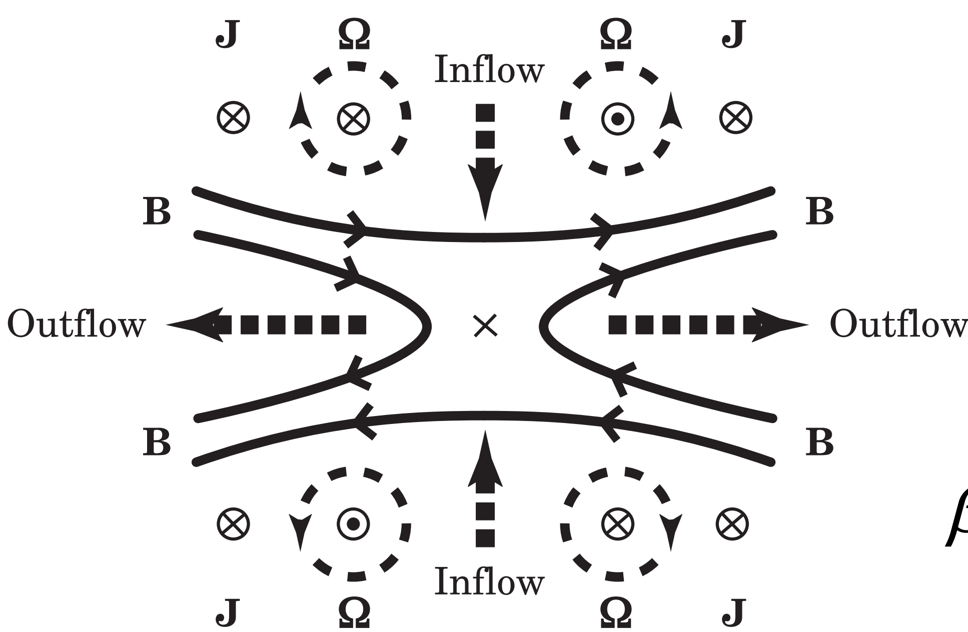
Kowal, et al. (2009)

Servidio, et al. (2009)

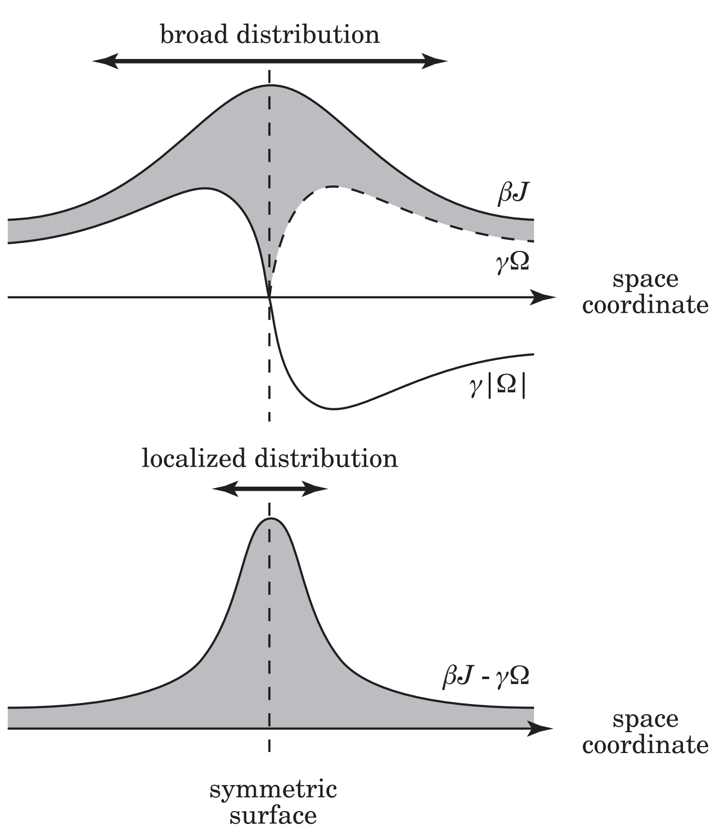
Loureiro, et al. (2009)

Lapenta & Lazarian (2011)

Inhomogeneous turbulence in reconnection



Yokoi & Hoshino, PoP 2011
Higashimori, Yokoi & Hoshino, PRL 2013
Yokoi, Higashimori & Hoshino, PoP 2013



β : Energy

$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\langle \mathbf{u}'^2 + \mathbf{b}'^2\rangle/2 = \cdots + \beta \mathbf{J}^2 - \varepsilon_K + \cdots$$

γ : Cross helicity

$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\langle \mathbf{u}'\cdot\mathbf{b}'\rangle = \cdots + \beta \mathbf{J}\cdot\mathbf{\Omega} - \varepsilon_W + \cdots$$

α : Residual helicity

$$\left(\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right)\langle \mathbf{b}'\cdot\mathbf{j}' - \mathbf{u}'\cdot\boldsymbol{\omega}'\rangle = \cdots + \frac{1}{\tau}\mathbf{B}\cdot\mathbf{J} - \varepsilon_H + \cdots$$

Basic equations to be solved

Mean fields

4th-order Runge-Kutta scheme in time
4th-order centered difference in space

$$\frac{\partial \overline{\rho}}{\partial t} = -\nabla \cdot (\overline{\rho} \mathbf{U})$$

$$\frac{\partial}{\partial t} (\overline{\rho} \mathbf{U}) = -\nabla \cdot \left[\overline{\rho} \mathbf{U}\mathbf{U} - \mathbf{B}\mathbf{B} + \left(p + \frac{1}{2} \mathbf{B}^2 \right) \boldsymbol{\mathcal{I}} \right]$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} + \frac{1}{2} \overline{\rho} \mathbf{U}^2 + \frac{1}{2} \mathbf{B}^2 \right) = -\nabla \cdot \left[\left(\frac{\gamma}{\gamma - 1} p + \frac{1}{2} \overline{\rho} \mathbf{U}^2 \right) \mathbf{U} + \mathbf{E} \times \mathbf{B} \right]$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

Effective transport due to the turbulent motions

$$\mathbf{E} = -\mathbf{U} \times \mathbf{B} + \eta \mathbf{J} + \beta \mathbf{J} - \gamma \mathbf{\Omega} = -\mathbf{U} \times \mathbf{B} + \eta \mathbf{J} + \tau (K \mathbf{J} - W \mathbf{\Omega})$$

$$\beta = \tau K$$

$$\gamma = \tau W$$

Turbulent quantities

Turbulence production due to large-scale inhomogeneities

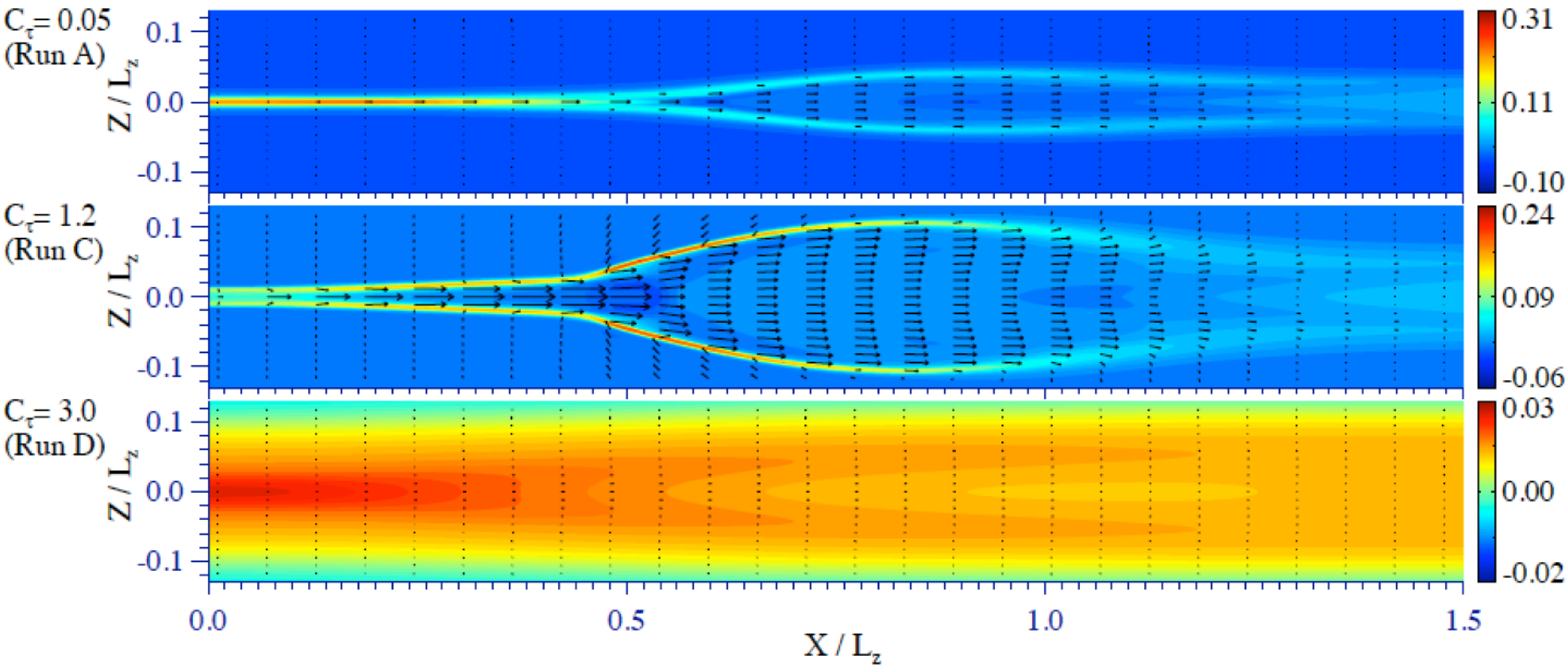
Turbulent energy

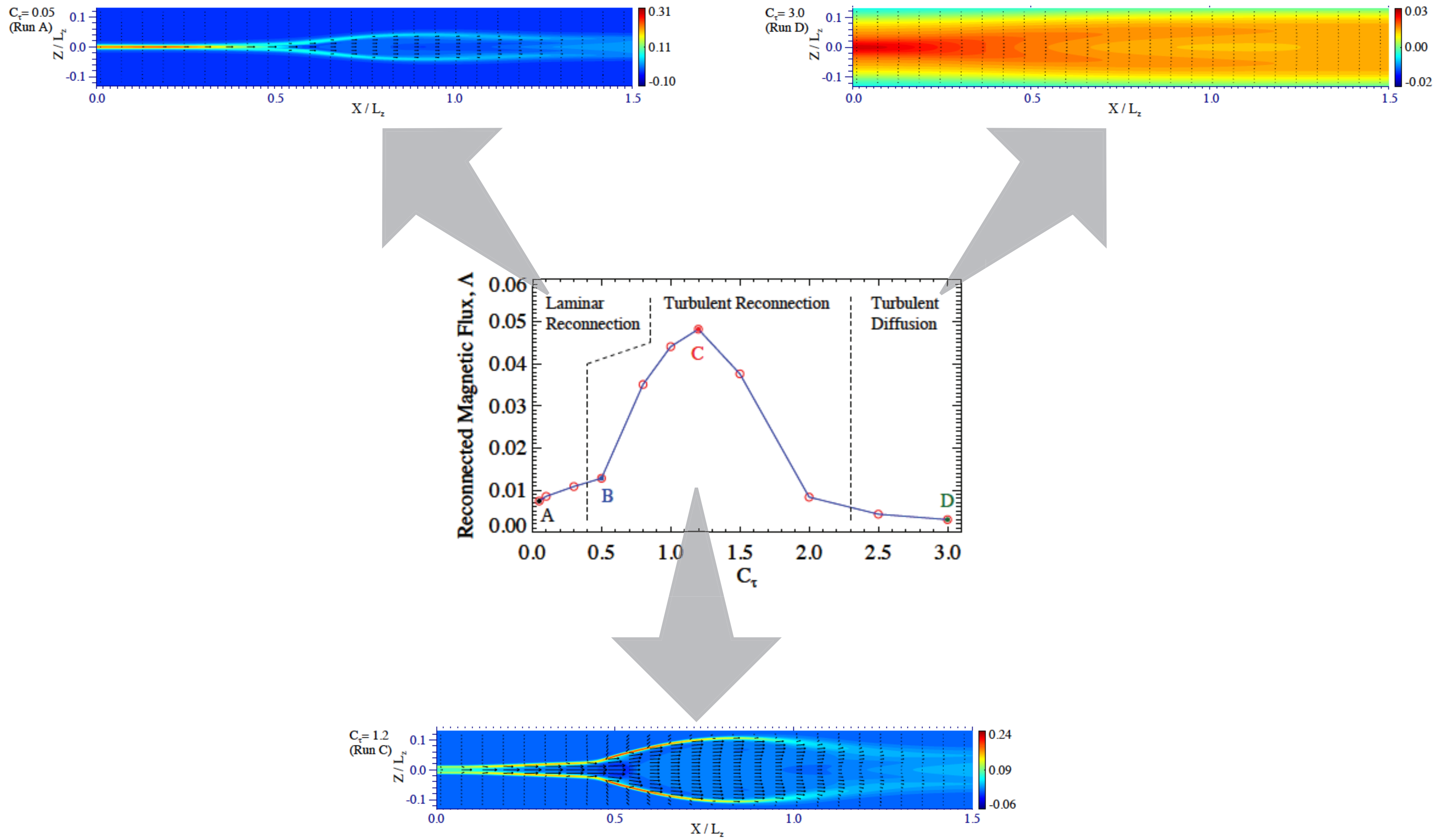
$$\frac{\partial K}{\partial t} = -\mathbf{U} \cdot \nabla K + \tau K \mathbf{J}^2 - \tau W \mathbf{\Omega} \cdot \mathbf{J} + \mathbf{B} \cdot \nabla W - \frac{K}{\tau}$$

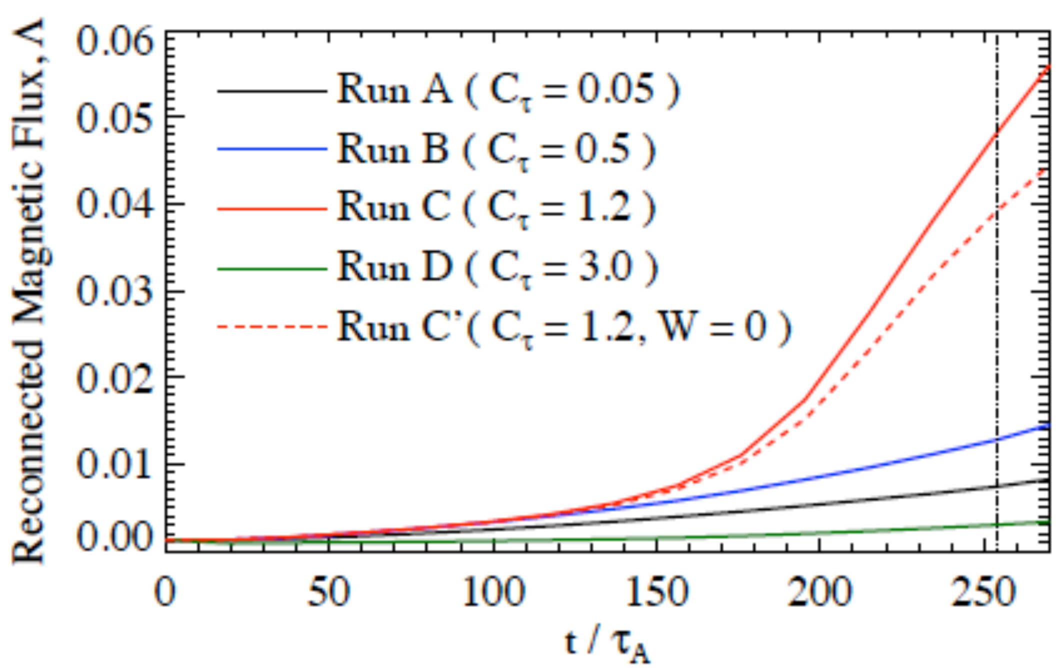
Turbulent cross helicity

$$\frac{\partial W}{\partial t} = -\mathbf{U} \cdot \nabla W + \tau K \mathbf{\Omega} \cdot \mathbf{J} - \tau W \mathbf{\Omega}^2 + \mathbf{B} \cdot \nabla K - C_W \frac{W}{\tau}$$

Electric-current and flow structures





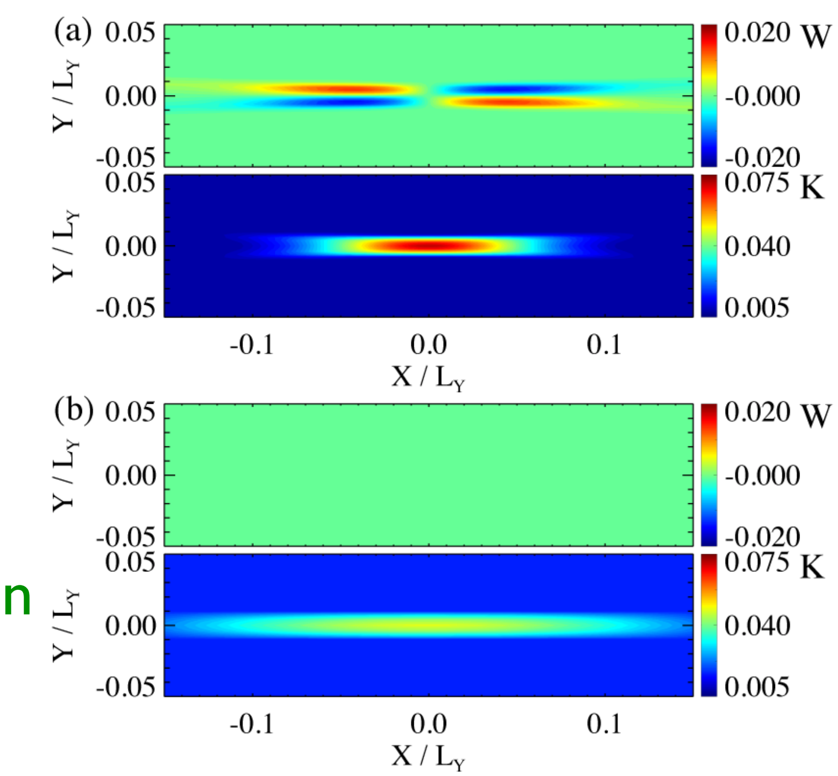


Turbulent reconnection

Turbulent reconnection without cross helicity

Laminar reconnection

Turbulent magnetic diffusion (too much turbulent)



V. Summary

- Inhomogeneities of large-scale fields
- Cross helicity in the turbulent electromotive force
- Applications
    - Solar-cycle dynamo
    - Turbulent magnetic reconnection
- Momentum transport

$$\mathcal{R}^{\alpha\beta} := -\nu_{\rm K}\mathcal{S}^{\alpha\beta} + [\mathbf{\Gamma}\mathbf{\Omega}]^{\alpha\beta},$$
$$\mathbf{E}_{\rm M} := -\beta\mathbf{J} + \alpha\mathbf{B}$$

$$\mathcal{R}^{\alpha\beta} := -\nu_{\rm K}\mathcal{S}^{\alpha\beta} + \nu_{\rm M}\mathcal{M}^{\alpha\beta},$$
$$\mathbf{E}_{\rm M} := -\beta\mathbf{J} + \gamma\mathbf{\Omega}$$

$$\mathcal{R}^{\alpha\beta} := -\nu_{\rm K}\mathcal{S}^{\alpha\beta} + \nu_{\rm M}\mathcal{M}^{\alpha\beta} + [\mathbf{\Gamma}\mathbf{\Omega}]^{\alpha\beta},$$
$$\mathbf{E}_{\rm M} := -\beta\mathbf{J} + \gamma\mathbf{\Omega} + \alpha\mathbf{B}$$

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Yokoi, N. & Balarac, G. (2011) J. Phys. Conf. Ser. **318**, 072039

★ Yokoi, N. (2013) Geophys. Astrophys. Fluid Dyn. **107**, 114

Yokoi, N., Schmitt, D. & Pipin, V. (2015) to be submitted

Details of eigenvalue analysis

omit index *y* on B^y, A^y, and U^y in the following

omit α term in the equation for *B*

$$\frac{\partial A}{\partial t} = \beta \left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial z^2} \right) + \alpha B$$

$$\frac{\partial B}{\partial t} = \beta \left(\frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial z^2} \right) - \frac{\partial^2 U}{\partial z^2}\gamma - \frac{\partial U}{\partial z}\frac{\partial \gamma}{\partial z} - \frac{\partial^2 U}{\partial x^2}\gamma - \frac{\partial U}{\partial x}\frac{\partial \gamma}{\partial x} - \frac{\partial U}{\partial z}\frac{\partial A}{\partial x} - \frac{\partial U}{\partial x}\frac{\partial A}{\partial z}$$

$$\frac{\partial \gamma}{\partial t} = \beta \left(\frac{\partial^2 \gamma}{\partial x^2}\frac{\partial^2 \gamma}{\partial z^2} \right) - \alpha\tau \left(\frac{\partial U}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial U}{\partial z}\frac{\partial A}{\partial z} \right) + \beta\tau \left(\frac{\partial U}{\partial x}\frac{\partial B}{\partial x} - \frac{\partial U}{\partial z}\frac{\partial B}{\partial z} \right)$$
$$- \gamma\tau \left(\left(\frac{\partial U}{\partial x} \right)^2 + \left(\frac{\partial U}{\partial z} \right)^2 \right)$$

non-dimensional variables

$$B = B_0\tilde{B},\ A = B_0 L\tilde{A},\ \gamma = \gamma_0\tilde{\gamma} = B_0 L\tilde{\gamma},\ x = L\tilde{x},\ t = \tfrac{L^2}{\beta_0}\tilde{t}$$

$$U = U_0\tilde{U},\ \alpha = \alpha_0\tilde{\alpha},\ \beta = \beta_0\tilde{\beta}$$

symmetry assumption with respect to the equator

$$\tilde{U} = \sin x,\ \tilde{\alpha} = \cos x,\ \tilde{\beta} = 1,\ \tilde{t} = 1$$

omit all ~ in the following

assume e^{ikz} dependence of *A*, *B*, γ,   assume $\dfrac{\partial U}{\partial z} = k_u U$

$$\frac{\partial A}{\partial t} = \frac{\partial^2 A}{\partial x^2} - k^2 A + R_\alpha \cos x B$$

$$\frac{\partial B}{\partial t} = \frac{\partial^2 B}{\partial x^2} - k^2 B - R_u\left(k_u^2 \sin x\gamma + ikk_u \sin x\gamma - \sin x\gamma + \cos x\frac{\partial \gamma}{\partial x}\right) + R_u\left(k_u \sin x\frac{\partial A}{\partial x} - ik\cos xA\right)$$

$$\frac{\partial \gamma}{\partial t} = \frac{\partial^2 \gamma}{\partial x^2} - k^2\gamma - R_\alpha R_u\left(\cos^2 x\frac{\partial A}{\partial x} - ikk_u \sin x\cos xA\right) + R_u\left(\cos x\frac{\partial B}{\partial x} - ikk_u \sin xB\right) - R_u^2\left(\cos^2 x + k_u^2 \sin^2 x\right)\gamma$$

for simplicity, assume now *k*=0, i.e., omit *z* derivatives of *A*, *B*, and γ, 1D, implying $B_x \simeq B_\theta = 0$

introduce $f_i = \{0, 1\}$ to switch individual terms on and off

$$\frac{\partial A}{\partial t} = \frac{\partial^2 A}{\partial x^2} - k^2 A + R_\alpha \cos x B$$

$$\frac{\partial B}{\partial t} = \frac{\partial^2 B}{\partial x^2} + R_u \left(-f_1 k_u^2 \sin x \gamma + f_2 \sin x \gamma - f_3 \cos x \frac{\partial \gamma}{\partial x} \right) + R_u \left(f_4 k_u \sin x \frac{\partial A}{\partial x} \right)$$

$$\frac{\partial \gamma}{\partial t} = \frac{\partial^2 \gamma}{\partial x^2} - R_\alpha R_u f_5 \left(\cos^2 x \frac{\partial A}{\partial x} \right) + R_u f_6 \cos x \frac{\partial B}{\partial x} - R_u^2 f_7 \left(\cos^2 x + k_u^2 \sin^2 x \right) \gamma$$

examples $f_1 = f_2 = f_3 = f_5 = f_6 = f_7 = 0, \; f_4 = 1$: α–Ω dynamo

$f_1 = f_4 = f_6 = f_7 = 0, \; f_2 = f_3 = f_5 = 1$: original version the following is the original (reduced) version

$$\frac{\partial A}{\partial t} = \frac{\partial^2 A}{\partial x^2} - k^2 A + R_\alpha \cos x B$$
$$\frac{\partial B}{\partial t} = \frac{\partial^2 B}{\partial x^2} + R_u \left(\sin x \gamma - \cos x \frac{\partial \gamma}{\partial x} \right)$$
$$\frac{\partial \gamma}{\partial t} = \frac{\partial^2 \gamma}{\partial x^2} - R_\alpha R_u \cos^2 x \frac{\partial A}{\partial x}$$

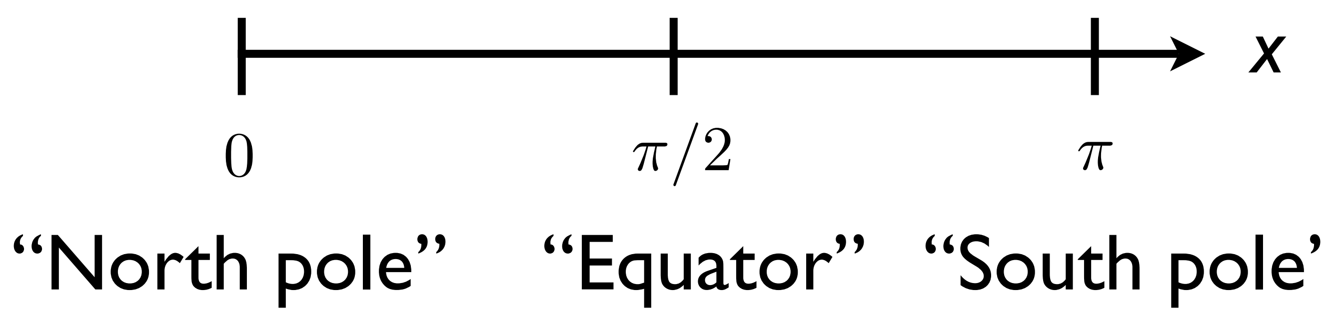
Reynolds numbers: $R_u = \frac{U_0 L}{\beta_0}$ and $R_\alpha = \frac{\alpha_0 L}{\beta_0}$

new variables: $\tilde{A} = R_\alpha R_u A, \ \tilde{B} = R_\alpha^2 R_u B,$ omit ~ again

$$\frac{\partial A}{\partial t} = \frac{\partial^2 A}{\partial x^2} - k^2 A + \cos x B$$
$$\frac{\partial B}{\partial t} = \frac{\partial^2 B}{\partial x^2} + P^2 \left(\sin x \gamma - \cos x \frac{\partial \gamma}{\partial x} \right) \qquad P = R_\alpha R_u$$
$$\frac{\partial \gamma}{\partial t} = \frac{\partial^2 \gamma}{\partial x^2} - \cos^2 x \frac{\partial A}{\partial x}$$

solution depends only on square of dynamo number: $P = R_\alpha R_u$

boundary conditions for solutions antisymmetric with respect to the equator



$$L = \frac{\pi}{2}$$

$$x = 0: \quad A = B = \frac{\partial \gamma}{\partial x} = 0$$

$$x = \frac{\pi}{2}: \quad \frac{\partial A}{\partial x} = B = \gamma = 0$$

free-decay mode

$$\frac{\partial A}{\partial t} = \frac{\partial^2 A}{\partial x^2}, \quad \frac{\partial B}{\partial t} = \frac{\partial^2 B}{\partial x^2}, \quad \frac{\partial \gamma}{\partial t} = \frac{\partial^2 \gamma}{\partial x^2}$$

$$A_n = e^{\omega_n t} \sin nx \quad \text{with} \quad \omega_n = -n^2, \quad n = 1, 3, 5, \cdots$$

$$B_n = e^{\omega_n t} \sin nx \quad \text{with} \quad \omega_n = -n^2, \quad n = 2, 4, 6, \cdots$$

$$\gamma_n = e^{\omega_n t} \cos nx \quad \text{with} \quad \omega_n = -n^2, \quad n = 1, 3, 5, \cdots$$

solution of dynamo equation as eigenvalue problem

discretization or better: expansion in decay modes (or any other complete, orthogonal system of functions which already satisfies the boundary conditions)

$$A(x,t) = e^{\omega t} \sum_{n=1,3,5,\cdots}^{N-1} a_n \sin nx$$

$$B(x,t) = e^{\omega t} \sum_{n=2,4,6,\cdots}^{N} b_n \sin nx$$

$$\gamma(x,t) = e^{\omega t} \sum_{n=1,3,5,\cdots}^{N-1} c_n \cos nx$$

$$\omega \sum_{n=1,3,\cdots} a_n \sin nx = -\sum_{n=1,3,\cdots} a_n n^2 \sin nx + \cos x \sum_{n=2,4,\cdot} b_n \sin nx$$

$$\omega \sum_{n=2,4,\cdots} b_n \sin nx = -\sum_{n=2,4,\cdots} b_n n^2 \sin nx$$

$$+ P^2 \left(\sin x \sum_{n=1,3,\cdot} c_n \cos nx + \cos x \sum_{n=1,3,\cdots} c_n n \sin nx \right)$$

$$\omega \sum_{n=1,3,\cdots} c_n \cos nx = -\sum_{n=1,3,\cdots} c_n n^2 \cos nx - \cos^2 x \sum_{n=1,3,\cdot} a_n n \cos nx$$

trigonometric relations

$$\cos x \sin nx = \frac{1}{2} \left[\sin(n+1)x + \sin(n-1)x \right]$$

$$\sin x \cos nx = \frac{1}{2} \left[\sin(n+1)x - \sin(n-1)x \right]$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$\cos 2x \cos nx = \frac{1}{2} \left[\cos(n+2)x + \cos(n-2)x \right]$$

2nd term on rhs of b_m equation

$$\frac{P^2}{2}\left[c_{m-1} - c_{m+1} + (m-1)c_{m-1} + (m+1)c_{m+1}\right] = \frac{P^2}{2}(mc_{m-1} + mc_{m+1})$$

2nd term on rhs of c_m equation

$$-\frac{1}{4}\left[(m-2)a_{m-2} + 2ma_m + (m+2)c_{m+2}\right]$$

orthogonality relations

$$\int_0^{\pi/2} \sin nx \sin mx dx = \int_0^{pi/2} \cos nx \cos mx = \frac{\pi}{4}\delta_{nm}$$

$$\int_0^{\pi/2} \sin nx \cos mx dx = 0$$

multiply equations by sin *mx* (a_n and b_n) and by cos *mx* (c_n), respectively, and integrate $\frac{\pi}{4}\int_0^{\pi/2} dx$

$$\omega a_m = -m^2 a_m + \frac{1}{2}\left(b_{m-1} + b_{m+1}\right) \qquad m = 1, 3, 5, \cdots, N$$

$$\omega b_m = -m^2 b_m + \frac{P^2}{2}\left(m c_{m-1} + m c_{m+1}\right) \qquad m = 2, 4, 6, \cdots, N+1$$

$$\omega c_m = -m^2 c_m - \frac{1}{4}\left[(m-2)a_{m-2} + 2m a_m + (m+2)a_{m+2}\right] \qquad m = 1, 3, 5, \cdots, N$$

$$\omega \mathbf{u} = \boldsymbol{\mathcal{M}}\mathbf{u}$$

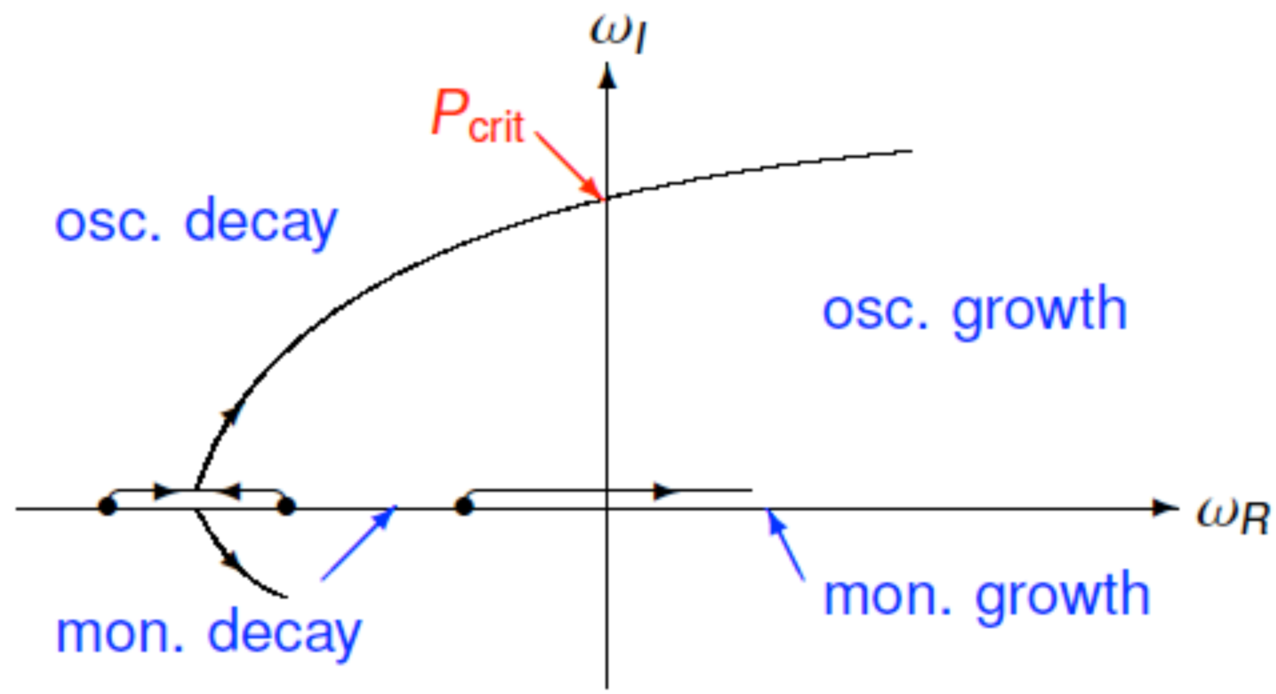
with $\mathbf{u} = (a_1, b_2, c_1, a_3, b_4, c_3, a_5, b_6, c_5, \cdots, a_N, b_{N+1}, c_N)^T$

$$\mathbf{M} = \left(\begin{array}{cccccccccccc}
-1 & 1/2 & & & & & & & & & & \\
 & -4 & 2P^2/2 & & & 2P^2/2 & & & & & & \\
-2/4 & & -1 & -3/4 & & & & & & & & \\
 & 1/2 & & -9 & 1/2 & & & & & & & \\
 & & & & -16 & 4P^2/2 & & & 4P^2/2 & & & \\
-1/4 & & & -6/4 & & -9 & -5/4 & & & & & \\
 & & & & & & \ddots & & & & & \\
 & & & & & & & \ddots & & & & \\
 & & & & & & & & \ddots & & & \\
 & & & & & & & 1/2 & & -N^2 & 1/2 & \\
 & & & & & & & & & & -(N+1)^2 & (N+1)P^2/2 \\
 & & & & & & -(N-2)/4 & & & -2N/4 & & -N^2
\end{array}\right)$$

Matrix eigenvalue problem

display $\omega(P)$ diagrams for $P > 0 \quad \text{and} \quad P < 0$ $\qquad P = R_\alpha R_u$

determine critical dynamo numbers where $\omega_{\rm R}(P_{\rm crit}) = 0$



check convergence with respect to number of expansion coefficients

display butterfly diagrams of $A(x;t) \quad B_z(x;t) \simeq B_r(x;t) \quad B(x;t) \quad \gamma(x;t)$