Convective Dynamos at Infinite Prandtl Number

David Hughes[1] and Fausto Cattaneo[2]

[1]School of Mathematics, University of Leeds, UK
[2]Department of Astronomy and Astrophysics, University of Chicago, USA

The length scale problem for the geodynamo

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The only possibility of a dynamo on these scales is a true, low Rm mean field dynamo.
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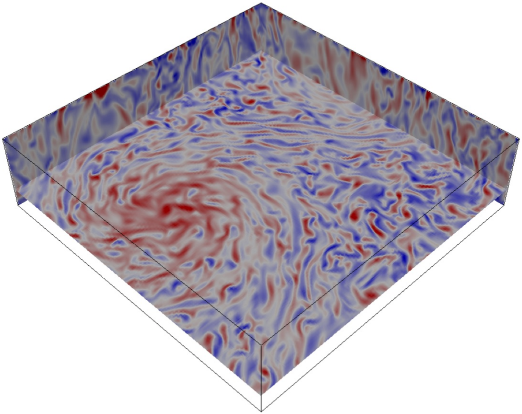
If this is not feasible, then an inviscid mechanism must be present to transfer energy from the small convective scales to larger scales where magnetic field can be generated.

Two possible solutions

- **Hydrodynamic:** Large scales may be generated *hydrodynamically*. These could then have an *Rm* sufficiently large to allow dynamo action.

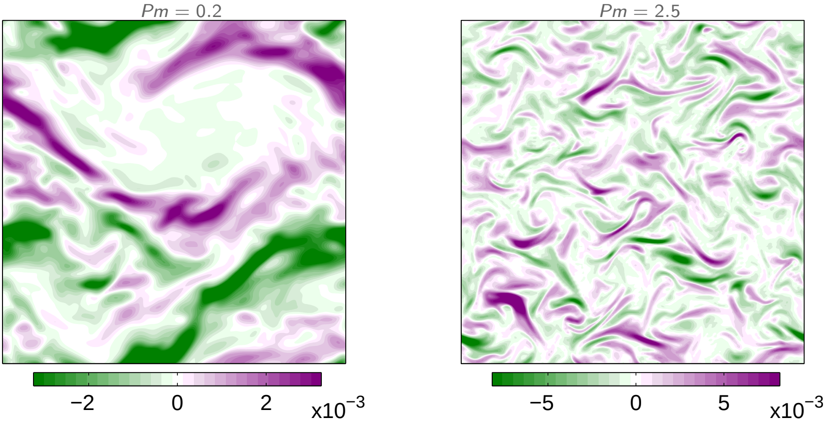


Hydrodynamic convection leading to large-scale vortices.
Guervilly, Hughes & Jones (*J. Fluid Mech.* 2014)

Dynamo action from large-scale vortices

Magnetic field (B_x shown here) can be large- or small-scale, depending on Pm (hence Rm).

At large enough Rm, small-scale dynamo action destroys the LSV. For smaller Rm, there is a range of Rm at which magnetic field is maintained on the scale of the LSV.



Guervilly, Hughes & Jones (*Phys. Rev. E* 2015)

Second Possibility

- **Magnetohydrodynamic:** The nonlinear state of the dynamo is such that the dominant balance in the momentum equation is between Magnetic, buoyancy (Archimedean) and Coriolis forces (so-called MAC balance).

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Here we consider the second of these possibilities and how we might find such a solution.

Neglecting Inertia

In the geodynamo, inertial terms are believed to be negligible ($Ro \approx 10^{-6}$).

One approach to geodynamo modelling, which has been very successful, is to solve the full equations at ever decreasing values of the Ekman number, with the idea that the inertial terms will decrease in importance as E decreases.

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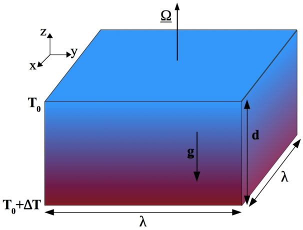
Alternatively, one can take more drastic action and simply discard the inertial terms from the outset. Formally, this may be considered as the limit of infinite Prandtl number.

Rotating Rayleigh-Bénard convection



- 3D Cartesian layer of Boussinesq fluid, depth d
- periodic in the horizontal directions
- rotating about the vertical (z) axis, rotation rate: Ω
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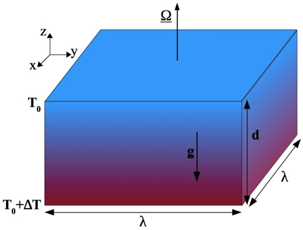

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Scale lengths with d, times with d^2/κ and magnetic field with $(2\Omega\kappa\mu_0\rho)^{1/2}$:

$$Pr^{-1}\left(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) + E^{-1}\mathbf{e}_z \times \boldsymbol{u} = -\nabla\tilde{p} + E^{-1}\boldsymbol{J}\times\boldsymbol{B} + Ra\,\theta\mathbf{e}_z + \nabla^2\boldsymbol{u},$$
$$(\partial_t - q^{-1}\nabla^2)\boldsymbol{B} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{u}\,,$$
$$(\partial_t - \nabla^2)\theta + \boldsymbol{u}\cdot\nabla\theta = \boldsymbol{u}\cdot\mathbf{e}_z,$$
$$\nabla\cdot\boldsymbol{B} = \nabla\cdot\boldsymbol{u} = 0.$$

Input parameters:

$$Ra = \frac{g\alpha\Delta T d^3}{\kappa\nu}, \quad E = \frac{\nu}{2\Omega d^2}, \quad Pr = \frac{\nu}{\kappa}, \quad q = \frac{\kappa}{\eta}.$$

Infinite Prandtl number limit

On letting $Pr \to \infty$, the momentum equation becomes

$$\mathbf{e}_z \times \boldsymbol{u} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B} + R\,\theta \mathbf{e}_z + E\nabla^2 \boldsymbol{u},$$

where the rotational Rayleigh number R is given by

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Note that under the conventional scaling of $\boldsymbol{B}$ with $(\mu_0\rho)^{1/2}\,\kappa/d$, the momentum equation is

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Note also that comparing magnetic and kinetic energies as $Pr \to \infty$ is not meaningful:

$$\frac{\text{Magnetic energy}}{\text{Kinetic energy}} = \frac{2\Omega\kappa}{\kappa^2/d^2}\frac{B^2}{U^2} = \frac{Pr}{E}\frac{B^2}{U^2}.$$

Waves supported

In the simplest system of linear waves on a static uniform field (no buoyancy) dispersion relation is

$$\omega^2 = \frac{k^2 \left(\boldsymbol{k} \cdot \boldsymbol{B}_0\right)^4}{4 \left(\boldsymbol{k} \cdot \boldsymbol{\Omega}\right)^2} = \frac{\omega_A^4}{\omega_I^2},$$

where $\omega_A = \left(\boldsymbol{k} \cdot \hat{\boldsymbol{B}}_0\right) V_A$ and $\omega_I = 2 \left(\boldsymbol{k} \cdot \hat{\boldsymbol{\Omega}}\right) / k$.
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With inertial terms included, the dispersion relation is

$$\omega^2 \pm \omega_I \omega - \omega_A^2 = 0.$$

In the case when $\omega_A / \omega_I \ll 1$,

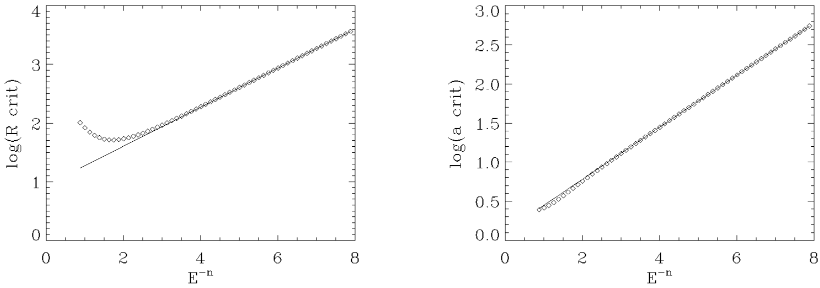
$$\omega^2 \approx \omega_I^2 \quad \text{and} \quad \omega^2 \approx \frac{\omega_A^4}{\omega_I^2}.$$

In our system, the inertial waves travel infinitely quickly and do not appear.

Hydrodynamic convection: linear theory

Convection sets in as a steady bifurcation. Neglect of inertia terms has no impact on the stability boundary.
As $E \to 0$, horizontal wavenumber a and critical Rayleigh number R_c given by

$$a \sim \left(\frac{\pi^2}{2}\right)^{1/6} E^{-1/3}, \qquad R_c \sim 3\left(\frac{\pi^2}{2}\right)^{2/3} E^{-1/3}.$$



R_c *vs* E dependence non-monotonic.

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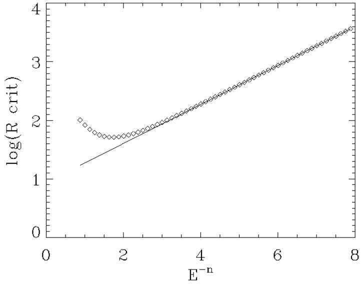
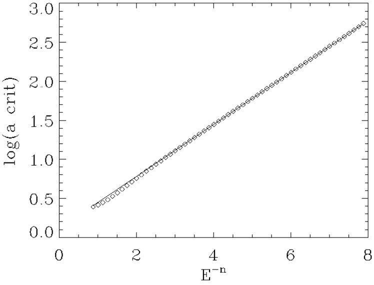
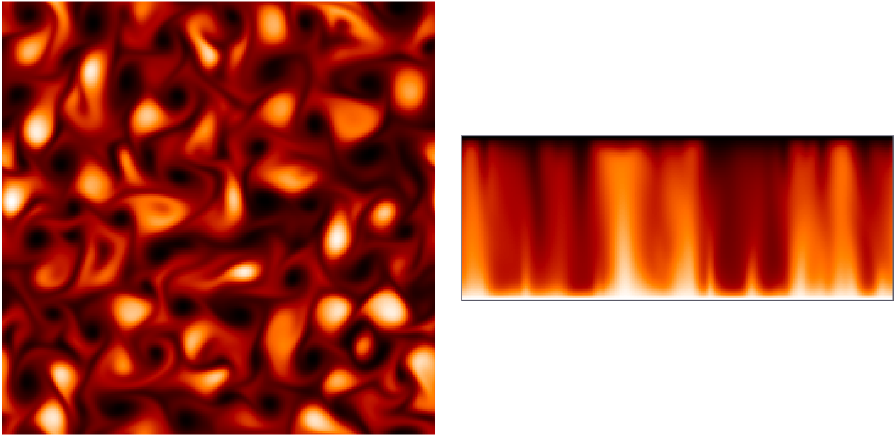




R_c *vs* E dependence non-monotonic.
Nonlinear problem fairly unexplored (Jones & Roberts 2000).

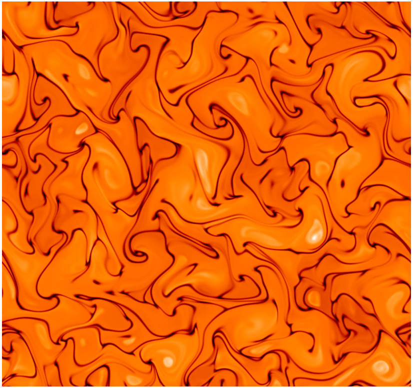
Nonlinear Convection: $E = 10^{-3}$

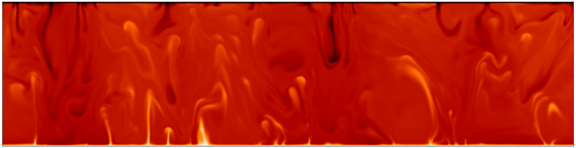
For $E = 10^{-3}$, $R_c \approx 87$.



Top and side planforms of temperature: $E = 10^{-3}$, $R = 500$, $\lambda = 5$.

More vigorous convection: $E = 10^{-3}$

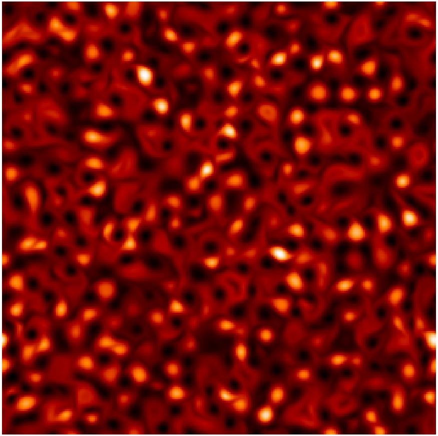


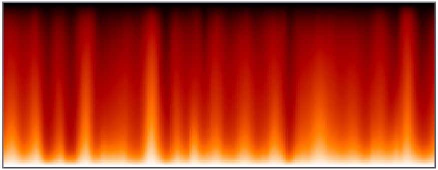


Top and side planforms of temperature: $E = 10^{-3}$, $R = 10^{5}$, $\lambda = 5$.

Faster rotation: $E = 10^{-4}$

For $E = 10^{-4}$, $R_c \approx 187$.





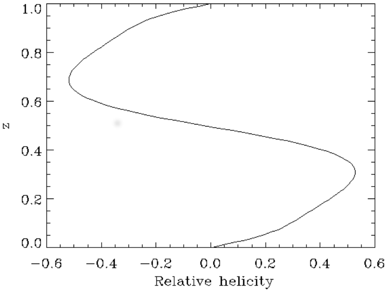
Top and side planforms of temperature: $E = 10^{-4}$, $R = 500$, $\lambda = 5$.

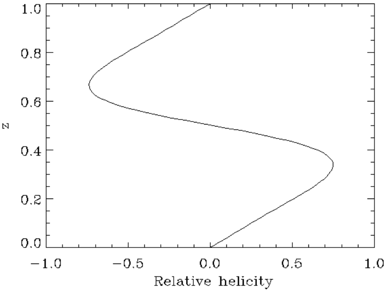
Convective Rossby number roughly 20 times smaller than for $E = 10^{-3}$, $R = 500$.

Kinetic helicity

Relative helicity, as a function of depth:

$$\frac{\langle \boldsymbol{u} \cdot \nabla \times \boldsymbol{u} \rangle}{\langle |\boldsymbol{u} \cdot \nabla \times \boldsymbol{u}| \rangle}$$



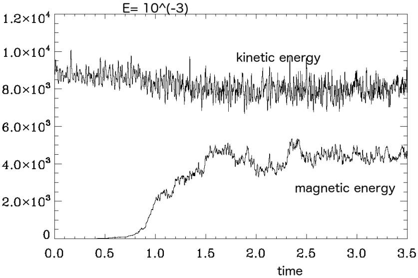


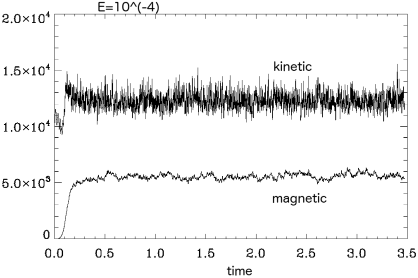
$R = 500$: $E = 10^{-3}$ $E = 10^{-4}$

Dynamo action

Consider the evolution, from the kinematic regime, of dynamos for the two cases:

$$(i)\ E = 10^{-3}, R = 500, q = 5.$$
$$(ii)\ E = 10^{-4}, R = 500, q = 20.$$





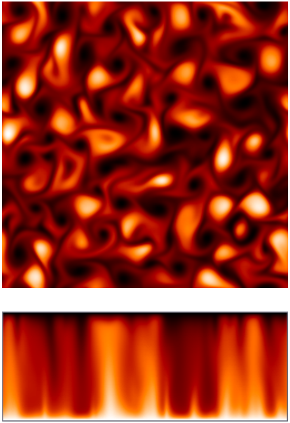
Magnetic energy multiplied by 5×10^3 to appear on same plot.

Kinetic energy increases as magnetic field grows — a suggestion of a 'strong field' dynamo.

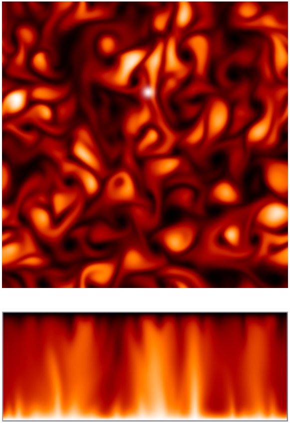
Dynamic regime: weak field dynamo

Two very different types of dynamo emerge:

- A 'weak field' dynamo, in which the convective patterns are modified only slightly as the dynamo saturates.



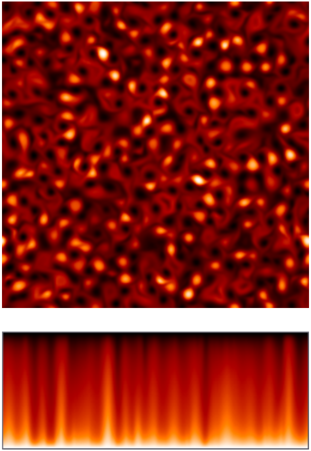
Kinematic regime



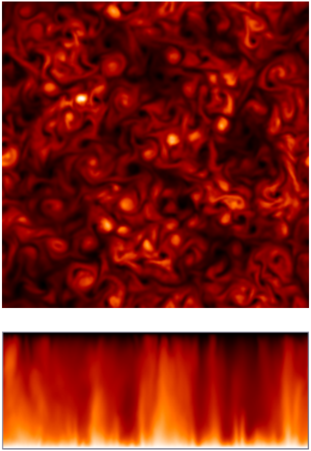
Dynamic regime

Dynamic regime: strong field dynamo

- A ‘strong field’ regime, in which a scale much larger than the convective scale emerges.

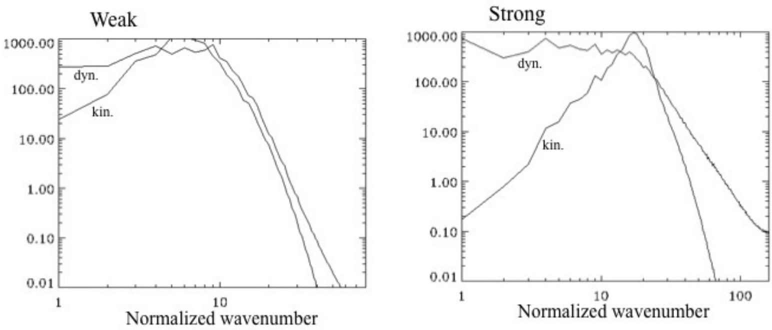


Kinematic regime



Dynamic regime

Kinetic energy spectra



Relatively little change in the KE spectrum of the weak field dynamo, between kinematic and dynamic regimes.

Marked change in the KE spectrum for the strong field dynamo. Flattening of the spectrum, leading to much more power at large scales, but also at smaller scales.

Decomposing the velocity

We may exploit the linearity of the momentum equation to decompose the velocity field into its thermal and magnetic components:

$$\boldsymbol{u} = \boldsymbol{u}_T + \boldsymbol{u}_M,$$

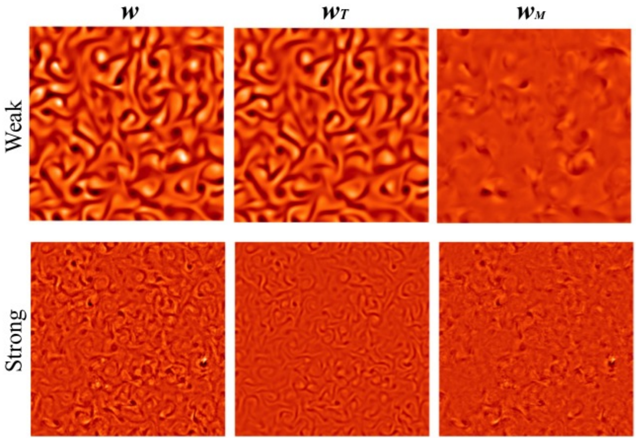
where $\boldsymbol{u}_T$ satisfies

$$\mathbf{e}_z \times \boldsymbol{u}_T = -\nabla p_T + R\,\theta \mathbf{e}_z + E\nabla^2 \boldsymbol{u}_T, \qquad \nabla \cdot \boldsymbol{u}_T = 0.$$

This allows us to visualise the thermal and magnetic contributions to the saturated velocity.

$\boldsymbol{u}$, $\boldsymbol{u}_T$ and $\boldsymbol{u}_M$

Vertical components of total, thermal and magnetic velocities near upper boundary.



- **Weak field dynamo:** total and thermal velocities almost identical, with very small change in magnetic velocity to saturate the dynamo.
- **Strong field dynamo:** thermal and magnetic velocities comparable in magnitude; features of each can be seen in total velocity.

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- We arrived at both solutions from a kinematic perspective. Important to consider the effects of hysteresis for the strong field solution.
- Can Taylor's constraint be used to characterise the solution? In plane layer geometry we have the exact result

$$\int_0^1 (\nabla \times (\boldsymbol{J} \times \boldsymbol{B})) \cdot \mathbf{e}_z \,\mathrm{d}z = -\int_0^1 E\nabla^2 (\boldsymbol{\omega} \cdot \mathbf{e}_z) \,\mathrm{d}z,$$

leading to Taylor's constraint (ignoring the diffusion term) in the form

$$\int_0^1 (\nabla \times (\boldsymbol{J} \times \boldsymbol{B})) \cdot \mathbf{e}_z \,\mathrm{d}z = 0.$$

for all x, y. Numerically this is not very nice, with the introduction of an extra derivative.