The Equatorial Ekman Layer

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GdR Dynamo
International Centre for Theoretical Sciences, Bangalore
8–12 June 2015

Outline

1. INTRODUCTION

Relative to a frame rotating with angular velocity $\boldsymbol{\Omega} = \Omega\widehat{\mathbf{z}}$,
we are interested in the slow steady flow of an incompressible fluid, viscocity ν, in the shell between two spheres.
The inner sphere, radius L, is at rest;
the outer sphere rotates with angular velocity $\varepsilon\boldsymbol{\Omega}$; $\varepsilon \ll 1$.

Our geometry differs slightly from Stewartson (1966):

$$r_j^{\mathrm{S}} = L\,.$$

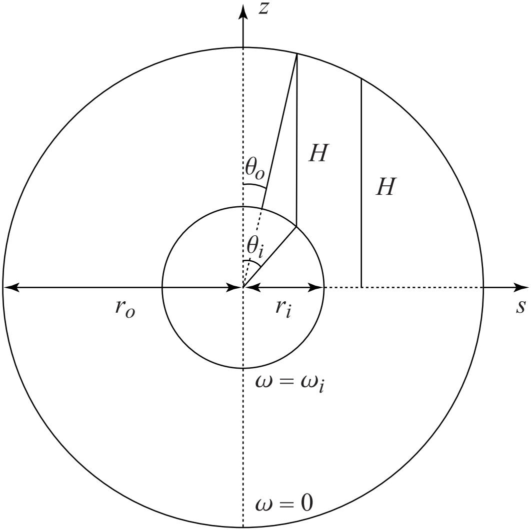
Relative to his frame rotating with angular velocity

$$\Omega^{\mathrm{S}} = (1+\varepsilon)\Omega\,,$$

his outer sphere is at rest, while his inner sphere rotates with angular velocity

$$\omega_j^{\mathrm{S}} = \Omega - \Omega^{\mathrm{S}} = -\varepsilon\Omega\,.$$

Stewartson's (1966) configuration, $r_j^s = L$, relative to a rotating $\Omega(1+\varepsilon)$ frame with $\omega_j^s = -\varepsilon\Omega$



Proudman solution (1956): Geostrophic flow and Ekman layers

The flow in the small Ekman number limit,

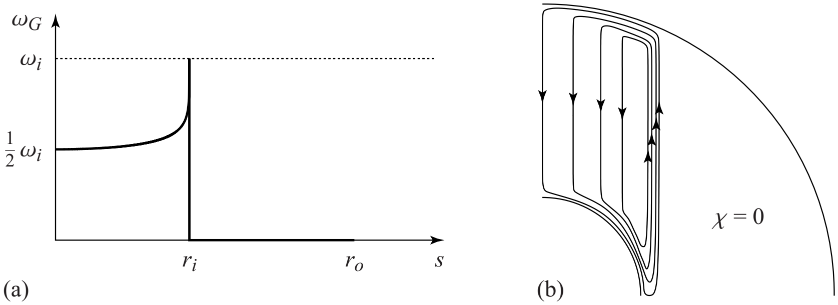
$$E = \nu/L^2\Omega \ll 1\,,$$

is characterised by mainstream geostrophic flow (azimuthal and z-independent) and boundary layer structures:

- The Ekman layers on the spheres width $(\nu/\Omega)^{1/2} = LE^{1/2}$, which largely control the mainstream flow.
- Outside the inner sphere tangent cylinder the fluid co-rotates with the outer sphere.
- Inside it rotates at an intermediate angular velocity which tends to rest as the tangent cylinder is approached.

The Proudman (1956) solution

(a) Geostrophic velocity ω^{s}_{G}; (b) Streamlines $\chi^{s} = \text{const.}$



Quasi-geostrophic flow with radial friction

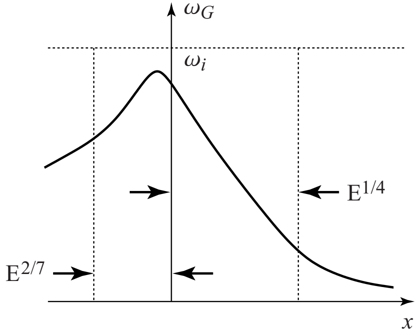
The discontinuity is smoothed out across nested shear layers on the tangent cylinder.

In units of L, Stewartson (1966) identified the "outer" layers

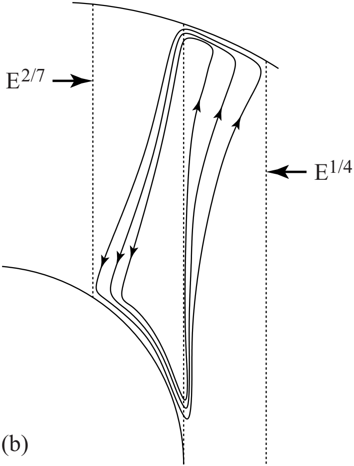
- an $E^{1/4}$-layer outside the tangent cylinder
  in which the flow continues to be geostrophic.
  Geostrophic degeneracy is resolved
  by Ekman suction and internal lateral friction;
- an $E^{2/7}$ inside the tangent cylinder with similar features.
  The different width arises because of
  the singularity of the Ekman layer
  as the equator is approached.

Quasi-geostrophic layers

(a) Geostrophic velocity ω^{s}_{G}; (b) Streamlines $\chi^{s} = \text{const.}$



(a)



(b)

The "inner" $E^{1/3}$ shear layer

The "outer" $E^{1/4}$ and $E^{2/7}$-layers embed
an "inner" $E^{1/3}$-layer, which ceases to be geostrophic,
as the shear is dependent on the axial co-ordinate $z^{\dagger}$.

A primary source for this layer is the inner core equator.
It thickens proportional to $(Ez^{\dagger})^{1/3}$,
while the Ekman layer thins proportional to $(E/z^{\dagger})^{1/2}$.
They are equal when

$$\delta = (Ez^{\dagger})^{1/3} = (E/z^{\dagger})^{1/2},$$

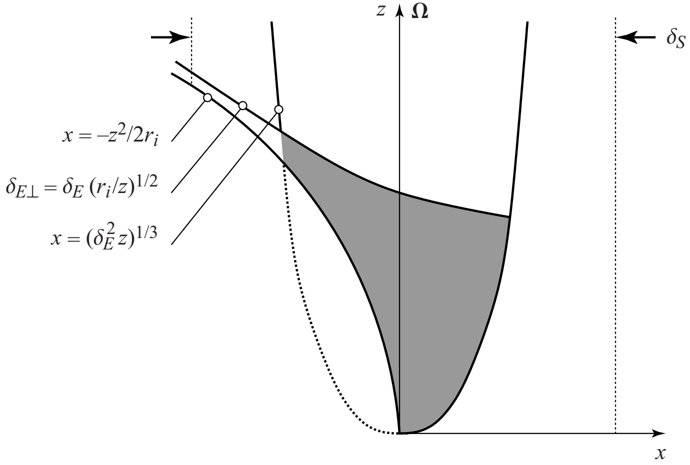
i.e.,

$$z^{\dagger} = E^{1/5}, \qquad \delta = E^{2/5},$$

which define the dimensions of the

$E^{2/5}$ Equatorial Ekman Layer

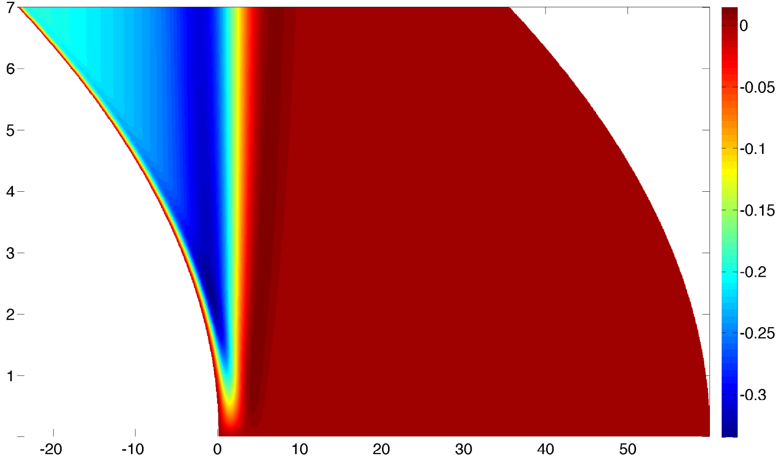
$E^{2/5}$ Equatorial Ekman Layer
$(Ez^\dagger)^{1/3}$ shear layer; $(E/z^\dagger)^{1/2}$ Ekman layer



Objectives

- We investigate the solution of Stewartson's (1966) reduced equations and boundary conditions (scaled independent of E) on an unbounded domain governing motion in the $E^{2/5}$ **Equatorial Ekman Layer**.
- In the absence of a far boundary, we find that, without some ingenuity, the numerical solution is dependant on the finite numerical box size! Our resolution is a "soft" boundary at finite $z^{\dagger}$, where we apply a non-local (integral) b.c..
- For large $z^{\dagger}$, we extend Stewartson's (1966) $E^{1/3}$ shear layer similarity solution valid near the equator to higher orders using matched asymptotic expansions. That analytic extension is in excellent agreement with the numerics.

Equatorial Ekman Layer: Meridional streamfunction



Outline

2. MATHEMATICAL FORMULATION

Geometry

Relative to local Equatorial Ekman layer Cartesian coordinates $x^\dagger$ (radial), "dummy $y^\dagger$" (azimuthal), $z^\dagger$ (axial) scaled as

$$x^\dagger = E^{2/5} L x \quad \text{(radial)} \qquad z^\dagger = E^{1/5} L z \quad \text{(axial)}\,,$$

our inner sphere is the severely flattened **oblate spheroid**

$$E^{2/5}\big(x + E^{-2/5}\big)^2 + z^2 = E^{-2/5}\,,$$

which for $(x, z) = \mathrm{O}(1)$ $(E \to 0)$ determines

$$2x + z^2 = 0\,.$$

Governing equations

Our unit of angular velocity $\Delta\Omega$ is not $\varepsilon\Omega$ but $CE^{1/28}\,\varepsilon\Omega$ (constant C of order unity; Stewartson 1966).

- The azimuthal and axial velocities are

$$u_y^\dagger\,, = (L\Delta\Omega)v(x,z)\,, \qquad u_z^\dagger = (L\Delta\Omega)w(x,z)\,;$$

  the radial velocity and the meridional flow streamfunction are

$$u_x^\dagger = E^{1/5}(L\Delta\Omega)u\,, \qquad \psi^\dagger = E^{2/5}(L^2\Delta\Omega)\psi\,,$$

  where $\quad u = -\partial\psi/\partial z\,, \qquad w = \partial\psi/\partial x\,.$

- Then the **governing equations** are

$$2\frac{\partial v}{\partial z} = \frac{\partial^4\psi}{\partial x^4}\,, \qquad \text{azimuthal vorticity}$$

$$2\frac{\partial \psi}{\partial z} = -\frac{\partial^2 v}{\partial x^2} \qquad \text{azimuthal velocity}$$

Boundary conditions

We adopt the frame with the fluid at rest far from the sphere.

- More precisely the **boundary conditions** are

$$\begin{aligned}
&\partial v/\partial z = 0\,, && && \psi = 0 && \text{on} \quad z = 0\,, && x > 0\,,\\
&v = 1\,, && w = 0\,, && \psi = 0 && \text{on} \quad z = \sqrt{-2x}\,, && x < 0\,,\\
&v \to 0\,, && && \psi \to 0 && \text{as} \quad x \uparrow \infty\,,
\end{aligned}$$

As $z\uparrow\infty$ an **Ekman layer** forms on the sphere boundary $x = -\frac{1}{2}z^2$ of width $\mathrm{O}(z^{-1})$, in which

$$v_0^{bl} = \mathrm{e}^{-\zeta}\cos\zeta\,, \qquad w_0^{bl} = -\,\mathrm{e}^{-\zeta}\sin\zeta\,;$$

Ekman layer coordinate $\zeta = z^{1/2}\left(x + \frac{1}{2}z^2\right) \Longrightarrow$ top b.c.

$$v \sim v_0^{bl}(\zeta)\,, \quad w \sim w_0^{bl}(\zeta)\,, \quad \psi \to 0 \qquad \text{for} \quad \zeta > 0\,.$$

Stewartson $E^{1/3}$-layer solution for $z \uparrow \infty$

- The **Mainstream region** is the entire region outside the Ekman boundary layer: $z^{-1/2}\zeta = x + \frac{1}{2}z^2 \gg z^{-1/2}$.
- Stewartson's (1966) **Mainstream similarity solution**

$$v = V_0(\Phi, z), \quad \psi = \Psi_0(\Phi, z), \quad w = W_0(\Phi, z), \qquad \Phi = x/z^{1/3};$$

for the **shear layer** of width $\mathrm{O}(z^{1/3})$, which forms in its interior on the tangent cylinder $x = 0$, is

$$V_0(\Phi, z) = \frac{2^{-1/4}\, z^{-5/12}}{\Gamma(1/4)} \int_0^\infty \varpi^{1/4} \cos\left(\varpi\Phi + \tfrac{3}{8}\pi\right) \exp\left(-\tfrac{1}{2}\varpi^3\right) \mathrm{d}\varpi\,,$$

$$\Psi_0(\Phi, z) = -\frac{2^{-1/4}\, z^{-1/12}}{\Gamma(1/4)} \int_0^\infty \varpi^{-3/4} \cos\left(\varpi\Phi + \tfrac{3}{8}\pi\right) \exp\left(-\tfrac{1}{2}\varpi^3\right) \mathrm{d}\varpi\,,$$

$$W_0(\Phi, z) = \frac{2^{-1/4}\, z^{-5/12}}{\Gamma(1/4)} \int_0^\infty \varpi^{1/4} \sin\left(\varpi\Phi + \tfrac{3}{8}\pi\right) \exp\left(-\tfrac{1}{2}\varpi^3\right) \mathrm{d}\varpi\,.$$

Properties and limitations

- The simplicity of the Stewartson $z \uparrow \infty$ solution hides his leading order mainstream assumption $\quad v = 0.$
- The Ekman layer jump to $v = 1$ at the sphere boundary drives the Ekman layer suction

$$\psi \sim -\tfrac{1}{2}(-2x)^{-1/4} \qquad \text{on} \qquad x = -\tfrac{1}{2}z^2$$

$$\Updownarrow$$

$$\Psi_0 \sim -\tfrac{1}{2}z^{-1/12}(-2\Phi)^{-1/4} \qquad \text{on} \qquad \Phi = -\tfrac{1}{2}z^{5/6}$$

providing the crucial b.c. on his lowest order solution.

- Here $z^{-1/12}$ sets the power law in the similarity solution satisfying the b.c. $\quad 2z^{1/12}\Psi_0 \sim -(-2\Phi)^{-1/4} \quad$ as $\quad \Phi \downarrow -\infty.$
- While $\psi = 0$ on $x > 0$, $z = 0$ implies the b.c. $\quad 2z^{1/12}\Psi_0 = \mathrm{o}\big(\Phi^{-1/4}\big) \quad$ as $\quad \Phi \uparrow \infty.$

- The corresponding similarity form $z^{5/12}V_0 =$ function of Φ, is largely slave to $z^{1/12}\Psi_0$.
- Indeed Stewartson's solution determines

$$V_0 \approx \begin{cases} \frac{1}{4} z^{-5/12}(-2\Phi)^{-5/4} & (\Phi \downarrow -\infty) \\ -2^{-3/2} z^{-5/12}(2\Phi)^{-5/4} & (\Phi \uparrow \infty) \end{cases}$$

$$\Downarrow$$

$$v \approx \begin{cases} \frac{1}{4}(-2x)^{-5/4} & (x < 0,\ z = 0) \\ -2^{-3/2}(2x)^{-5/4} & (x > 0,\ z = 0) \end{cases}$$

- The symmetry condition $\partial v/\partial z = 0$ on $x > 0$, $z = 0$ is met.
- **Importantly** the value

$$v \approx \frac{1}{4} z^{-5/2}$$

at the edge of the sphere Ekman layer $-2x = z^2$
does **not** meet the assumed $v = 0$.

Outline

3. SERIES SOLUTION

- **Shear layer solution** $\Phi = x/z^{1/3} = \mathrm{O}(1)$:

$$\begin{bmatrix} V \\ W \\ z^{-1/3}\Psi \end{bmatrix} \sim z^{-5/12}\begin{bmatrix} \widehat{V}_0 \\ \widehat{W}_0 \\ \widehat{\Psi}_0 \end{bmatrix}(\Phi) + z^{-5/6}\begin{bmatrix} \widehat{V}_1 \\ \widehat{W}_1 \\ \widehat{\Psi}_1 \end{bmatrix}(\Phi) + z^{-5/4}\begin{bmatrix} \widehat{V}_2 \\ \widehat{W}_2 \\ \widehat{\Psi}_2 \end{bmatrix}(\Phi) + \cdots .$$

- **Ekman layer solution** $\zeta = z^{1/2}\left(x + \frac{1}{2}z^2\right) = \mathrm{O}(1)$:

$$\begin{bmatrix} v \\ w \\ z^{1/2}\psi \end{bmatrix} \sim \begin{bmatrix} v_0 \\ w_0 \\ \psi_0 \end{bmatrix}(\zeta) + z^{-5/2}\begin{bmatrix} v_1 \\ w_1 \\ \psi_1 \end{bmatrix}(\zeta) + z^{-5}\begin{bmatrix} v_2 \\ w_2 \\ \psi_2 \end{bmatrix}(\zeta) + \cdots .$$

which splits into shear layer ^{sl} (mainstream)
and Ekman (boundary) layer ^{bl} parts, e.g., $v = v^{bl} + v^{sl}$.

- The **complete solution** is provided by the **composite**

$$v = V + v^{bl}, \qquad w = W + w^{bl}, \qquad \psi = \Psi + \psi^{bl} .$$

Shear layer solutions

- Set

$$\widehat{V}_n + \mathrm{i}\widehat{W}_n = \mathrm{Y}_n'(\Phi)\,,$$

where Y_n solves

$$\mathrm{Y}_n''' + \tfrac{2}{3}\mathrm{i}\big[\Phi \mathrm{Y}_n' + \tfrac{1}{4}(5n+1)\mathrm{Y}_n\big] = 0\,.$$

- The solutions, bounded as $|\Phi| \to \infty$, are

$$\mathrm{Y}_n(\Phi) = \mathrm{A}_n \exp\big[-\mathrm{i}(5n+1)\pi/8\big]\int_0^\infty \varpi^{(5n-3)/4}\exp\Big(\mathrm{i}\varpi\Phi - \tfrac{1}{2}\varpi^3\Big)\,\mathrm{d}\varpi$$

where $\mathrm{Im}\{\mathrm{A}_n\} = 0$ to meet

- the symmetry condition $w = 0$ on $z = 0,\ x > 0$

$$\Longrightarrow \qquad \mathrm{Im}\{\mathrm{Y}_n\} = \mathrm{o}\Big(\Phi^{-(5n+1)/4}\Big) \qquad \text{as} \quad \Phi \uparrow \infty\,.$$

- The **real** constants A_n are fixed by matching with the **Ekman layer solution**.

Ekman layer solutions

- Set

$$v_n - \mathrm{i}w_n = \mathcal{W}_n(\zeta)\,,$$

$$\psi_n = \int_0^\zeta w_n\,\mathrm{d}\zeta = -\int_0^\zeta \mathrm{Im}\{\mathcal{W}_n\}\,\mathrm{d}\zeta\,,$$

where $\mathcal{W}_n$ solves

$$\mathcal{W}_n''' + 2\mathrm{i}\mathcal{W}_n' = \begin{cases} 0 & (n=0)\,, \\ -\mathrm{i}\big[\zeta\mathcal{W}_{n-1}' - 5(n-1)\mathcal{W}_{n-1}\big] & (n \geq 1) \end{cases}$$

subject to the boundary condition

$$\mathcal{W}_n = \begin{cases} 1 & (n=0)\,, \\ 0 & (n \geq 1) \end{cases}$$

and matching conditions as $\zeta \uparrow \infty$.

The 0^{th}-order solution:

$$\mathcal{W}_0 = E(\zeta) \equiv \exp\big[-(1-\mathrm{i})\zeta\big]\,.$$

The 1^{st}-order solution

$$\mathcal{W}_1(\zeta) = \tfrac{1}{4}\{(1+\mathrm{i}\alpha) - \left[(1+\mathrm{i}\alpha) + \tfrac{3}{2}\zeta + \tfrac{1}{2}(1-\mathrm{i})\zeta^2\right]E(\zeta)\}\,.$$

- Here we have **fixed** the real part of the constant $1+\mathrm{i}\alpha$ of integration to meet the matching condition

$$v_1 = \mathrm{Re}\{\mathcal{W}_1\} \to \tfrac{1}{4} \qquad \text{as} \qquad \zeta\uparrow\infty.$$

- Also

$$\psi_1 = -\int_0^{\zeta} \mathrm{Im}\{\mathcal{W}_1\}\,\mathrm{d}\zeta \sim \tfrac{1}{4}\left[\tfrac{1}{4}(7+2\alpha) - \alpha\zeta\right] \qquad \text{as} \qquad \zeta\uparrow\infty\,.$$

- Matching the term in $z^{-3}\psi_1 \propto \alpha\zeta$ with the 0^{th}-order shear layer solution $z^{-1/12}\Psi_0$ **fixes** $\alpha = 1$.
- $\therefore$ The remaining constant term becomes $\frac{1}{4}(7+2\alpha) = 9/4$.
- In turn, matching $(9/16)z^{-3}$ with the 1^{st}-order shear layer contribution $z^{-1/2}\widehat{\Psi}_1$ **fixes** the value of A_1.

- With $\alpha = 1$, the ensuing

$$\mathcal{W}_1(\zeta) = \tfrac{1}{4}\left\{(1+\mathrm{i}) - \left[(1+\mathrm{i}) + \tfrac{3}{2}\zeta + \tfrac{1}{2}(1-\mathrm{i})\zeta^2\right]E(\zeta)\right\}.$$

determines the boundary and shear layer contributions

$$\begin{aligned}
v_1^{bl} &= -\tfrac{1}{4}\left[\left(1+\tfrac{3}{2}\zeta+\tfrac{1}{2}\zeta^2\right)\cos\zeta + \left(1-\tfrac{1}{2}\zeta^2\right)\sin\zeta\right]\mathrm{e}^{-\zeta},\\
w_1^{bl} &= \tfrac{1}{4}\left[\left(1-\tfrac{1}{2}\zeta^2\right)\cos\zeta - \left(1+\tfrac{3}{2}\zeta+\tfrac{1}{2}\zeta^2\right)\sin\zeta\right]\mathrm{e}^{-\zeta},\\
\psi_1^{bl} &= -\tfrac{1}{16}\left[\left(9+5\zeta\right)\cos\zeta + \left(5\zeta+2\zeta^2\right)\sin\zeta\right]\mathrm{e}^{-\zeta},
\end{aligned}$$

$$\begin{aligned}
v_1^{sl} &= \tfrac{1}{4},\\
w_1^{sl} &= -\tfrac{1}{4},\\
\psi_1^{sl} &= \tfrac{1}{4}\left(\tfrac{9}{4}-\zeta\right).
\end{aligned}$$

- Noting that $\psi_0^{sl} = -\frac{1}{2}$, $v_0^{sl} = 0$,
correct to first order the **entire** shear layer contributions are

$$\begin{aligned}
\psi^{sl} &\approx -\tfrac{1}{2}z^{-1/2} + \tfrac{1}{4}z^{-3}\left(\tfrac{9}{4}-\zeta\right),\\
v^{sl} &\approx \tfrac{1}{4}z^{-5/2}.
\end{aligned}$$

Outline

4. TOP BOUNDARY CONDITION

- The top boundary condition

$$v \sim v_0^{bl}(\zeta)\,, \quad w \sim w_0^{bl}(\zeta)\,, \quad \psi \to 0 \qquad \text{as} \quad z \uparrow \infty\,, \quad \zeta > 0\,.$$

  is problematic to implement at finite (but largish) $z = H$.

- The very thin Ekman layer can be managed but difficulties are encountered with the mainstream.

- Since the governing equations are 2^{nd}-order in z, we expect one bottom b.c. and one top b.c..

- The natural mainstream top b.c. is $v = 0$.

  In practice that is far to severe for the moderate H usable numerically. The solution is seriously influenced by that choice and so varies with box size!!!!

- Use of the similarity expansion

$$v(x,H) \sim H^{-5/12}\widehat{V}_0(\Phi_H) + H^{-5/6}\widehat{V}_1(\Phi_H) + H^{-5/4}\widehat{V}_2(\Phi_H) + \cdots,$$

where $\Phi_H = x/H^{1/3}$ fairs little better!

- The problem is its approximate nature and the realised solution is sensitive to the discrepancy.

Fourier Transform of the mainstream solution

Defining

$$\left[\widehat{\psi}, \widehat{v}\right](\varpi, z) = \int_{-\infty}^{\infty}\left[\psi, \ v\right]\psi(x,z)\exp(-\mathrm{i}\varpi x)\,\mathrm{d}x,$$

the Fourier transforms of the governing equations are

$$2\frac{\partial\widehat{v}}{\partial z} = \varpi^4\widehat{\psi}, \qquad 2\frac{\partial\widehat{\psi}}{\partial z} = \varpi^2\widehat{v}.$$

- The solution that tends to zero as $z \uparrow \infty$ is

$$\widehat{v}(\varpi, z) \;=\; a(\varpi)\exp\!\left(-\tfrac{1}{2}|\varpi|^3 z\right),$$

$\Longrightarrow$

$$\widehat{\psi}(\varpi, z) \;=\; \frac{2}{\varpi^4}\frac{\partial\widehat{v}}{\partial z}(\varpi, z) \;=\; -\,\frac{1}{|\varpi|}\,\widehat{v}(\varpi, z)\,.$$

- At $z = H$ the inversion of $\widehat{v}(\varpi, H) = -|\varpi|\widehat{\psi}(\varpi, H)$ determines the convolution integral

$$v(x, H) \;=\; \mathcal{F}_H\{\psi\} \;\equiv\; -\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{x - x'}\,\frac{\partial\psi}{\partial x}(x', H)\,\mathrm{d}x'\,,$$

the basis of our top “soft” (cf. acoustics) mainstream b.c..

Outline

5. NUMERICAL METHOD

We make the change of variable $y = \frac{1}{2}z^2 + x$ and solve

$$2\left(\frac{\partial v}{\partial z} + z\frac{\partial v}{\partial y}\right) = \frac{\partial^4 \psi}{\partial y^4}\,, \qquad 2\left(\frac{\partial \psi}{\partial z} + z\frac{\partial \psi}{\partial y}\right) = -\frac{\partial^2 v}{\partial y^2}\,.$$

subject to

$$v = 1 \quad \text{and} \quad \psi,\ \frac{\partial \psi}{\partial y} = 0 \qquad \text{on} \quad y = 0,$$

$$v = 0 \quad \text{and} \quad \psi,\ \frac{\partial \psi}{\partial y} = 0 \qquad \text{on} \quad y = L,$$

$$\psi = 0 \qquad \text{on} \quad z = 0,$$

Together with the implementation of the soft b.c. at $z = H$, which pretends that the boundary is absent (a familiar acoustic problem).

The box width L is chosen dependant on the box height H, so that the tangent cylinder crosses the top boundary reasonably far from both the $y = 0$ and $y = L$ edges.

The finite difference discretization of the governing equations uses a symmetric, second-order scheme for all the y-derivatives and a third-order backward (respectively forward) scheme for the approximation of ψ (respectively v) z-derivatives.

Iterative method

Rather than apply the top soft b.c. directly we iterate and consider the sequence of solutions v_n, ψ_n ($n = 0, 1, 2 \cdots$) subject to

$$v_n^H = \begin{cases} 0 & (n = 0)\,, \\ \mathcal{F}_H\{\psi_{n-1}^H\} & (n \geq 1)\,, \end{cases}$$

where $v_n^H(x) \equiv v_n(x, H)$, $\psi_n^H(x) \equiv \psi_n(x, H)$.

The soft boundary condition

Though $v^H = \mathcal{F}_H\{\psi^H\}$ was derived for the mainstream solution, we ignore the Ekman layer and simply apply

$$v_n^H(x) = \begin{cases} 0 & (-\frac{1}{2}H^2 < x < a)\,, \\ v_n^H(x_-)\dfrac{x-a}{x_- - a} & (a < x < x_-)\,, \\ -\dfrac{1}{\pi}\displaystyle\int_a^b \dfrac{1}{x-x'}\,\dfrac{\mathrm{d}\psi_{n-1}^H}{\mathrm{d}x}(x')\,\mathrm{d}x' & (x_- < x < x_+)\,, \\ v_n^H(x_+)\dfrac{b-x}{b-x_+} & (x_+ < x < b)\,, \\ 0 & (b < x < -\frac{1}{2}H^2 + L)\,, \end{cases}$$

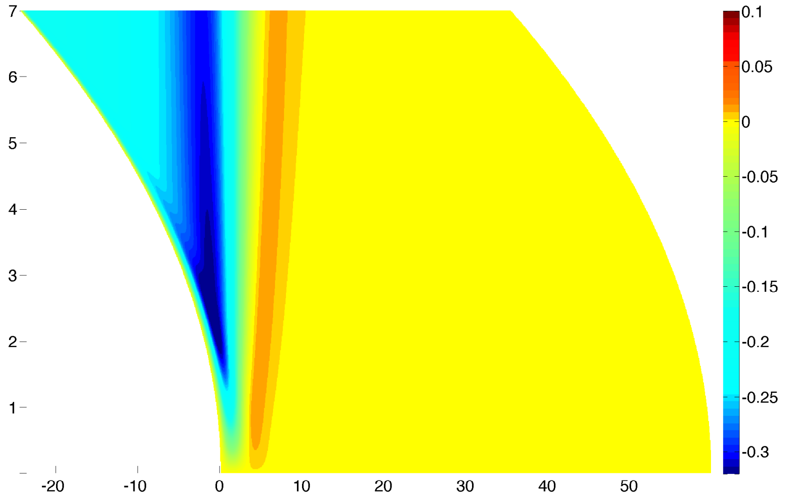
where thin region $[-\frac{1}{2}H^2\,,\,a]$ contains the Ekman layer,
the wider region $[x_-\,,\,x_+]$ contains the shear layer,
another thin region $[b\,,\,-\frac{1}{2}H^2 + L]$,
and the overlap regions $[a\,,\,x_-]$, $[x_+\,,\,b]$.

Outline

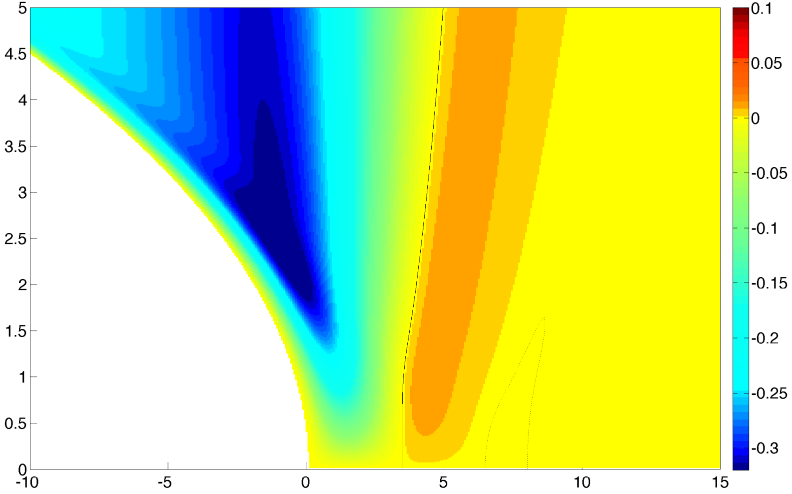
Meridional streamfunction ψ

$L = 60$, $H = 7$

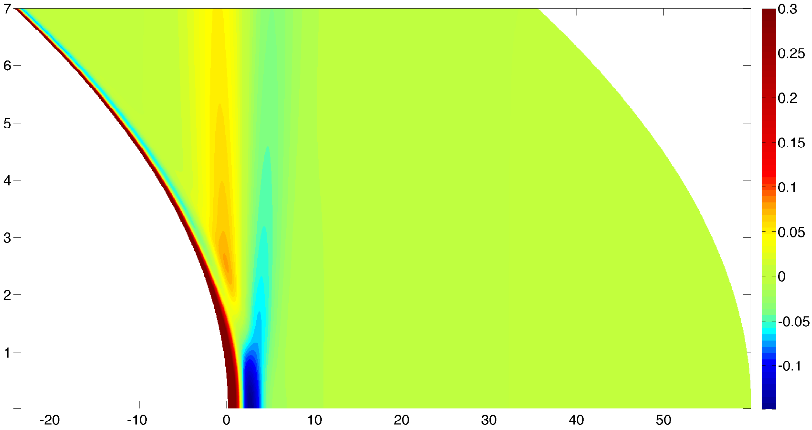
950×700 gridpoints



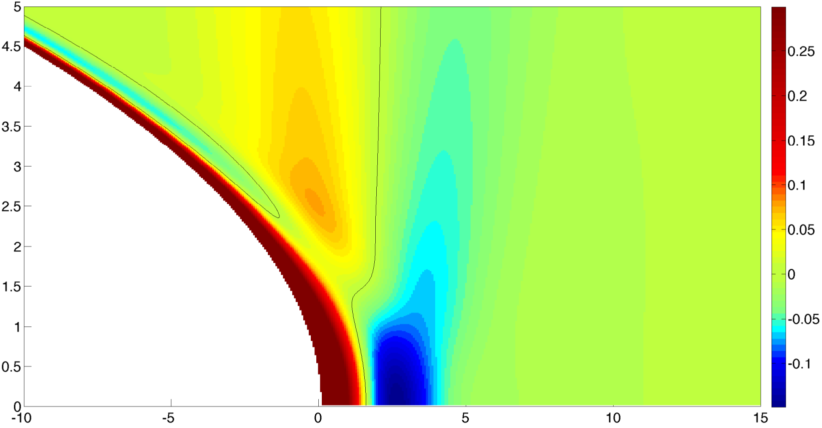
Zoom ψ $L = 60,\ H = 7$



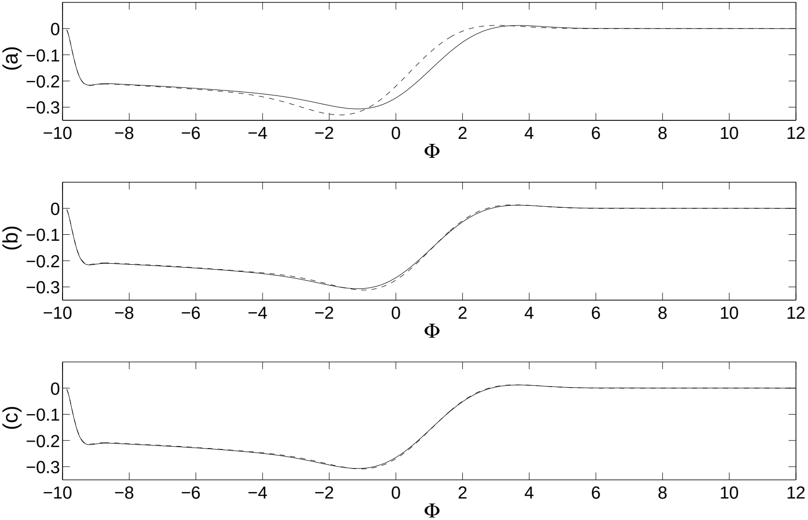
Azimuthal velocity v $L = 60,\ H = 7$



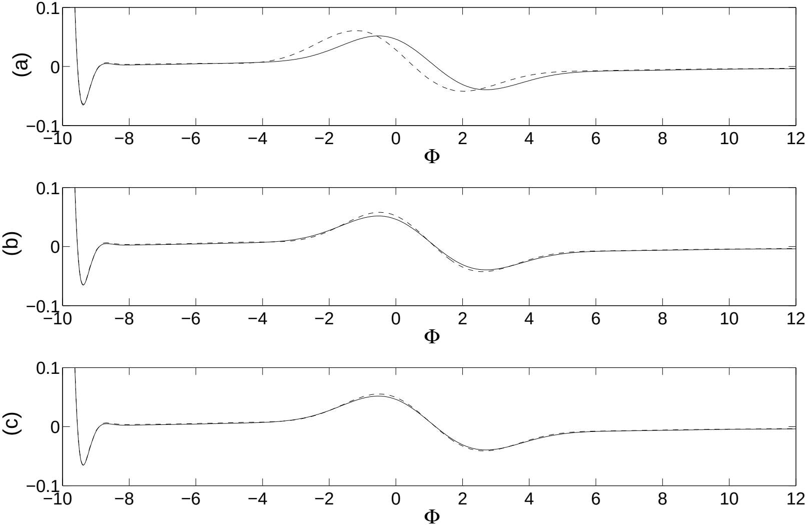
Zoom v $L = 60,\ H = 7$



ψ *vs.* Φ at $z = 6$ ($H = 7$): Numerics —— Asymptotics - - -, (a,b,c) $0^{\rm th}$, $1^{\rm st}$, $2^{\rm nd}$-order respectively



v *vs.* Φ at $z = 6$ $(H = 7)$: Numerics —— Asymptotics - - -, (a,b,c) 0^{th}, 1^{st}, 2^{nd}-order respectively



Outline

7. A CAUTIONARY NOTE

- Our far field ($z \gg 1$) mainstream solution

$$\psi \sim z^{-1/12}\widehat{\Psi}_0(\Phi) + z^{-1/2}\widehat{\Psi}_1(\Phi) + z^{-11/12}\widehat{\Psi}_2(\Phi) + \cdots$$

is not unique (cf. an o.d.e. Particular Integral).

- There is another "free" solution

$$\psi \sim \widehat{\Psi}^0(\Phi, z) + z^{-1/3}\widehat{\Psi}^1(\Phi, z) + z^{-2/3}\widehat{\Psi}^2(\Phi, z) + \cdots$$

(Moore and Saffman, 1969; cf. an o.d.e. Complimentary Function), where each term ($n = 0,\ 1,\ 2 \cdots$) has an expansion

$$\widehat{\Psi}^n \sim \widehat{\Psi}^n_0(\Phi) + z^{-5/12}\widehat{\Psi}^n_1(\Phi) + z^{-11/6}\widehat{\Psi}^n_2(\Phi) + \cdots$$

- Resonances, occur when the z-powers coincide $\Longrightarrow$ $\ln z$ terms.

- The modes $n = 0,\, 1,\, 2\, \cdots$ are "free" because $z^{-n/3}\widehat{\Psi}_0^n(\Phi) = 0$ on $z = 0$ for all x.

- They correspond to sources at the origin (Moore and Saffman 1969), and ultimately their respective magnitudes are outputs from the complete solution.

- The point source solution $\widehat{\Psi}^0$ (for the anti-symmetric spilt discs; Stewartson 1957) is prohibited (as in the symmetric spilt discs) by the boundary condition as $z \uparrow \infty$.

- The others are dipole, $z^{-1/3}\widehat{\Psi}^1$, quadrupole, $z^{-2/3}\widehat{\Psi}^2$, etc.. That they do not seem visible presumably reflects their weak size.

- However, the Stewartson (1966) solution $z^{-1/12}\widehat{\Psi}_0(\Phi)$ dominates at large z and so the"free" modes might never be significant.

Outline

8. CONCLUSIONS

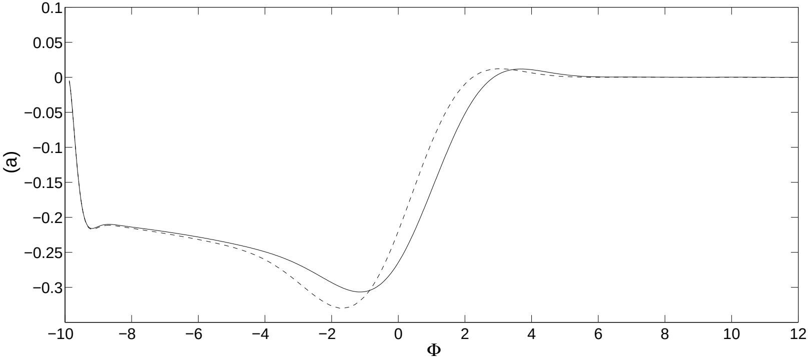
Numerical solution

- We have provided a numerical solution of the Equatorial Stewartson layer formulated by Stewartson (1996) by a combination of asymptotic and numerical methods.
- The key step in overcoming the solution on an unbounded domain has been the application of a **soft** boundary condition at large z consisting of
  - a mainstream non-local integral,
  - the Ekman layer ignored (simply $v = 0$),
  - and a linear interpolation across the overlap regions ($v \approx 0$).

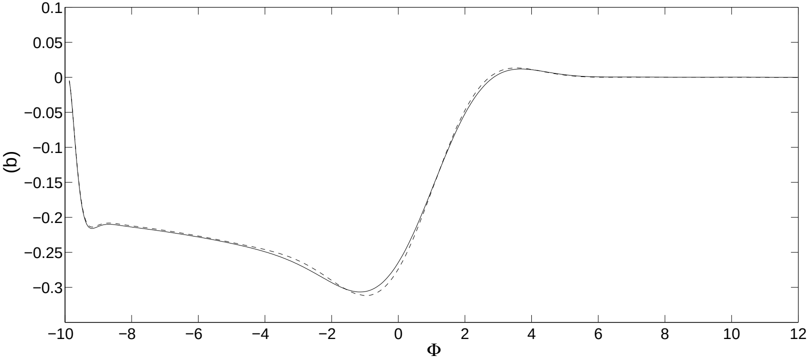
Comparison with asymptotics at large z

- We extended Stewartson's similarity solution valid for $z \gg 1$ to higher orders using matched asymptotic expansions.
- For largish z, the comparisons with the asymptotic results at
  - 0^{th}-order were reasonable;
  - $0^{\text{th}} + 1^{\text{st}}$-order were excellent;
  - $0^{\text{th}} + 1^{\text{st}} + 2^{\text{nd}}$-order were marginally improved.
- The apparent absence of any significant free "interlocking" mode at moderately large z was intriguing.

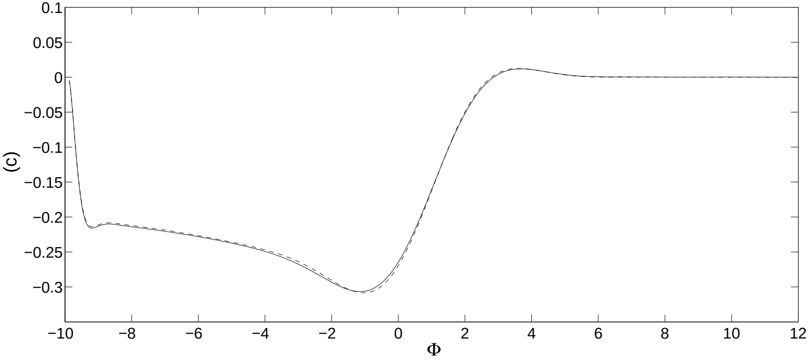
ψ *vs.* Φ. $0^{\rm th}$-order asymptotics - - - -



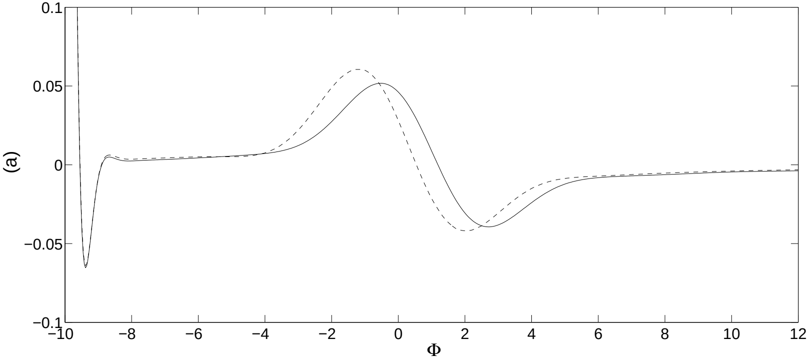
ψ *vs.* Φ. $1^{\rm st}$-order asymptotics - - - -



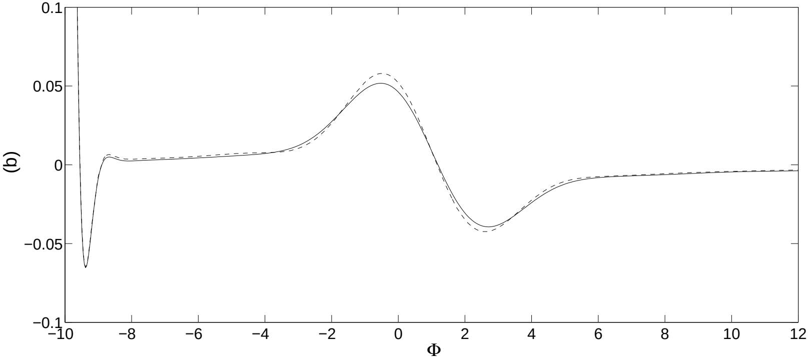
ψ *vs.* Φ. 2^{nd}-order asymptotics - - - -



v vs. Φ. 0^{th}-order asymptotics - - - -



v vs. Φ. 1^{st}-order asymptotics - - - -



v *vs.* Φ. 2[nd]-order asymptotics - - - -