Schurs exponent Conjecture

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Modified Question 1: Let G be an odd order group. Does the exponent of $H_2(G,\mathbb{Z})$ divide the exponent of G?

Validity of the Conjecture

Theorem (M. R. Jones, Some inequalities for the multiplicator of a finite group, Proc. Amer. Math. Soc 45 1974)

Proves the conjecture for p groups of Class 2.

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Prove the conjecture for powerful p groups.

Theorem (P. Moravec, Schur multipliers and power endomorphisms of groups, J. Algebra 308 (2007))

Proved the conjecture for metabelian p groups of exponent p

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Theorem (P. Moravec, The exponents of nonabelian tensor products of groups, J. Pure Appl. Algebra 212 (2008))

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Theorem (P. Moravec, On pro-p groups with potent filtrations, J. Algebra 322 (2009))

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Proves that $exp(H_2(G,\mathbb{Z}))|(expG)^2$.

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- *Finite p groups of class 5 with p odd.*

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- *Finite p groups of class at most p.*
- *Finite p-central metabelian p groups*
- *Finite groups satisfying* $\gamma_p(G) \subset G^{p^2}$.

Bounds depending on nilpotency class

Theorem (G. Ellis, On the relation between upper central quotients and lower central series, Transactions of the AMS 353 2001)

Let G be a finite p group of nilpotency class c. Then $exp(H_2(G,Z))|(expG)^{\frac{c}{2}}$.

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Theorem (N. Sambonet, Bounds for the exponent of the Schur multiplier, J. Pure Appl. Algebra 221 2017)

$exp(M(G)) \mid (exp(G))^m$, *where* $m = \log_{p-1} c + 1$

Bounds depending on nilpotency class

Theorem (A. Antony, P. Komma, V. Z.T 2019)

Let G be a finite group with nilpotency class $c > 1$ and set $n = \log_3(\frac{c+1}{2})$. If $exp(G)$ is odd, then $exp(G \wedge G) \mid (exp(G))^n$. In particular, $exp(M(G)) \mid (exp(G))^n$.

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Theorem (A. Antony, P. Komma, V.Z.T 2019)

Let p be an odd prime and G be a finite p-group of nilpotency class $c \geq p$. Then $exp(G \wedge G) \mid exp(G)^n$, where $n = 1 + \log_{p-1}(\frac{c+1}{p+1})$. In particular, $exp(M(G)) \mid (exp(G))^n$.

Bounds depending on derived length

Theorem (N. Sambonet, The unitary cover of a finite group and the exponent of the Schur multiplier, J. Algebra 426 (2015))

- *If p is odd, then* $exp(M(G)) \mid (exp(G))^d$.
- *if* $p = 2$, *then* $exp(M(G)) \mid 2^{d-1}(exp(G))^d$.

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Theorem (A. Antony, P. Komma, V.Z.T , 2019)

Let G be a solvable group of derived length d.

- *If $exp(G)$ is odd, then* $exp(G \otimes G) \mid (exp(G))^d$. *In particular,* $exp(M(G)) \mid (exp(G))^d$.
- *If $exp(G)$ is even, then* $exp(G \otimes G) \mid 2^{d-1}(exp(G))^d$. *In particular,* $exp(M(G)) \mid 2^{d-1}(exp(G))^d$.

Groups of Arganbright and Andrei Jaikin-Zapirain

Definition (Groups of Arganbright, Power Commutator Structure of Finite p groups,Pacific Journal Vol 29 1969)

Let G be a finite p group such that $\gamma_n(G) \subseteq G^p$, $1 < n < p$.

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Definition (Groups of A Jaikin-Zapirain, Potent p groups, J. Algebra 276 2004 193-209)

Let G be a finite p satisfying $\gamma_{p-1}(G) \subseteq G^p$ for $p > 2$ and $[G, G] \subseteq G^4$ if $p = 2$.

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Notice that this generalizes the definition of powerful 2 groups for all primes.

Results of G. Ellis and P. Moravec

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Let G be a finite p group of nilpotency class c. Then $exp(H_2(G, Z)) | (expG)^{\frac{c}{2}}$.

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Comparison of our results with Ellis' and Moravec's

Theorem (A. Antony, K. Patali, V. Thomas)

Let G be nilpotent group of odd order with nilpotency class c. If $c < (3 \times 2^n) - 1$, then $exp(H_2(G, Z)) | (expG)^n$.

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2) For $c = 95$, Ellis obtains $n = 48$, Moravec obtains $n = 12$ and we have $n = 6$.

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Theorem (N. Sambonet, J. Alg 2015 Unitary Covers of Finite Groups and Exponent...)

If G is a finite p group of derived length d, then $exp(H_2(G, Z)) | (expG)^d$ when $p > 2$

Further results by P. Moravec's

Theorem (Special Classes of Regular Groups)

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If G be an odd nilpotent group of class less than or equal to 4, then Schurs conjecture holds.

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Theorem

If G be an odd nilpotent group of class less than or equal to 4, then Schurs conjecture holds.

Theorem

If G is a metabelian group of prime exponent, then Schurs conjecture holds

Basic setup

Definition (Compatibility Condition)

Let G and H be groups acting on each other and acting on themselves via conjugation. The mutual actions are said to be compatible if ${}^{({}^{g}h)}g' = {}^{ghg^{-1}}g'$ and ${}^{({}^{h}g)}h' = {}^{hgh^{-1}}h'$

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Definition (Non-abelian tensor product)

The nonabelian tensor product $G \otimes H$ is the group generated by $g \otimes h$, $g \in G$ $h \in H$ and relations $gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h)$ and $g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$ for all $g, g' \in G$, $h, h' \in H$.

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Definition (Non-abelian tensor square)

When $G = H$ and all mutual actions are via conjugation, then the group is called non-abelian tensor square.