CENTRAL UNITS OF INTEGRAL GROUP RINGS

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Let G throughout be a **finite** group and $\mathrm{U}(\mathbb{Z}G)$ the unit group of its integral group ring.

Problem

(1) Describe $\mathrm{U}(\mathbb{Z}G)$.

(2) Describe $\mathrm{Z}(\mathrm{U}(\mathbb{Z}G)) \leqslant \mathrm{U}(\mathbb{Z}G)$.

(3) Which properties of G can we recover from $\mathbb{Z}G$?

$$\mathrm{Z}(\mathrm{U}(\mathbb{Z}G)) \simeq \pm\mathrm{Z}(G) \times C_\infty^r.$$

Definition (Bakshi-Maheshwary-Passi, 2017)

A finite group G is called *cut*[a] $\Leftrightarrow$ $\mathrm{Z}(\mathrm{U}(\mathbb{Z}G))$ is finite.

[a] cut stands for "central units trivial"

Besides the intrinsic interest in cut groups here are two further motivations why cut groups are relevant:

- When describing the structure of the unit group, one frequently aims for a "generic" subgroup of finite index:

$$\left[\mathrm{U}(\mathbb{Z}G) : \left\langle\ (\mathbb{Z}G)^1, \mathrm{Z}(\mathrm{U}(\mathbb{Z}G))\ \right\rangle\right] < \infty$$

$$\nearrow \qquad\qquad \nwarrow$$

often up to f.i. by "bicyclic units" | covered by "bi-cyclic units" & "Bass units"

  (see Jespsers-del Río, *Group Ring Groups*, Chapter 11). So for cut groups, the Bass units are superfluous.

- G being cut has a strong bearing on fixed point properties like Serre's property (FA) and Kazhdan's property (T) for the group $\mathrm{U}(\mathbb{Z}G)$; see B-Janssens-Jespers-Kiefer-Temmerman, *Abelianization and fixed point properties of linear groups and units in integral group rings*, Chapter III.

Let G throughout be a **finite** group.

$$G \textit{ rational} \quad :\Longleftrightarrow \quad \mathrm{CT}(G) \in \mathbb{Q}^{h \times h}$$

$$\Longleftrightarrow \quad \forall\, x, y \in G\colon \quad \langle x \rangle = \langle y \rangle \ \Rightarrow \ x \sim y$$

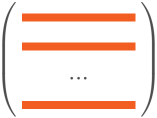
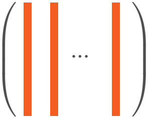
$$G \textit{ cut} \quad :\Longleftrightarrow \quad \mathrm{Z}(\mathrm{U}(\mathbb{Z}G)) = \pm \mathrm{Z}(G)$$

$$\Longleftrightarrow \quad \forall\, \chi \in \mathrm{Irr}(G)\colon \mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-d_\chi}),\ d_\chi \in \mathbb{Z}_{\geqslant 0}$$

$$\Longleftrightarrow \quad \forall\, x \in G\colon \quad \mathbb{Q}(x) = \mathbb{Q}(\sqrt{-d_x}),\ d_x \in \mathbb{Z}_{\geqslant 0}$$

$$\Longleftrightarrow \quad \forall\, x, y \in G\colon \quad \langle x \rangle = \langle y \rangle \ \Rightarrow \ x \sim y \text{ or } x \sim y^{-1}$$

Where: $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) : g \in G)$,

$\mathrm{CT}(G) =$  $\mathrm{CT}(G) =$  $\mathrm{CT}(G) =$ 

Examples.

- S_n is rational.
- $P \in \mathrm{Syl}_p(S_n)$.
  P rational $\Leftrightarrow$ $p = 2$.
  P cut $\Leftrightarrow$ $p \in \{2, 3\}$.
- $P \in \mathrm{Syl}_p(\mathrm{GL}(n, p^f))$.
  P rational $\Leftrightarrow$ $p = 2$ and $n \leqslant 12$.
  P cut $\Rightarrow$ $p = 2$ and $n \leqslant 24$ or $p = 3$ and $n \leqslant 18$.

Theorem (Bakshi-Maheshwary-Passi, 2017)

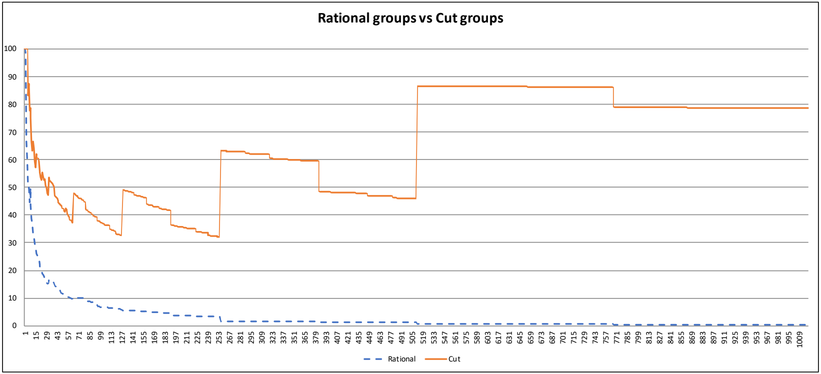
$G \neq 1$ cut group, then $\quad 2 \mid |G|$ *or* $3 \mid |G|$.

$f(r)$: number of all groups of order r,
$c(r)$: number of cut groups of order r.

Proposition (B-Caicedo-Jespers-Maheshwary, 2018)

$$\lim_{n\to\infty} \frac{\ln c(p^n)}{\ln f(p^n)} = 1, \qquad \textit{for } p \in \{2, 3\}.$$

Percentage of rational and cut groups in all groups



An old conjecture:

Conjecture

G rational, $P \in \mathrm{Syl}_2(G) \quad \Rightarrow \quad$ *P rational?*

In 2012, I.M. Isaacs and G. Navarro presented counterexamples of order 1536 to this conjecture. Yet, they also proved:

Theorem (Isaacs-Navarro, 2012)

Let G be a solvable rational group and $P \in \mathrm{Syl}_2(G)$ *has nilpotency class at most 2. Then P is rational.*

Problem A

G cut, $P \in \mathrm{Syl}_3(G) \quad \Longrightarrow \quad P$ cut?

Why might $p = 3$ really be different?

An element $x \in H$ is called *inverse semi-rational (isr) in* G iff for all $y \in H$ s.t. $\langle x \rangle = \langle y \rangle$: $x \sim y$ or $x \sim y^{-1}$.

Hence:
H cut $\quad \Longleftrightarrow \quad \forall\, x \in H$: x inverse semi-rational in H.

Lemma

Let $x \in G$ be a 3-element. Then
x is isr in G $\quad \Longleftrightarrow \quad$ x is isr in P for some $P \in \mathrm{Syl}_3(G)$ with $x \in P$.

Problem A

G cut, $P \in \mathrm{Syl}_3(G) \implies P$ cut?

Theorem (B-Caicedo-Jespers-Maheshwary, 2018)

Let G be a cut group and $P \in \mathrm{Syl}_3(G)$. Then P is also cut, provided one of the following holds:

1. *G is supersolvable,*
2. *G is a Frobenius group,*
3. *G is (almost) simple,*
4. *G is of odd order and $\mathrm{O}_3(G)$ is abelian,*
5. *$|G| \leqslant 2000$ or $|G| \in \{ 2^2 \cdot 3^6, 2^3 \cdot 3^6, 2^2 \cdot 3^7 \}$.*

Theorem (Grittini, 2019; Navarro-Tiep, 2019)

Let p be an odd prime, let G be a (p-solvable) cut group,
$P \in \mathrm{Syl}_p(G) \implies P/P'$ elementary abelian.

DEFINITION. $\pi(G) = \{p \text{ prime} : p \mid |G|\}$, the *prime spectrum of G*.

REMARK. $|\pi(S_n)| \longrightarrow \infty$ for $n \to \infty$.

Theorem (Gow, 1976)

Let G be a solvable rational group. Then $\pi(G) \subseteq \{2, 3, 5\}$.

Theorem (Chillag-Dolfi, 2010; B, 2017)

Let G be a solvable cut group. Then $\pi(G) \subseteq \{2, 3, 5, 7\}$.

G solvable. If G is rational, then $\pi(G) \subseteq \{2, 3, 5\}$.
If G is cut, then $\pi(G) \subseteq \{2, 3, 5, 7\}$.

Every $\{2, 3\}$-group can be emebdded in a rational $\{2, 3\}$-group.

Theorem (Hegedűs, 2005)

If G is a solvable rational group and $P \in \mathrm{Syl}_5(G)$.
Then $P \trianglelefteq G$ and $\exp P \mid 5$.

Remark

Let G be a solvable cut group and $p \in \{2, 3, 5, 7\}$, $P \in \mathrm{Syl}_p(G)$.
The p-length of G and the exponent of P can be arbitrarily large.

Problem B

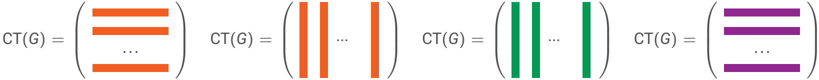
Let G be a solvable cut group.
Is it true that $\exp \mathrm{O}_5(G) \mid 5$ and $\exp \mathrm{O}_7(G) \mid 7$?

G *cut* $\iff \forall\, \chi \in \mathrm{Irr}(G)$: $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-d_\chi}),\ d_\chi \in \mathbb{Z}_{\geqslant 0}$.

$\iff \forall\, x \in G$: $\mathbb{Q}(x) = \mathbb{Q}(\sqrt{-d_x}),\ d_x \in \mathbb{Z}_{\geqslant 0}$.

G *semi-rational* $:\iff \forall\, x \in G$: $[\mathbb{Q}(x) : \mathbb{Q}] \leqslant 2$.

G *quadratic rational* $:\iff \forall\, \chi \in \mathrm{Irr}(G)$: $[\mathbb{Q}(\chi) : \mathbb{Q}] \leqslant 2$.



Denote $\mathbb{Q}(G) = \mathbb{Q}(\chi(g) : g \in G,\ \chi \in \mathrm{Irr}(G))$. Obviously, $[\mathbb{Q}(G) : \mathbb{Q}] = 1$, if G is rational. Can we bound $[\mathbb{Q}(G) : \mathbb{Q}]$ for an interesting class containing the rational groups?

Theorem (Tent, 2012)

Let G be a solvable group, that is semi-rational or quadratic rational. Then $[\mathbb{Q}(G) : \mathbb{Q}] \leqslant 2^7$.

In particular, this holds for solvable cut groups (there $\leqslant 2^5$)!

Robinson-Thompson (1995): $[\mathbb{Q}(A_n) : \mathbb{Q}] \sim 2^{\pi(n)}$ (π prime counting function), so $[\mathbb{Q}(G) : \mathbb{Q}]$ can get arbitrarily large for semi-rational and quadratic rational groups G.

Yet, only 5 simple alternating groups are cut and $A_n \times \ldots \times A_n$'s "tend" to become rational as chief factors of cut groups.

Problem C

Is there $c > 0$ such that for all cut groups G, $[\mathbb{Q}(G) : \mathbb{Q}] \leqslant c$?

Theorem (Bakshi-Passi-Maheshwary, 2017)

Description of all (46) metacyclic cut groups.

Theorem (B., 2018)

Let G be a Frobenius cut groups.

1. *If the Frobenius complement is of even order, then G is in one of 6 infinite series or one of 4 "sporadic" such groups.*
2. *If the Frobenius complement K is of odd order, then $K \simeq C_3$ and the Frobenius kernel is a 2- or 7-group of nilpotency class at most 2 and exponent dividing 4 or 7, respectively.*

Theorem (Trefethen, 2019)

The non-abelian composition factors of cut groups are all alternating + 16 of Lie type + 12 sporadics.

Theorem (B.-Caicedo-Kiefer-del Río, 2019)

The Prime Graph question holds for all cut groups not mapping onto the Monster.

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Thank you for your attention!