On torsion units of ZG and related questions

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$\mathrm{V}(RG)$	group of normalized units of RG, i.e.

$$\mathrm{V}(RG) = \left\{ \sum_{g \in G} u_g g \in U(RG) : \sum_{g \in G} u_g = 1 \right\}$$

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It may be investigated with respect to each coefficient ring R. But it is clear that the „best "answers are obtained looking at the integers $\mathbb{Z}$.

Because G lives naturally in the units of RG, $\mathbb{Z}G$ rsp. it is natural to expect answers considering the units and there the normalized units $V(\mathbb{Z}G)$ contain all informations.

Basic notions and results on torsion units

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- H is a group basis iff $|H| = G|$.

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But ZC 3 and so all three Zassenhaus conjectures hold for nilpotent groups (A.Weiss 1991) and ZC1 for cyclic-by-abelian groups (M.Caicedo,A.delRio,L.Margolis 2013).

Replacements for ZC1

Several questions have been posed which are weaker than ZC 1, e.g.

SP Does the order of a torsion element of $V(\mathbb{Z}G)$ coincide with the order of a group element of G (the so-called Spectrum question SP) ?

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- Positive answer to Question R1
- u is conjugate to an element of G in $\mathbb{Q}S_G$.
- (Conjecture of A.A.Bovdi 1987) Let $u = \sum_{g \in G} z_g g \in \mathbb{Z}G$. Then for each $m \in \mathbb{N}$ with $m \neq o(u)$ the coefficients of elements of order m of u sum up to zero , i.e.

$$\sum \quad z_g = 0.$$

Known results on Bovdi's conjecture

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- G soluble, all Sylow subgroups are abelian and u has prime power order. (S.O.Juriaans 1994)
- G arbitrary , u has prime order p. (M.Hertweck 2006)

Recent Results

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Proposition 1 (A.Bächle, W.K. - M.Serrano)

Suppose that G has a nilpotent Hall subgroup N such that G/N is abelian. Then there is a group H containing G as subgroup such that ZC1 holds for $\mathbb{Z}H$.

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Note. In the special case when the Hall subgroup is a p-group M.Hertweck showed 2006 that then even ZC1 holds.
The statement of Proposition 1 is slightly stronger than an affirmative answer to Question R1.

Recent Results ctd

Proposition 2 (A.Bächle, W.K. - M.Serrano)

Suppose that G has a normal Sylow p - subgroup P such that Bovdi's conjecture has an affirmative answer for $\mathbb{Z}G/P$. Then it has also an affirmative answer for $\mathbb{Z}G$.

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Corollary

Suppose that G has a supersoluble normal Hall - subgroup H such that Bovdi's conjecture has an affirmative answer for $\mathbb{Z}G/H$. Then it has also an affirmative answer for $\mathbb{Z}G$.

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Corollary

Suppose that G has a supersoluble normal Hall - subgroup H such that Bovdi's conjecture has an affirmative answer for $\mathbb{Z}G/H$. Then it has also an affirmative answer for $\mathbb{Z}G$.

Note. With respect to supersoluble groups ZC 1 is still open.

Comparison with ZC 1

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So the result of M. Dokuchaev and S. K. Sehgal shows that for these groups Bovdi's conjecture holds and Proposition 1 that these groups may be even embedded into larger groups for which ZC 1 holds.

Ingredients of the proof

Theorem (Hertweck 2013)

Let N be a normal p - subgroup of G. Then any torsion unit in $\mathbb{Z}G$ which maps to the identity under the natural map $\mathbb{Z}G \longrightarrow \mathbb{Z}G/N$ is conjugate to an element of N by a unit in $\mathbb{Z}_pG$, where $\mathbb{Z}_p = p$ - adic integers.

Proposition (Hertweck 2008)

Let G be a finite group and u a normalized torsion unit of $\mathbb{Z}_pG$. Suppose that the p - part of u is conjugate to an element $x \in G$ within $\mathbb{Z}_pG$. Then the partial augmentation

$$\varepsilon_y(u) = 0 \quad (\text{i.e.} \quad \sum_{g \in y^G} z_g = 0, \text{ when } u = \sum_{g \in G} z_g g)$$

for every $g \in G$ whose p - part is not conjugate to x.

Sketch of the proof of Proposition 1

Using the two results of Hertweck one can show that ZC 1 holds for the group

$$G = (P_1 \cdot A_1) \times \ldots \times (P_k \cdot A_k),$$

where each A_j is abelian and $P_1 \times \ldots \times P_k$ form a normal nilpotent Hall subgroup N of G and P_j are the Sylow subgroups of N.

Recall that Hertweck showed ZC 1 for $P \cdot A$, i.e. for „Sylow -by- abelian “. So the first step just generalizes the proof of Hertweck to „nilpotent Hall -by- sufficiently large abelian. “

If now H is given with a nilpotent normal Hall subgroup N such that H/N is abelian, then let A be a complement of N in H and let

$$G = N \rtimes (A_1 \times \ldots \times A_k),$$

where $A_i \cong A$ for each i, $N = P_1 \times \ldots \times P_k$ is the Sylow decomposition and each A_i acts on P_i as A and trivially on the other factors P_j.
Then H embeds into G and ZC 1 holds for G by the first step.

Metabelian Groups

Theorem

Let G be a finite metabelian group. Then there is a group E (e.g. $E = S_G$, containing G such that the following holds.

(1) Each torsion unit of $V(\mathbb{Z}G)$ is conjugate within $\mathbb{Q}E$ to an element of G.

(2) Each group basis of $\mathbb{Z}G$ is conjugate within $\mathbb{Q}E$ to G.

(3) Each subgroup of $V(\mathbb{Z}G)$ is conjugate within $\mathbb{Q}E$ to a subgroup of G.

NilpotentHall - by - Abelian Groups

Theorem

Let G be a finite nilpotentHall-by-abelian group. Then there is a group E containing G such that the following holds.

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(3) holds even with $\mathbb{Q}G$ instead of $\mathbb{Q}E$ and is due to M.Dokuchaev.

Supersoluble Groups

Theorem

Let G be a finite supersoluble group. Then there is a group E containing G such that the following holds.

(1) Each torsion unit of $V(\mathbb{Z}G)$ is conjugate within $\mathbb{Q}E$ to an element of G.

(2) Each group basis of $\mathbb{Z}G$ is conjugate within $\mathbb{Q}E$ to G.

(3) Each p - subgroup of $V(\mathbb{Z}G)$ is conjugate within $\mathbb{Q}E$ to a subgroup of G.

The proof of all these three results uses the following.

Lemma

Suppose (R1) holds for the finite group G.
Let H be a subgroup of $V(\mathbb{Z}G)$ and assume that H is isomorphic to a subgroup of G.
Let E be the symmetric group on G and consider G as subgroup of E via multiplication on itself.
Then H is conjugate within $\mathbb{Q}E$ to a subgroup of G.

The preceeding results suggest to consider for **certain** finite groups and certain subgroups the following questions

Question R2 Are two group bases of $V(\mathbb{Z}G)$ conjugate within $\mathbb{Q}S_G$, i.e. a normalized automorphism of $\mathbb{Z}G$ is given by such a conjugation followed by a group automorphism.

Question R3 Given a torsion subgroup U of $V(\mathbb{Z}G)$. Is there a finite group E containing G such that U is conjugate within $\mathbb{Q}E$ to a subgroup of G?

Of course the questions make only sense for special classes, e.g. for p - subgroups. This class leads to the following topic.

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A Sylowlike theorem in $V(\mathbb{Z}G)$ may have the following form

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So G would determine the finite p - subgroups of $V(\mathbb{Z}G)$.

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Of course one could also try to prove as a first goal weaker statements (so-called weak Sylow like theorems), weak means isomorphism instead of conjugacy

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and with respect to specific primes, if

- $G/O_{p'}(G)$ has a normal Sylow p - subgroup (A. Weiss 1993)
- $G = PSL(2, r^f)$ if $p \neq r$ or $p = r = 2$ or $f = 1$. (Hertweck - Höfert - Ki. 2009, Margolis 2016)

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A weak Sylowlike theorem holds, if

- G has cyclic Sylow p - subgroups. (Ki. for p=2 2007, Hertweck for p odd 2008)
- 2 - subgroups of $V(\mathbb{Z}G)$ are isomorphic to subgroups of G if Sylow 2 - subgroups of G are abelian, quaternion or dihedral. (Bächle-Ki. 2011, Ki. 2015, Margolis 2017)

Sylow numbers for group bases

M.Hertweck showed 1997 that group bases of $\mathbb{Z}G$ are not always isomorphic. Thus it is not clear whether different group bases have the same number of Sylow p - subgroups.

Let G be finite. Let X be a group basis of $V(\mathbb{Z}G)$.
Denote by $n_p(G), n_p(X)$ rsp. the number of Sylow p - subgroups of G,X rsp.
Is $\quad n_p(G) = n_p(H) \;\; ?$

Characters and Sylow numbers

For a finite group U denote by $\mathbb{X}(U)$ its ordinary character table and by $Spec(U)$ its spectral table, i.e. the character table including the head line.

G.Navarro 2003

Let G and H be finite groups with $\mathbb{X}(G) = \mathbb{X}(H)$. Does it follow for each prime p that

$$n_p(G) = n_p(H)$$

Note that $\mathbb{Z}G \cong \mathbb{Z}H \Longrightarrow Spec(G) = Spec(H) \Longrightarrow \mathbb{X}(G) = \mathbb{X}(H)$. Thus results on character tables yield results for group rings.

Properties from Character Tables

The character table $\mathbb{X}(G)$ determines

- The length of the conjugacy classes.
- The normal subgroup lattice. For each normal subgroup N it determines $\mathbb{X}(G/N)$.
- The chief series of G, i.e. in particular the composition factors up to isomorphism (W.K. 1989).
- It does not determine the orders of the representatives of the conjugacy classes but the primes dividing the order of a representative (G.Higman).
- It determines whether G has abelian Sylow subgroups and if so their isomorphism type (W.K. and R.Sandling 1989).

Theorem (G.Navarro and N.Rizo 2016).

$$Spec(G) = Spec(H) \Longrightarrow n_p(G) = n_p(H) \ \forall p$$

provided G is p - soluble.

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Theorem (W.K. and I.Köster 2017)

$$\mathbb{X}(G) = \mathbb{X}(H) \Longrightarrow n_p(G) = n_p(H) \ \forall p$$

provided G is nilpotent - by - nilpotent or a Frobenius group.

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Theorem S

Let G be a p - constrained group. Let q be a prime not dividing $O_{p'}(G)$ and let X be a group basis of $\mathbb{Z}G$.

a) (W.K.-K.W.Roggenkamp 1993) A q - subgroup U of X is conjugate within $\mathbb{Q}G$ to a subgroup of G.
b) (W.K.-I.Köster 2016) The number of Sylow q - subgroups of X and G coincide.

Note. p - soluble groups are p - constrained. Thus a **Sylow like theorem for group bases** follows for integral group rings of arbitrary **finite soluble groups**.

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In a first moment one might think that there may be no general results on these problems. **However**

ZC2 and therefore IP are almost true

Theorem A

Let G be an arbitrary finite group then ZC2 (and thus also the isomorphism problem IP) has a positive answer for $\mathbb{Z}(F_pG \cdot G)$.

Corollary

Let G be an arbitrary finite group then

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Theorem A is a special case of Theorem B.

Some Ingredients of the proof of Theorem S and related results

- Rational conjugacy for finite q - subgroups theorem holds for $V(\mathbb{Z}G)$ if it holds for $V(\mathbb{Z}G/O_{p'}(G))$.

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- Theorem B , i.e. if $O_{p'}(G) = 1$ then $\mathbb{Z}G$ determines G up to isomorphism because G is by assumption p - constrained.
- (I. Köster, 2017) Suppose that the finite group G has normal subgroups M and N such that $M \cap N = 1$. Then the number of Sylow p - subgroups may be computed from the number of Sylow p - subgroups of G/M, G/N and $G/(M \cdot N)$,

$$n_p(G) = \frac{n_p(G/M) \cdot n_p(G/N)}{n_p(G/MN)}.$$

Summarizing open questions

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Moreover the following questions are open.

- **SP** Are finite cyclic subgroups of $V(\mathbb{Z}G)$ isomorphic to a subgroup of G? In other words have $V(\mathbb{Z}G)$ and G the same spectrum ?
- **Sylowlike** Does in $V(\mathbb{Z}G)$ a Sylowlike theorem hold ? Does $\mathbb{Z}G$ determine the Sylow numbers ?

For the proof of the Sylowlike result for $PSL(2, r^f)$ (as in the most results concerning insoluble groups)
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the so-called HeLP - method is used (explained in the workshop last week).
The origin for this computational method and for the many resulting articles is the paper

Zassenhaus conjecture for A_5

by

I.S.Luthar and I.B.S.Passi

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Thank you for your attention