Polynomial Groups, Polynomial 2- cocycles, Dimension Subgroups and Related Problems

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Let G be a group, ZG the integral group ring of G and $\Delta(G)$ its augmentation ideal. For $n \geqq 1$, let $\Delta^n(G)$ denote the nth associative power of $\Delta(G)$ with $\Delta^1(G) = \Delta(G)$ and let $\gamma_n(G)$ denote the nth term in the lower central series of G with $\gamma_1(G) = G$ and $\gamma_2(G) = G'$ the derived group of G.Let $T = Q/Z$ denote the additive group of rationals $mod 1$ regarded as a trivial $G-$ module. For G Abelian, let $SP^n(G)$ denote the nth symmetric power of G. If R is a commutative ring with identity, let RG denote the group ring of G with coefficients in R and let $\Delta_R(G)$ denote its augmentation ideal.

In this talk we give a report/ review the work done by Professor I. B. S. Passi upto the year 1990. Later work of Professor Passi will be reprted upon by the next speakers.

1. Polynomial groups

For an Abelian group A, a map $f : G \longrightarrow A$ is called a polynomial map of degree $\leq n$ if the linear extension $f^{*29} : ZG \longrightarrow A$ of f vanishes on $\Delta^{n+1}(G)$.The quotient groups $P_n(G) = \Delta(G)/\Delta^{n+1}(G), Q_n(G) = \Delta^n(G)/\Delta^{n+1}(G)$ are called polynomial groups. Passi [29] proves that

Theorem 1.1. (a) If G is finite, then $P_n(G)$ and, so also, $Q_n(G)$ is finite for all $n \geqq 1$;

(b) If G is cyclic, then $Q_n(G) \equiv G$;

(c) If G is a free group of finite rank, then $P_n(G), Q_n(G)$ are free Abelian for all $n \geq 1$.

(d) If G is a free Abelian group of rank m,then $P_n(G)$ is a free Abelian group of rank $\binom{n+m}{m} - 1$ and $Q_n(G)$ is a free Abelian group of rank $\binom{n+m-1}{n-1}$.

He also obtains the structure of $P_n(G)$ and $Q_n(G)$ when G is a finite elementary Abelian $p-$group.

Passi [31] proves that P_n and Q_n are nonadditive functors from the category of Abelian groups to itself.

The map $\theta_n(G) : SP_n(G) \longrightarrow Q_n(G)$, given by $\theta_n(G)(\sum_{x_{i_j} \epsilon G} x_{i_1} \otimes^\wedge ... \otimes^\wedge x_{i_n}) = \sum_{x_{i_j} \varepsilon G}(x_{i_1} - e)...(x_{i_n} - e) + \Delta^{n+1}(G), \quad x_{i_j} \in G,$

which is an epimorphism induces a natural transformation $\theta_n : SP^n \longrightarrow Q_n$ which is always surjective and is an isomorphism on free Abelian groups. Consequently it follows that the derived functors of Q_n coincide with those of SP^n. He also proves

Theorem 1.2. θ_2 is always an isomorphism.

G. Losey tried to extend the above result of Passi and proved

Theorem 1.3.[23]. If G is any finitely generated group, then $Q_3(G) \cong \gamma_2(G)/\gamma_3(G) \oplus SP^3(G/\gamma_2(G))$.

Recall that if G is an Abelian group, the group $SP^n(G) = F/R$, where F is the free Abelian group generated by all symbols $u(x_1, x_2, \ldots, x_n)$, $x_i \in G$ and R is the subgroup of F generated by all elements of the form (1) $u(x_1, ..., x_{i-1}, x_i x_i', x_{i+1} ..., x_n) - u(x_1, ... x_{i-1}, x_i, x_{i+1}, ..., x_n) - u(x_1, ..., x_{i-1}, x_i', x_{i+1}, ..., x_n)$ and (2) $u(x_!, ..., x_n) - u(x_{\pi(1)}, ..., x_{\pi(n)})$, where π denotes any permutation of $1, 2, ..., n$.The coset $u(x_1, ..., x_n) + R$ is denoted by $x_1 \hat{\otimes} ... \hat{\otimes} x_n$. Vermani [51] proved that $\phi_n : G \longrightarrow SP^n(G)$ given by $\phi_n(x) = x \hat{\otimes} ... \hat{\otimes} x$ (n factors, $x \varepsilon G$) is a polynomial map of degree $\leqq n$.The map ϕ_n induces a natural trnsformation : $Q_n \longrightarrow SP^n$. Vermani further proves that the composition $\phi_n \theta_n$ is multiplication by $n!$ (factorial n). Consiquently

Theorem 1.4. [52]. (a) If G is $n!$ tortionfree, then θ_n is an isomorphism.
(b) If G is torsionfree, then θ_n is an isomorphism for all $n \geqq 1$.

For an Abelian group G, let $S(G) = \sum_n SP^n(G)$ be the symmetric algebra of G and $grZG = \sum_n Q_n(G)$ be the associated graded ring of ZG. The natural epimorphisms θ_n then induce a natural epimorphism $\theta(G) : S(G) \longrightarrow grZG$. F. Bachman and L. Grunenfelder prove

Theorem 1.5[3]. For a finite Abelian group $G, \theta(G)$ is an isomorphism if and only if G is cyclic.

Passi and Vermani prove that

Theorem 1.6 [44]. (1) If G is an elementary Abelian $p-$group of rank k, then $grZG \equiv Z[x_1, ..., x_k]/ < px_i, x_j^p x_i - x_j x_i^p >$.

(2) If $G \neq (1)$ is of type $(p^{m_1}, p^{m_2}, ..., p^{m_k})$ with p prime and $m_i \geqq 1$, then θ_n is an isomorphism if and only if $n < (r+1)(p-1)+2$, where $r = \infty$ for $k = 1$ and $r = \min\{|m_i - m_j| | i \neq j\}$ for $k > 1$.

Bachman and Grunenfelder prove that

Theorem 1.7 [5]. If the group G is finite Abelian, then there exists an integer $N(G)$ such that $Q_n(G) \cong Q_{n+1}(G)$ for all $n \geqq N(G)$.

It is then immediate that

Theorem 1.8 [44]. For the $p-$group G as in Theorem 1.6(2), $N(G) \geqq (r+1)(p-1)+1$ for $k > 1$.

Hales [17] has described the structure of $Q_n(G)$ for finite Abelian G for large n. Later, Hales and Passi [20] gave a simplified proof of the following:

Theorem 1.9. If G is a finite Abelian $p-$group, p prime, then for sufficient large $n, Q_n(G)$ is independent of n and its order is p^{d-1}, where d is the number of cyclic subgroups of G.

Finding the structure of the polynomial groups $Q_n(G)$ generated a lot of interest. The structure of $Q_2(G)$, is given by an exact sequence

$$o \longrightarrow \gamma_2(G)/\gamma_3(G) \longrightarrow Q_2(G) \longrightarrow SP^2(G/\gamma_2(G)) \longrightarrow 0 \quad (1.1)$$

in which the maps are natural. Sandling [47] proved that the sequence (1.1) splits (although not naturally) if G is finite and Losey [24] proved the splitting of this sequence for G finitely generated. Hales and Passi [19] imroved upon this result and proved

Theorem 1.10. The sequence (1.!) splits in the following cases:

(i) $G/\gamma_2(G)$ and, hence, G is finitely generated; (the same as Losey's result)

(ii) $G\gamma_2(G)$ is direct sum of (a) direct sum of cyclic groups, or (b) is divisible, or (c) completely decomposable torsion- free, or (d) torsion.

They also show that the sequence (1.1) does not always split and construct an example to prove their claim.

If R is a radical ring, then R with composition defined by $a \star b = a + b + a \times b, a, b \in R$, is a group called the circle group of R. Hales and Passi [19] also consider the question: which nilpotent groups of class 2 arise as circle groups of nilpotent rings of index 3? They prove

Theorem 1.11. A nilpotent group G of class 2 is a circle group of a nilpotent ring of index 3 if $G/\gamma_2(G)$ is a direct sum of cyclic groups, or is divisible, or is torsion, or is torsion-free and completely decomposable.

This result corresponds to a result of Ault and Watters [2] who proved it with $G/\gamma_2(G)$ replaced by $G/\zeta(G), \zeta(G)$ the center of G. They then prove that the conjecture of Ault and Watters namely "every nilpotent group of class 2 is a circle group of a nilpotent ring of index 3" is false.

2. Polynomial 2−cocycles and dimension subgroup

For an ideal J of $ZG, G \cap (1 + J) = \{x \in G | x - 1 \in J\}$ is a normal subgroup of G. Also G

is commutative. Let $indH^2(G,T)$ denote the subgroup of $H^2(G,T)$ which is generated by the elements of $H^2(G,T)$ which correspond to induced cental extensions. Passi [32] proved

Theorem 2.5. Dimension conjecture is equivalent to the following statement:

For all finite $p-$groups G, every element of $indH^2(G,T)$ is of degree $\leq$ class of G.

He then characterized induced central extensions and proved

Theorem 2.6. If G is nilpotent of class n, then a central extension $0 \longrightarrow T \xrightarrow{i} M \xrightarrow{\alpha} G \longrightarrow 1$ is an induced central extension if and only if the central extension $0 \longrightarrow T/\gamma_{n+1}(M) \longrightarrow M/\gamma_{n+1}(M) \longrightarrow G \longrightarrow 1$ is a split extension.

He then deduced that

(a) every central extension of T by any Abelian group is an induced central extension;

(b) every central extension of T by a free nilpotent group is an induced cental extension.

Improving upon an earlier result of his, he roved

(c) For any Abelian group $G, \deg H^2(G,T) \leq 1$.

As another characterization of induced central extensions, he proved

Theorem. **2.7** If G is a nilpotent group of class $n \geqq 2$, then every cental extension of T by G is an induced central extension if and only if the inflation homomorphism $\inf : H^2(G/\gamma_n(G), T) \longrightarrow H^2(G, T)$ is zero.

Vermani [53] gave yet another characterization of induced central extensions. Induced central extensions could, in fact, also be defined with T replaced by any Abelian group A regarded as a trivial $G-$module. Let A be an Abelian group regarded as a trivial $G-$module and

$$E : 0 \longrightarrow A \xrightarrow{i} M \xrightarrow{\alpha} G \longrightarrow 1$$

be a central extension of A by a nilpotent group G of class $n \geqq 1$. Let ξ be the element of $H^2(G, A)$ which corresponds to the central extension E, $\delta_E : H_2(G, Z) \longrightarrow A$ be the cotransgression associated with E and $\tau : H_2(G, Z) \longrightarrow H_2(G/\gamma_n(G), Z)$ be the coinflation homomorphism. Let B be image of δ_E. Vermani[53] gave the following characterization.

Theorem 2.8. The central extension E is induced if and only if

(i) $H_2(G, Z) = \ker \delta_E + \ker \tau$,and (ii) $\xi \in \ker\{H^2(G, A) \longrightarrow H^2(G, A/B)\}$, where $H^2(G, A) \longrightarrow H^2(G, A/B)$ is the homomorphism induced by the natural projection $A \longrightarrow A/B$.

He also obtained certain necessary conditions for E to be Baer sum of an induced central extension and a central extension of class n. As a particular case , he proved

Every central extension of an elementary Abelian $p-$group by a nilpotent group of class n is Baer sum of an induced central extension and a central extension of class n.

In view of Theorems 2.5 and 2.7, we are lead to investigate conditions under which the inflation homomorhism $H^2(G/\gamma_n(G), T) \longrightarrow H^2(G, T)$ is zero. Observe that when $G = H \oplus K$, with H a finite $p-$group of class 2 and K cyclic of order p,then $\inf : H^2(G/\gamma_2(G), T) \longrightarrow H^2(G, T)$ is not zero. Let $Z^2_m(G, A)$ denote the the subgroup of $Z^2(G, A)$ consisting of $2-$cocycles of G to A which are of degree $\leq m$. As a first step Passi and Vermani [43] prove that the subgroup $P_nH^2(G, T)$ is given by the following exact sequence

$$0 \longrightarrow Hom(\Delta^2(G)/\Delta^{n+2}(G), T) \xrightarrow{u_n} Z^2_n(G, T) \xrightarrow{v_n} P_n(G, T) \longrightarrow 0$$

where u_n and v_n are suitably defined homomorphisms. They prove that if $H^2(G/\gamma_n(G), T)$ is of degree $\leq n - 1$, then $\inf : H^2(G/\gamma_n(G), T) \longrightarrow H^2(G, T)$ is zero if and only if the homorphism u_{n-1} is an isomorphism. As a consequence of this, it can be proved

Proposition 2.9. For any group G, $\inf : H^2(G/\gamma_2(G), T) \longrightarrow H^2(G, T)$ is zero if and only if $\Delta^2(G)/\Delta^3(G) \cong G/\gamma_2(G) \otimes G/\gamma_2(G)$.

Then they prove

Theorem 2.10. If G is a finite group, then inf:$H^2(G/\gamma_2(G), T) \longrightarrow H^2(G, T)$ is zero if and only if $|\gamma_2(G)/\gamma_3(G)||SP^2(G/\gamma_2(G)| = |G/\gamma_2(G) \otimes G/\gamma_2(G)|$.

If G is nilpotent of class n and $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$ is a free presentation of G, they prove

Theorem 2.11. inf : $H^2(G/\gamma_n(G), T) \longrightarrow H^2(G, T)$ is
(a) zero if and only if $R \cap \gamma_2(F) = [F, R]\gamma_{n+1}(F)$;
(b) an epimorphism if and only if $\gamma_{n+1}(F) \subseteq [F, R]$.

3. Filtration of the Schur multiplicator

(a) **Schur multiplicator**

In view of the relationship between the dimension property and the study of the subgroup $P_nH^2(G, T)$, of $H^2(G, T)$ (refer Theorem 2.2) it is of interest to study the increasing filtration $0 = P_0H^2(G, T) \subseteq P_1H^2(G, T) \subseteq ... \subseteq P_nH^2(G, T) \subseteq P_{n+1}H^2(G, T) \subseteq ...$

of the multiplicator $H^2(G, T)$ of G. Passi and Stammbach [40] construct a

short exact sequence $0 \longrightarrow Hom(\Delta^{n+1}(G)/\Delta^{n+2}(G), T) \longrightarrow Hom(\Delta(G)/\Delta^{n+1}(G), T) \longrightarrow P_nH^2(G, T) \longrightarrow 0$.

Let $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$ be a free presentation of G. Set $\bar{R} = R/[R, F], \bar{F} = F/[R, F]$,and$\{w(g)\}_{g\in G}$ be a set of representatives of G in $\bar{F}$ with $W : G \times G \longrightarrow \bar{R}$ the corresponding 2−cocycle. For $\alpha \in Hom(\bar{R}, T)$,let $\theta(\alpha)$ be the element of $H^2(G, T)$ determined by the 2−cocycle $\alpha W : G \times G \longrightarrow T$. The map θ is an epimorphism : $Hom(\bar{R}, T) \longrightarrow H^2(G, T)$.The subgroup $P_n(G, T)$ is then characterized as follows

Theorem 3.1.[40] $P_nH^2(G, T) = \{\theta(\alpha)|\alpha \in Hom(\bar{R}, T), \alpha|_{\bar{R}\cap(1+\Delta^{n+2}(\bar{F})+\Delta(\bar{F})\Delta(\bar{R}))=0.}\}$

Using this characterization they prove

Theorem 3.2. $P_nH^2(G, T) = H^2(G, T)$ if and only if $\bar{R} \cap (1 + \Delta^{n+2}(\bar{F}) + \Delta(\bar{F})\Delta(\bar{R})) = 1$.

A group G is called parafree if G is residually nilpotent and G has the same lower cental sequence as some free froup.

Parafree groups are then characterized by

Theorem 3.3. A residually nilpotent group G is parafree if and only if
(i) $G/\gamma_2(G)$ is free Abelian and (ii) $P_nH^2(G, T) = 0$ for all $n \geqq 0$.

Continuing with the relationship between dimension subgroups and filtration of the Schur multiplicator,

Passi and Vermiani [45] first give the following necessary and sufficient condition for an element cooresponding to a central extension to be of degree $\leq n$.

Proposition 3.4. Let A be a divisible Abelian group regarded as a trivial $G-$ module, $\xi \in H^2(G, A)$ and $0 \longrightarrow A \longrightarrow \Pi \longrightarrow G \longrightarrow 1$ be a central extension corresponding to ξ. Then $\xi \in P_nH^2(G, A)$ if and only if $A \cap (1 + \Delta^{n+2}(\Pi) + \Delta(\Pi)\Delta(A)) = (0)$.

If the group G is a torsion group, then the above result becomes (with the notations as in Theorem 3.4)

Proposition 3.5. $\xi \in P_n H^2(G, A)$ if and only if $A \cap D_{n+2}(\Pi) = (0)$.

Passi and Vermani then prove

Theorem 3.6. The following statements are equivalent

(i) For every nilpotent group G, there is an integer $n \geq 1$ such that $D_n(G) = 1$.

(ii) For every nilpotent group G, there exists an integer $n \geq 1$ such that $P_n H^2(G, T) = H^2(G, T)$.

Write $\gamma_\omega(G) = \cap_n \gamma_n(G)$ and $D_\omega(G) = \cap_n D_n(G)$.

Theorem 3.7. The following statements are equivalent.

(i) For every group $G, D_\omega(G) = \gamma_\omega(G)$.

(ii) For every nilpotent $p-$ group G without elements of infinite $p-$ height, $\cup_n P_n H^2(G, T_p) = H^2(G, T_p)$.

If $\zeta(G)$ denotes the center of G, let $I_n(G) = G \cap (1 + \Delta^n(G) + \Delta(G)\Delta(\zeta(G)))$ for $n \geqq 1$. Then $I_n(G)$ is called the *nth* generalized dimension subgroup of G. Let $I_\omega(G) = \cap_n I_n(G)$. We have

Theorem 3.8. The following statements are equivalent.

(i) For every group $G, I_\omega(G) = \gamma_\omega(G)$.

(ii) For every nilpotent group G and prime p, $\cup_n P_n H^2(G, T_p) = H^2(G, T_p)$.

As some positive results, Passi and Vermani [45] prove

Theorem 3.9. Let the group G be nilpotent which is either finitely generated or torsionfree. Then there exists an integer $n \geqq 1$ such that $P_n H^2(G, T) = H^2(G, T)$.

Theorem 3.10. If G is a nilpotent $p-$group of class $n < p$, then $P_n H^2(G, T) = H^2(G.T)$.

Sjogren [49] has given constants $c_1, c_2, ..., c_{n,}, ...$ such that for every group $G, D_n^{c_n}(G) \leq \gamma_n(G)$ for all $n \geqq 1$. This result is equivalent to a result about $\{P_n H^2(G, T)\}$. Explicitly

Theorem 3.11[45].The following statements are equivalent:

(i) There exist constants $c_1, c_2, ..., c_n, ...$such that for every group $G, D_n^{c_n}(G) \leq \gamma_n(G)$ for all $n \geqq 1$.

(ii) There exist constants $d_1, d_2, ..., d_n, ...$such that , for every nilpotent group G of class $\leq n, d_n H^2(G, T) \leq P_n H^2(G, T)$.

When G is a torsionfree nilpotent group of class c, then the n in Theorem 3.9 is $c^2 + 4c + 1$. Improving upon the argument of Passi- Vermani, Passi and Sucheta[41] prove that this $n = 3c + 1$. Passi and Sucheta also prove that if G is nilpotent of class c such that the torsion subgroup of G is central, then $D_{2c+1}(G) = 1$.

Looking for classes of groups for which the dimension property holds, Passi and Sucheta[41] first prove

Theorem 3.12. If G is a divisible nilpotent group of class c, then $P_cH^2(G,T) = H^2(G,T)$.

Using this result, they prove

Theorem 3.13. If G is a divisible group, then $I_n(G) = \gamma_n(G)$ for all $n \geqq 1$.

Observe that this result in particular implies that dimension property holds for divisible groups.

In connection with Theorem 2.3(c) and (d), M. Goyal [11] proved

Theorem 3.14. If G is finite nilpotent of class 2 with $\gamma_2(G)$ cyclic, then $P_2H^2(G,T) = H^2(G,T)$.

while Passi, Sucheta and Tahara [42] prove

Proposition 3,15.. If the group G is finite $2-$group of class 2 with $G/\gamma_2(G)$ direct sum of three cyclic groups, then $P_2H^2(G,T) = H^2(G,T)$.

Regarding dimension subgroupps, they give an alternative proof of the following result of Tahara.

Proposition 3.16. [51] If G is a finite $2-$group of class 3 with $G/\gamma_2(G)$ direct sum of three cyclic groups, then $D_4(G) = 1$.

For a subgroup H of G, let $\sqrt{H} = \{x \in G | x^m \in H$ for some integer $m \geqq 1\}$

Working with the coefficients in the field Q, Passi, Sucheta and Tahara [42] prove

Theorem 3.16. For any group $G, I_{n,Q}(G) = \sqrt{\gamma_n(G)}$ for all $n \geqq 1$.

As a consequence of this, it is deduced

Corollary 3.17. If G is torsionfree nilpotent of class c, then $P_cH^2(G,Q) = H^2(G,Q)$.

Corollary 3.18. If G is nilpotent of class c such that the second integral homology group H_2G is torsionfree, then $P_cH^2(G,T) = H^2(G,T)$.

(b) **Varietal multiplicator**

Let $\mathbf{V}$ be a variety, G a group in $\mathbf{V}$ and A an Abelian group regarded as a trivial $G-$module. If $q : F \longrightarrow G$ is a $\mathbf{V}-$free presentation of G, then the varietal multiplicator of G with coefficients in A is $\widetilde{V}(G,A)$ = kernel of the inflation map $q^* = \inf : H^2(G,A) \longrightarrow H^2(F,A)$ (cf. U. Stammbach, Homology in Group Theory, Lecture Notes in Math. no. 359, Springer Verlag, 1973). Define $P_n\widetilde{V}(G,A) = \widetilde{V}(G,A) \cap P_nH^2(G,A)$. This leads to a filtration of the varietal multiplcator $\widetilde{V}(G,A)$. Then $P_n\widetilde{V}(G,A) = \ker(\ q^* : P_nH^2(G,A) \longrightarrow P_nH^2(F,A))$. Passi, Sharma and Vermani [39] prove that if $\mathbf{Ab}$ denotes the category of Abelian groups, then $P_n\widetilde{V} : \mathbf{V} \times \mathbf{Ab} \longrightarrow \mathbf{Ab}$ is a bifunctor, cntravariant in the first variable and covariant in the second variable. Then they prove that $P_n\widetilde{V}(G,A) \leqq Im(\inf : \widetilde{V}(G/\gamma_{n+1}(G),A) \longrightarrow \widetilde{V}(G,A))$.

Generalizing an earlier result of Passi, they prove

$P_1\widetilde{V}(G,T) = Im(\inf : \widetilde{V}(G/\gamma_2(G) \longrightarrow \widetilde{V}(G,T)$ for any $G \in \mathbf{V}$.

If $\mathbf{V} = \mathbf{N}_m$ is the variety of nilpotent groups of class $\leqq m$, then for any $G \in \mathbf{V}, \inf : \widetilde{V}(G/\gamma_m(G),T) \longrightarrow \widetilde{V}(G,T)$ is an epimorphism.

Let $1 \longrightarrow N \longrightarrow G \longrightarrow K \longrightarrow 1$ be an exact srquence with G in $\mathbf{V}$. Let $E_n(N)$ be the subgroup of G given by $E_n(N)/[G,N] = G/[G,N] \cap (1 + \Delta^n(G/[G,N]) + \Delta(G/[G,N])\Delta(N/[G,N]))$. Then $\{E_n(N)\}$ is a decreasing sequence of normal subgroups G. Corresponding to the $5-$term exact sequence of Hochschild and Serre, we have

Theorem 3. 19. For every $n \geqq 0$, there exists a natural exact sequence $0 \longrightarrow Hom(K/\gamma_n(K),T) \longrightarrow Hom(G/\gamma_n(G),T) \longrightarrow Hom(N/N \cap E_{n+2}(N),T) \longrightarrow P_n\widetilde{V}(K,T) \stackrel{\inf}{\longrightarrow} P_n\widetilde{V}(G,T)$.

Using the above natural exact sequence, they prove

Theorem 3.20. Let $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$ be a $\mathbf{V}-$free presentation of $G \in \mathbf{V}$ and n a positive integer. Then $P_n\widetilde{V}(G,T) \cong Hom(R \cap \gamma_n(F)/R \cap E_{n+2}(R),T)$.

Cosequently, we have

Proposition 3. 21. (i) For any positive integer n, $P_n\widetilde{V}(G,T) = \widetilde{V}(G,T)$ if and only if $R \cap E_{n+2}(R) = [F,R]$. **Proposition 3.22.** If F is a $\mathbf{V}-$free group, then $P_n\widetilde{V}(F/\gamma_i(F),T) = 0$ for $n \leq i-2$ and $P_{i-1}\widetilde{V}(F/\gamma_i(F),T) \cong Hom(\gamma_i(F)/\gamma_i(F) \cap D_{i+1}(F),T)$..

Proposition 3.23. (i) If F is a $\mathbf{V}-$free group, then every element of $\widetilde{V}(F/\gamma_{n+1}(F),T)$ is of degree precisely n if and only if $\gamma_{n+2}(F) = \gamma_{n+1}(F) \cap D_{n+2}(F)$.

(ii) If $\mathbf{V}$ is a variety free groups have the dimension property, then every element of $\widetilde{V}(F/\gamma_{n+1}(F),T)$ is of degree n.

The authors prove the following interesting result

Proposition 3.24. If $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$ is a $\mathbf{V}-$free presentation of a group G in $\mathbf{V}$, then $R\cap\gamma_n(F)/R\cap\gamma_n(F)[F,R]$ is a $\mathbf{V}-$presentation invariant of G.

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The authors then extend the exact sequence of Theorem 3.19 by one more term and prove

Theorem 3.25. If the group G is in $\mathbf{V}$,N is a normal subgroup of G, and $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1, 1 \longrightarrow S \longrightarrow F \longrightarrow G/N \longrightarrow 1$ are $\mathbf{V}-$presentations of $G, G/N$ respectively, then for every $n \geqq 0$ there exists a natural exact sequence $0 \longrightarrow Hom(G/\gamma_2(G)N,T) \longrightarrow Hom(G/\gamma_2(G),T) \longrightarrow Hom(N/N \cap E_{n+2}(N),T) \stackrel{t}{\longrightarrow} P_n\widetilde{V}(G/N,T) \stackrel{\inf}{\longrightarrow} P_n\widetilde{V}(G,T) \stackrel{\theta}{\longrightarrow}$
$Hom(R \cap E_{n+2}(S)/R \cap E_{n+2}(R),T) \longrightarrow 0$.

A group G is called $V-$parafree if G is residually nilpotent and there exists a $\mathbf{V}-$free group F and a homomorphism $f : F \longrightarrow G$ which induces an isomorphism $F/\gamma_n(F) \longrightarrow G/\gamma_n(G)$ for every natural number n.

Passi, Sharma and Vermani then give a characterization of $V-$ parafree groups in terms of $P_n\widetilde{V}$ for the variety V in which the free groups have the dimension property. Explicitly

Theorem 3.27. Let $\mathbf{V}$ be a variety such that $\mathbf{V}-$free groups satisfy the dimension property. If $G \in \mathbf{V}$ is

(a) residually nilpotent,

(b) $G/\gamma_2(G)$ is free in $\mathbf{V} \cap \mathbf{Ab}$,and

(c) $P_n\widetilde{V}(G, T) = 0$ for all $n \geqq 1$,

then G is $\mathbf{V}-$parafree.

Observe that free groups have the dimension property in (i) the variety of all groups, (ii) the variety in which the free groups have torsion-free lower central sequences.

If $\mathbf{V}$ is a variety and $G \in \mathbf{V}$ is $\mathbf{V}-$parafree, is it true that $P_n\widetilde{V}(G, t) = 0$ for all $n \geqq 1$?

Finally, it is proved

Theorem 3.28. Let $\mathbf{V}$ be a variety, G a finitely generated group in $\mathbf{V}$ and p a prime. Then $P_n\widetilde{V}(G, Z_p) = 0$ for all $n \geqq 1$ if and only if there exists a free group F in $\mathbf{V}$ and a homomorphism $f : F \longrightarrow G$ which induces isomorphisms $f_n : F/(\gamma_n(F))^{(p)} \longrightarrow G/(\gamma_n(G))^{(p)}$ for all $n \geqq 1$.

4. Relation modules

Let $1 \longrightarrow R \longrightarrow F \stackrel{\alpha}{\longrightarrow} G \longrightarrow 1$ be a non-cyclic free presentation of G. Let $n \geqq 1$. For $a \in \gamma_n(R), g \in G$,define $a\gamma_{n+1}(R).g = x^{-1}ax\gamma_{n+1}(R)$, where $g = \alpha(x)$. This is a well defined action of G on $\gamma_n(R)/\gamma_{n+1}(R)$ and makes it a right $ZG-$module. For $n = 1$, this module is called a relation module while for $n \geqq 2$, these modules are called higher relation modules. Recall that if S is a ring with identity, a right $S-$module M is called faithful if $\mathrm{Ann}M = \{s \in S | a.s = 0$ for all $a \in M\} = 0$. Using free differential calculus, Mital and Passi [27] prove that if $R/\gamma_2(R)$ is a faithful $ZG-$module, then so are $\gamma_n(R)/\gamma_{n+1}(R)$ for all $n \geqq 2$. In [33] Passi prved

Theorem 4.1. $R/\gamma_2(R)$ and, hence, $\gamma_n(R)/\gamma_{n+1}(R)$ for all $n \geqq 1$ are faithful $ZG-$modules.

A generalization of this result of Passi namely: If R, S are normal subgroups of a noncyclic free group F, then when is the $Z(F/R)-$ module $R \cap S/[R, S]$ faithful has been considered by Passi and Mikhailov but we do not consider this here.

Let G be a group, R be a commutative ring with identity, $U(R)$ the multiplicative group of invertible elements (units) of R and $t : G \times G \longrightarrow U(R)$ be a twisting fuction i.e. a funtion which satisfies $t(x, yz)t(y, z) = t(x, y)t(xy, z), x, y, z \in G$.

Observe that the twisting fuction is just like a 2−cocycle (wth trivial $G-$action on $U(R)$.

Let R^tG be the twisted group ring of G over R with the twisting function t. observe that R^tG is a free $R-$ module freely generated by the set $\bar{G} = \{\bar{x}|x \in G\}$ with multiplication defined by $\bar{x}.\bar{y} = t(x,y)\bar{\bar{xy}}$, $x, y \in G$ and linearity.

An ideal A of R^tG is said to be controlled by a normal subgroup H of G if $A = (A \cap R^tH)R^tG$. For a subset S of R^tG, let $l(S) = \{\alpha \in R^tG|\alpha S = o\}$ which is a left ideal of R^tG. A two sided ideal A of R^tG is called an annihilator ideal if $A = l(S)$ for some non-empty subset S of R^tG. Let $\delta^+(G)$ be the subgroup of G generated by those torsion elemnts of G which have only finitely many conjugates. Arora and Passi [1] proved

Theorem 4.2. If the order of every element of $\delta^+(G)$ is regular in R, then the annihilator ideals of R^tG are controlled by $\delta^+(G)$.

This leads, in particular, to the follwing

Corollary 4.3. If RG is a semiprime group ring, then the annihilator ideals of RG are cotrolled by $\delta^+(G)$.

The above result is a generalization of a result of Smith [49] who proved that if k is a field such that kG is semiprime, then the annihilator ideals of kG are controlled by $\delta^+(G)$.

5. Polynomial groups, dimension subgroups mod n and modular group algebras

Let R be a commutative ring with identity. Let $\Delta_R(G)$ denote the augmentation ideal of the group ring RG, $Q_{n,R}(G) = \Delta^n_R(G)/\Delta^{n+1}_R(G)$ and the Brauer-Jennings-Zassenhaus series $\{M_i(G)\}$ defined by $M_1(G) = G$, $M_i(G) = [G, M_{i-1}(G)]M_{(i/p)}(G)^p$ for $i \geqq 2$, where (i/p) is the largest integer $\geqslant i/p$. If N is a normal subgroup of G, let $\Delta_R(G,N)$ denote the kernel of the homomorphism $RG \longrightarrow RG/N$ which is induced by the natural projection $G \longrightarrow G/N$.

Passi and Sehgal [37] prove

Theorem 5.1. If G and H are two groups with $Z_pG \cong Z_pH$ for some prime p, then the quotients of the $M-$series with respect to the prime p satisfy

(i) $M_i(G)/M_{i+1}(G) \cong M_i(H)/M_{i+1}(H)$ and (ii) $M_i(G)/M_{i+2}(G) \cong M_i(H)/M_{i+2}(H)$ for all $i \geqq 1$.

They deduce

Proposition 5.2. If $M_3(G) = 1$, then $Z_pG \cong Z_pH$ implies that $G \cong H$.

They then prove

Theorem 5.3. If division by 2 is uniquely defined in $Q_{2,R}(G/\gamma_2(G))$, then the exact sequence

$0 \longrightarrow \Delta_R(G,\gamma_2(G)) + \Delta^3_R(G)/\Delta^3_R(G) \longrightarrow Q_{2,R}(G) \xrightarrow{\alpha} Q_{2,R}(G/\gamma_2(G)) \longrightarrow 0$,

where α is the homomorphism induced by the natural epimorphism $G \longrightarrow G/\gamma_2(G)$, splits.

When $R = Z_n$ and n is an odd integer or 0, then the above exact sequence takes the form

$$0 \longrightarrow \gamma_2(G)/(\gamma_2(G) \cap D_3(G, Z_n)) \longrightarrow Q_{2,Z_n}(G) \xrightarrow{\alpha} Q_{2,Z_n}(G/\gamma_2(G)) \longrightarrow 0. \quad (5.1)$$

Finally, Passi and Sehgal prove

Theorem 5.4. If n is an odd integer or G is a group of odd exponent and $Z_nG \equiv Z_nH$ for some group H. then $G/D_3(G, Z_n) \equiv H/D_3(H, Z_n)$.

Corollary 5.5. If G and H are $p-$groups of exponent $p^n, p \neq 2$ and class 2 with $Z_{p^n}G \cong Z_{p^n}H$, then $G \cong H$.

Passi and Sharma [28] consider the above results for n even and modify the exact sequence (5.1) to include the case of n even and prove

Theorem 5.6. If G is a finitely generated group and n is an integer $\geqq 0$, the the sequence

$$0 \longrightarrow \gamma_2(G)/(\gamma_2(G) \cap G^n D_3(G, Z_n)) \xrightarrow{i} \Delta^2_{Z_n}(G)/\Delta^3_{Z_n}(G) + \Delta_{Z_n}(G, G^n) \xrightarrow{j} Q_{2,Z_n}(G/G^n\gamma_2(G)) \longrightarrow 0$$

where i is induced by $x \longrightarrow x - 1 + \Delta^3_{Z_n}(G) + \Delta_{Z_n}(G, G^n), x \in G$ and j is induced by the natural projection $G \longrightarrow G/G^n\gamma_2(G)$, is a split exact sequence.

Corresponding to Theorem 5.4 and Corollary 5.5, they prove

Corollary 5.7. (i) If $G^n \leq D_3(G, Z_n), G$ finitely generated, then $G/D_3(G, Z_n)$ is an isomorphism invariant for such groups.

(ii) If G and H are finite $2-$groups of exponent 2^n and class 2 with $Z_{2^n}G \cong Z_{2^n}H$, then $G \cong H$.

Let $n \geqq 0$ and if n is even, let $n = 2qm$ where q is a power of 2 and m is odd. Let

$$K_n(G) = \left\{ \begin{array}{c} G^n\gamma_3(G) \quad \text{for } n \text{ odd or } n = 0 \\ G^m\gamma_3(G) \cap \left\langle x^{2q} | x^q \in G^{2q}\gamma_2(G) \right\rangle \gamma_3(G) \ \ \text{for } n \text{ even,} \end{array} \right\}$$

and $N/K_n(G)$ =the subgrop of the centre of $G/K_n(G)$ consisting of elements of order dividing n.For n odd or $n = 0, N/K_n(G)$ is the centre of $G/K_n(G)$.

Passi and Sharma prove

Proposition 5.8. (i) $G \cap \{1 + \Delta^3_{Z_n}(G) + \Delta_{Z_n}(G)\Delta_{Z_n}(N)\} = K_n(G)$ for n odd or $n = 0$.

(ii) $G \cap \{1 + \Delta^3_{Z_n}(G) + \Delta_{Z_n}(G)\Delta_{Z_n}(N)\} = K_n(G) \langle x^n | x^{qm} \in N \rangle$ for n even.

(iii) $G \cap \{1 + \Delta^3_{Z_n} G)] = K_n(G)$ for all n.

6. Polynomial ideals in group rings and Lie solvable group rings

Parmentor, Passi and Sehgal [28] considered polynomial ideals which are a generalization of polynomial maps. Let $f(x_1, x_2, ..., x_n)$ be a polynomial in n non-commutating variables $x_1, x_2, ..., x_n$ and their inverses with coefficients in the ring Z of integers. Let R be a commutative ring with identity 1_R. If $g_1, g_2, ..., g_n$ are elements of the group G, then $f(g_1, g_2, ..., g_n)$ can be regarded as an element of the group ring RG. The $2-$sided ideal of RG generated by $f(g_1, g_2, ..., g_n)$, $g_1, g_2, ..., g_n \in G$ denoted by $A_{f.,R}$ is called a polynomial ideal given by the polynomial f. For ideals A, B of RG, let $[A, B]$

be the $R-$submodule of RG spanned by the Lie products $[a,b] = ab - ba$. $a \in A, b \in B$. For any A, the Lie powers $A^{(i)}$ are defined by $A^{(1)} = A$, and for $i \geqq 2, A^{(i)} = [A, A^{(i-1)}]RG$. Ideals $\Delta_R^n(G), \Delta_R^{(n)}(G)$ and $\Delta_R(G, \gamma_n(G))$ are polynomial ideals. Let $i_R : ZG \longrightarrow RG$ be the homomorphism induced by the map $n \longrightarrow n1_R, n \in Z$. Generalizing the concept of polynomial maps, they define what are called f_R-polynomial maps. Explicitely, if f is a polynomial in n non- commuting variables $x_1, ..., x_n$ and their inverses with coefficients in Z, then a map $\theta : G \longrightarrow M, M$ an $R-$module, is called an f_R-polynomial map if the linear extension θ^* of θ to RG vanishes on the polynomial ideal $A_{f,,R}$ determined by f.

Parmentor, Passi and Sehgal [28] study the elements of ZG which are mapped into $A_{f,R}$, both when the characteristic of R is 0 and > 0. They then obtain dimension subgroups $D_{n,R}(G)$ and Lie dimension subgroups $D_{(n),R}(G)$ in terms of $D_{n,Z}(G), D_{n,Z?p^rZ}(G)$ and $D_{(n),Z}(G), D_{(n),Z//p^rZ}(G)$ respectively. These results for charecteristic of R non-zero being particlarly simple, we have

Theorem 6.1. If charecteristic of R is $r > 0$, then
(i) $D_{n,R}(G) = D_{n,Z_r}(G)$ for all $n \geqq 1$;
(ii) $D_{(n),R}(G) = D_{(n),Z_r}(G)$ for all $n \geqq 1$.

They then consider Lie powers of the augmentation ideal and prove

Theorem 6.2. $\Delta_R^{(n)}(G) = 0$ for some $n > 2$ and $\Delta_R^{(2)}(G) \neq 0$ if and only if G is nilpotent, $\gamma_2(G) \neq 1$ is a finite $p-$group and p is nilpotent in R.

Theorem 6.3. When the characteristic of R is is a power of p, p prime, then $\cap_n \Delta_R^{(n)}(G) = 0$ if and only if G is a residually nilpotent group with $\gamma_2(G)$ a $p-$group of finite exponent.

They also give necessary and sufficient conditions for $\cap_n \Delta_R^{(n)}(G) = 0$ in the case when characteristic of R is 09. The result of Theorem 6.3 for finite G is due to Sandling [47(a)]

Passi, Passman and Segal [36] consider group rings KG where K is a field and obtain necessary and sufficient condions for KG to be Lie nilpotent and to be Lie solvable.

When p is a prime > 0, a group A is called $p-$Abelian if $\gamma_2(A)$ is a finite $p-$group. Let R be a $K-$algebra. Lie central and Lie derived series of R are defined inductively by

$\gamma^0 R = R, \quad \gamma^{n+1} R = [R, \gamma^n R]$ (central series) and $\delta^0 R = R, \quad \delta^{n+1} R = [\delta^n R, \delta^n R]$ (derived series). (The brackets are Lie brackets/products.)

The $K-$algebra R is called Lie nilpotent if $\gamma^n R = 0$ for some integer n and R is Lie solvable if $\delta^n R = 0$ for some integer n.Then Passi, Passman and Segal obtain the following result about Lie nilpotence and Lie solvability of KG.

Theorem 6.4. Let the characteristic of K be $p \geqq 0$. Then
(i) KG is Lie nilpotent if and only if G is $p-$Abelian and nilpotent;
(ii) for $p \neq 2, KG$ is Lie solvable if and only if G is $p-$Abelian;
(iii) for $p = 2, KG$ is Lie solvable if and only if G has a $2-$Abelian subgroup of index at most 2.

7. Fox subgroups of free groups

Let F be a noncyclic free group and R a normal subgroup of F. For $n \geqq 0$, the subset $F(n, R) = F \cap (1 + \Delta^n(F)\Delta(R))$ is a normal subgroup of F and is called the *nth* Fox subgroup of F relative to R. Identification of the Fox subgroups is a problem that dates back to Fox [9]. Observe that for $R = F, F(n, R)$ is just the dimension subgroup $D_{n+1}(F) = \gamma_{n+1}(F)$ and it is clear that $F[1, R] = \gamma_2(R)$. The results for $n = 2$ and for $R = \gamma_2(F), n \geqq 1$ are due to Enright (cf. [14]) and Gupta and Gupta (cf.[14]) respectively. Gupta and Passi ([14], [17]) investigate the quotients $F/F(n, R)$ and $\overline{F(n, R)} = F(n, R)/F(n + 1, R)$. The quotients $\overline{F(n, R)}$ are free Abelian groups and can be regarded as right $F/R-$ modules via conjugation in F. Passi and Gupta prove that are faithful. The case $n = 0$ of this result which is due to Passi and has already been discussed. Let $\zeta_k(G)$ denote the *kth* term of the upper central series of G. Passi and Gupta[16] prove

Theorem 7.1.(i) $\zeta_{n+1}(F/F(n, R)) = \zeta_n(F/F(n, R))$ for all $n \geqq 1$ and

(ii) $\zeta_1(F/F(n, R))$ is nontrivial if and only if F/R is finite.

(iii) If $R \leq \gamma_c(F)$ but $R \nleqq \gamma_{c+1}(F)$, then $\zeta_1 F/F(n, R)) \leq \zeta_1(R/F(n, R)) = F(n - c, R)/F(n, R)$ for all $n \geqq c$.

Regarding the residual proerties of $F/F[n, R]$, they prove

Theorem. 7.2. If F/R is residually torsion free nilpotent, then so is $F/F[n, R]$. Conversely, if $F/[n, R]$ is residually nilpotent, then $\cap_k \Delta^k(F/R) = 0$.

In [15] Gupta and Passi cosider the structure of $U_k = \overline{F(k, R)} \otimes Q$ as a right $Q(F/R)-$module, where Q is the field of all rational numbers.Explicitely, they prove (using basic commotators and Witt's formula)

Theorem. 7.3. $U_k = Q \oplus ... \oplus Q \oplus \Delta(F/R) \oplus ... \oplus \Delta(F/R)$ with ρ_{k+1} copies

of Q and $m^k(m - 1)$ copies of $\Delta(F/R)$,

where m is the rank of F and ρ_{k+1} is the rank of $\gamma_{k+1}(F)/\gamma_{k+2}(F)$.

Among applications of this theorem, they prove

Theorem 7.4. (i)The centre $\zeta(F/F(k + 1, R))$ of $F/F(k + 1, R)$ is the isolator of $\gamma_{k+1}(C)F(k + 1, R)/F(k + 1, R)$, where $C/\gamma_2(R)$ is the center of $F/\gamma_2(R)$.

(ii) If F/R is finite, then $\zeta_2(F/F(k+1, R)) = \zeta(F/F(k+1, R))$ for all $k \geqq 0$.

8. Sugroups of G determined by ideals of ZG of the form $\Delta^n(G) + \Delta(G)\Delta(H)$

There has been considerable interest in determining generalized dimension subgroups $I_n(G) = G \cap (1 + \Delta^n(G) + \Delta(G)\Delta(\zeta(G))$ and subgroups of G determined by ideals of the form $\Delta^n(G) + \Delta(G)\Delta(H)$ of ZG for some suitable subgroups H of G. For example (among many other researchers) Sandling [47] proved

Theorem 8.1. (a) Abelian- by- cyclic groups have dimension property.

(b) A split extension of an Abelian group by a group with dimension property itself has dimension property.

(c) $I_3(G) = \gamma_3(G)$.

Let $1 \longrightarrow A \stackrel{i}{\longrightarrow} \Pi \longrightarrow G \longrightarrow 1$ be an exact sequence with A Abelian and i the inclusion map. Let M be an Abelian group and $f : A \longrightarrow M$ a homomorphism. Passi [34] obtains necessary and sufficient conditions for extending f to a polynomial map $\phi : \Pi \longrightarrow M$ the linear extension of which to $Z\Pi$ vanishes on $\Delta^{n+1}(\Pi) + \Delta(\Pi)\Delta(A)$. As an application of this result he proves

Theorem 8.2. If A is an Abelian normal Abelian subgroup of G with G/A cyclic, then $G \cap (1 + \Delta^n(G) + \Delta(G)\Delta(A)) = \gamma_n(G)$.

An immediate consequence of this is Theorem 8.1(a) of Sandling.

Theorem 8.3. If A, H are subgroups of G with H normal such that $G = AH$ and $A \cap H = 1$, then $G \cap (1 + \Delta^n(G) + \Delta(G)\Delta(A)) = D_n(H)[A, (n-1)G]$, where $[A, kG] = [...[[A, G], G], ..., G], G$ repeated k times.

This result as a corollary gives Theorem 8.1(b) of Sandling.

Passi also gives an alternative proof of Theorem 8.1(c) of Sandling. As another application of his technical result on the extension of a homomorphism to a polynomial map, he proves

Proposition 8.4. If G is a $p-$group of odd order, then $I_4(G) = \gamma_4(G)$.

Continuing search for classes of groups which have the dimension property, Guta, Hales and Passi [16] first prove

Theorem 8.5. If F is a free group of finite rank and S is a normal subgroup of F containing $\gamma_2(F)$, then there exists a natural number $n_0(F/S)$ depending only on F/S such that for all $n \geqq n_0(F/S)$, $F \cap (1 + \Delta^{n+1}(F) + \Delta(F)\Delta(S)) = \gamma_{n+1}(F)\gamma_2(S)$.

Then they deduce

Theorem 8.6. Let H be a normal Abelian subgroup of a finitely generated metabelian group G with G/H Abelian. Then there exists a natural number $n_0(G/H)$ depending only on G/H such that for all $n \geqq n_0(G/H)$, $G \cap (1 + \Delta^{n+1}(G) + \Delta(G)\Delta(H)) = \gamma_{n+1}(G)$.

In case the finitely generated metabelian group G is such that $G/\gamma_2(G) \cong C_{p^k}^{(r)}$ for some prime p and integers $r, k \geqq 1$, they prove that the natural number $n_0(G/\gamma_2(G)) = p^k + p^{k-1}$. Finally they prove

Proposition. 8.**7**. If G is a finitely generated metabelian group such that $G/\gamma_2(G)$ is elemetary Abelian, then $D_n(G) = \gamma_n(G)$ for all $n \geqq 1$.

For a treatment of the subgroups of G determined by some ideals of the form $\Delta^n(G) + \Delta(H_1)...\Delta(H_k)$, for some normal subgroups H_i of G upto the end of 1998 refer to [54]. As a particular example, Vermani [55] proved

Theorem 8.8. For any normal subgroup H of G, $G \cap (1 + \Delta(H)\Delta(G)\Delta(H)) = \gamma_3(H)$.

9. Residual nilpotence of augmentation ideal and residual solvability of units group

Let P a property. Recall that a group G is said to be residually P if for every $x \in G, x \neq 1$, there exists a normal subgroup H_x of G such that $x \notin H_x$ and the group G/H_x has the property P. Augmentation ideal $\Delta(G)$ of G is said to be residually nilpotent if $\Delta^{\omega}(G) = \cap_{m=1}^{\infty}\Delta^m(G) = 0$. Lichtman [22] proved

Theorem 9.1. For a group $G, \Delta^{\omega}(G) = 0$ if and only if either (a) G is residually torsion-free nilpotent or (b) G is discriminated by the class of nilpotent groups of finite prime power exponenets.

He, then, also proved

Theorem 9.2. If F is a noncyclic free group with a normal subgroup R, then $\Delta^{\omega}(F/R) = 0$ if and only if $\Delta^{\omega}(F/\gamma_2(R)) = 0$.

The above result of Lichtman was improved upon by Hartley [21] who proved

Theorem 9.3. With F, R as in Theorem 9.2, let $V(R)$ be a fully invariant subgroup of R with $R/V(R)$ torsion- free nilpotent. Then $\Delta^{\omega}(F/R) = 0$ if and only if $\Delta^{\omega}(F/V(R)) = 0$.

Let F be a noncyclis free group and R be a normal subgroup of F. Let $\acute{r} = ZF\Delta(R)$ i.e. the ideal of ZF generated by $\Delta(R)$. Define $\acute{r}^q, q \in \{0,1\}$ to be $\acute{r}$ if $q = 1$ and $\Delta(F)$ if $q = 0$. Let $(q_1, ..., q_n)$ be an $n-$ tuple with every $q_i \in \{0,1\}$, and let $M(q_1, ..., q_n) = F \cap (1 + \acute{r}^{q_1} ... \acute{r}^{q_n})$. $M(q_1, ..., q_n)$ is said to be minimally expressed if $M(q_1, ..., q_n)$ is a proper subgroup of both $M(q_{1,} ..., q_{n-1})$ and $M(q_2, ..., q_n)$.

Generalizing one side of the result of Lichtman which is different from that of Hartley, Gupta and Passi [15] prove (with the notations as above)

Theorem 9.4. If $\Delta^{\omega}(F/R) = 0$, then $\Delta^{\omega}(F/M(q_1, ..., q_n)) = 0$ for any $n-$tuple $(q_1, ..., q_n) \neq (0, ..., 0)$.

They also prove

Theorem 9.5. If F/R is periodic and the group $F/M(q_1, ..., q_n)$ is residually nipotent wih $n \geqq 2$, then $\Delta^{\omega}(F/R) = 0$ provided $(q_1, q_n) \neq (0, 0)$.

In the reverse direction they prove

Theorem 9.6. If $\delta^+(F/R) = 1$ and $M(q_1, ..., q_n)$ is minimally expressed, then the residual nilpotence of $F/M(q_1, ..., q_n)$ implies that $\Delta^{\omega}(F/R) = 0$ provided $(q_1, q_n) \neq (0, 0)$.

As a corollary it follows

Proposition 9.7. If F/R is torsion-free and $M(q_1, ..., q_n)$ is minimally expressed, then the residual nilpotence of $F/M(q_1, ..., q_n)$ implies that $\Delta^{\omega}(F/R) = 0$ provided $(q_1, q_n) \neq (0, 0)$.

Goncalves [10] proved that for G finite, the group $U(\mathbb{Q}G)$ of units of the group algebra $\mathbb{Q}G, \mathbb{Q}$ the field of rational numbers, is residually nilpotent if and only if it is nilpotent and that $U(\mathbb{Q}G)$ is solvable if and only if G is Abelian. Bhandari and Passi consider and charcterize the residual solvability of $U(KG)$, when K is either (i) a $P-$adic field of charecteristic 0 or (ii) an algebraic number field. Explicitely they ([6], [7]) prove

Theorem 9.8. When G is a finite group, then $U^{(\omega)}(\mathbb{Q}G)$ is the identity group or an elementary Abelian $2-$group if and only if either (i) G is Abelian or (ii) G is a Hamiltonian group of order $2^n m, m$ odd, such that the multiplicative order of 2 modulo m is odd.

They [7] also prove

Theorem 9.9. The unit group $U(\mathbb{R}G), \mathbb{R}$ the field of real numbers, is residually solvable if and only if G is Abelian.

10. Algebraic elements in group rings

Definition. Two torsion units u and v of ZG, G a finite group are rationally conjugate if in each irreducible representation the matrices of u, v have the same charcteristic polynomials.

Let C be a conjugacy class of G. For an element $\alpha = \sum \alpha(g)g$ in KG, K a field, define its partial augmentation $\varepsilon_C(\alpha)$ over C by $\varepsilon_C(\alpha) = \sum_{g \in C} \alpha(g)$. This leads for a fixed conjugacy class C to a map $\varepsilon_C : KG \longrightarrow K$ which is $K-$linear. Since for $g, h \in G, gh$ is a conjugate of hg, we have $\varepsilon_C(\alpha\beta) = \varepsilon_C(\beta\alpha)$ for all $\alpha, \beta \in KG$. The map ε_C is thus a trace map. The partial augmentation $\varepsilon_C(\alpha)$ is also called the $C-$trace of α. Two torsion units u and v of $ZG$are said to be rationally conjugate if and only if in each irreducible representation the matrices of u, v have the same characteristic polynomial.

tLuthar and Passi [25] prove

Theorem 10.1. Let the group G be finite. Let u and v be units in ZG with $u^k = 1 = v^k, k \geqq 1$. Then u and v are rationally conjugate if and only if $\varepsilon_C(u^d) = \varepsilon_C(v^d)$ for every divisor d of k and every conugacy class C of G.

In particular, if the units u, v are of order p, p a prime, then they are rationally congruent if and only if $\varepsilon_C(u) = \varepsilon_C(v)$ for every conjugacy class C of G.

A conjecture of Zaqsssenhaus asserts that a torsion unit of augmentation 1 in ZG is conjugate in $\mathbb{Q}G$ to a group element. Equivalently, that every partial augmentation of a torsion unit of augmentation 1 in ZG is 0 or 1.

Using Theorem 10.1, they prove the Zassenhaus conjecture for the alternating group A_5 of degree 5. Explicitely they prove

Theorem 10.2. Every normalized torsion unit in ZA_5 is rationally conjugate to a group element.

In [18] Hales, Luthar and Passi make a detailed study of partial augmentations. Let $C_1, ..., C_t$ be the conjugacy classes of G. As a first result, they prove

Theorem 10.3. Let G be a finite group and $\alpha \in \mathbb{C}G$. Let λ be the maximum of the absolute values of the roots of the minimal polynomial $m(x)$ of α over $\mathbb{C}$. Then $\sum_{i=1}^{t} |\varepsilon_i(\alpha)|^2/|C_i| \leq \lambda^2$. Equality holds if and only if α is the sum of a central and a nilpotent element and all roots of $m(x)$ are of absolute value λ.(Here $\varepsilon_i(\alpha\}$ stands for $\varepsilon_{C_i}(\alpha)$ and $|C_{i|}|$ is the orderof C_i.)

Corollary 10.4. Every torsion unit u as also every idempotent u in $\mathbb{C}G$ satisfies $\sum_{i=1}^{t-1} \varepsilon_i(u)^2/|C_i| \leq 1$.

Following are two other conseqences of Theorem 10.3.

(i) A central torsion unit of augmentation 1 in ZG is a group element.

(ii) If u is a torsion unit of augmentation 1 in ZG and $u(g) \neq 0$ for some central element g in G,then $u = g$.

Some other applications of partial augmrntations (which include an alternative proof of a theorem of Bass) are also proved. Then they consider the problem of Jordan decomposition.

Let K be a field of characteristic $0, V$ be a finite dimensioal vector space over K. An endomorphism $\phi \in End_K(V)$ is called *semisimple* if its minimal polynomial over K has no repeated roots in the algebraic closure $\bar{K}$ of K. Also an element α of KG is said to be *semisimple* if its minimal polynomial over K has no repeated roots in $\bar{K}$. Jordan decomposition theorem states:

(1) $\phi \in End_K(V)$ can be uniquely expressde as $\phi = \phi_s + \phi_n$, where $\phi_s, \phi_n \in End_K(V)$, with ϕ_s semisiple and ϕ_n nilpotent,

(2) ϕ_s and ϕ_n can be expressed as polynomials in ϕ with coefficients in K.

(3) $\alpha \in KG$ can be uniquely expressed as $\alpha = \alpha_s + \alpha_n$, where $\alpha_s, \alpha_n \in KG$ and α_s is semisimple, α_n is nilptent.

(4) α_s and α_n can be expressed as polynomials in α with coeffficients in K.

One of the results on Jordan decomposition provedis:

Theorem 10.5. Let $\mathbf{A}$ be the ring of algebraic integers of K. If Jordan decomposition holds in $\mathbf{A}G$, then G is Abelian.

Towards a contribution on determining finite groups so that Jordan decomposition holds nontrivially in ZG,

Hales, Luthar and Passi prove

Theorem 10.6. If G is a dihedral group of order $2p, p$ a prime, then Jordan decomposition holds in ZG.

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Continuing this study of partial augmentations, Passi and Passman [35] prove

Theorem 10.7 Let ξ be an element of the group algebra $\mathbb{C}G, \mathbb{C}$ the field of complex numbers, G any group (not necessarily finite) and suppose that $f(\xi) = 0$ for some nonzero polynomial $f(x) \in \mathbb{C}[x]$. If λ denotes the maximum of the absolute values of the complex roots of f, then $\sum_C \varepsilon_C(\xi)\overline{\varepsilon_C(\xi)}/|C| \leq \lambda^2$. Here $\bar{\alpha}$ for $\alpha \in \mathbb{C}$ denotes the complex conjugate of α.

For G finite, this result reduces to that of the inequality inTheorem 10.3. In case ξ is an idempotent, the above theorem extends a result of Weiss [56]. We mention just two interesting corollaries of the above theorem proved by Passi and Passman.

Corollary 10.8. Let $\xi \in \mathbb{C}G$ be algebraic over $\mathbb{C}$ and suppose that its minimal polynomial has distinct roots, say $\lambda_1, ..., \lambda_n$. If λ is the maximum of the absolute values of the λ_i, then $\sum_C \varepsilon_C(\xi)\overline{\varepsilon_C(\xi)}/|C| \leq \lambda^2$ with equality if and only if ξ is central and $|\lambda_i| = \lambda$ for all i.

Corollary 10.9. Let ξ be a unit of finite order in $\mathbb{C}G$. Then $\sum_C \varepsilon_C(\xi)\overline{\varepsilon_C(\xi)}/|C| \leq 1$ with equality if and only if ξ is central. Furthermore if ξ is in ZG, then equality occurs if and only if $\pm\xi$ is acentral torsion element of G.

In conclusion, let me admit that the presentation has been very sketchy and that I have not done justice to the other researchers who have worked on the problems considered above. I appologise to those authors. The problems considered here are not considered by Passi and his coauthors in isolation but are an active area of research. The only thing is that the history of these problems could not be traced in one lecture. To do that each topic considered would need one lecture. Also I have restricted myself to the work done by Professor Passi only upto the year 1990. Later developments and work done by Professor Passi will be taken care of by next speakers.

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