On the pioneering works of Professor I.B.S. Passi

Sugandha Maheshwary

Indian Institute of Science Education & Research (IISER), Mohali
INDIA
sugandha@iisermohali.ac.in

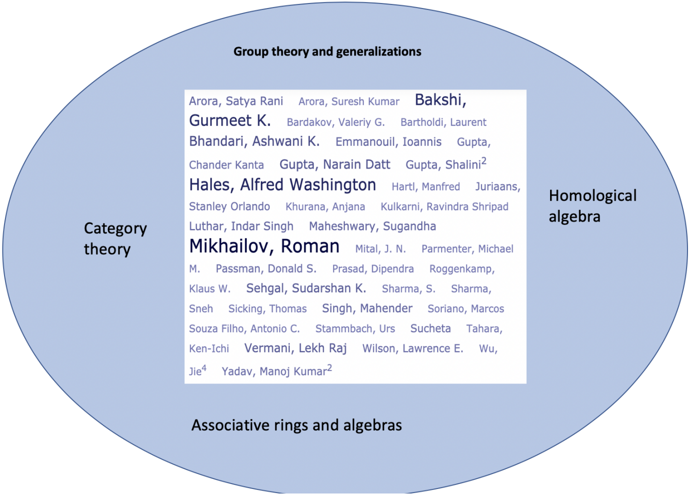
21/10/2019- Group Ring day

Few words About Prof Passi

The study of group rings, has been keeping Prof. Passi (and others) busy for many decades. He has contributed significantly, by investigating this algebraic structure, from various aspects. While some results give information about the structure of these group rings, the others acquaint us with the properties of elements in them. In fact, almost each work initiated by him in this direction has created a new line of research.

A fascinating Group ring

A fascinating Group ring



Group ring RG-some basic notions

Definition (Group ring)

- $G :=$ a group.
- $R :=$ a commutative unital ring.

Group ring RG-some basic notions

Definition (Group ring)

- $G :=$ a group.
- $R :=$ a commutative unital ring.
- $RG := \{\sum_{g \in G} \alpha_g g,\ \alpha_g \in R\}$, where only finitely many α_g are non-zero, addition and multiplication being defined in natural manner is an R-algebra, called the group ring RG of G over R.

Definition (Augmentation ideal)

Group ring RG-some basic notions

Definition (Group ring)

- $G :=$ a group.
- $R :=$ a commutative unital ring.
- $RG := \{\sum_{g \in G} \alpha_g g,\ \alpha_g \in R\}$, where only finitely many α_g are non-zero, addition and multiplication being defined in natural manner is an R-algebra, called the group ring RG of G over R.

Definition (Augmentation ideal)

- $\epsilon : RG \to R$, augmentation homomorphism $(g \to 1)$.

Group ring RG-some basic notions

Definition (Group ring)

- $G :=$ a group.
- $R :=$ a commutative unital ring.
- $RG := \{\sum_{g\in G}\alpha_g g,\ \alpha_g \in R\}$, where only finitely many α_g are non-zero, addition and multiplication being defined in natural manner is an R-algebra, called the group ring RG of G over R.

Definition (Augmentation ideal)

- $\epsilon : RG \to R$, augmentation homomorphism $(g \to 1)$.
- The two sided ideal $\Delta_R(G) := ker(\epsilon)$ is called the augmentation ideal.

Group ring RG-some basic notions

Definition (Group ring)

- $G :=$ a group.
- $R :=$ a commutative unital ring.
- $RG := \{\sum_{g\in G} \alpha_g g,\ \alpha_g \in R\}$, where only finitely many α_g are non-zero, addition and multiplication being defined in natural manner is an R-algebra, called the group ring RG of G over R.

Definition (Augmentation ideal)

- $\epsilon : RG \to R$, augmentation homomorphism $(g \to 1)$.
- The two sided ideal $\Delta_R(G) := ker(\epsilon)$ is called the augmentation ideal.
- $\Delta_{\mathbb{Z}}(G) := \Delta(G)$.

In this talk[1]

The broad themes that I want to talk about are as follows:

[1] The talk is not (and cannot be) exhaustive compilation of his work in group rings.

In this talk[1]

The broad themes that I want to talk about are as follows:

- Augmentation ideals and dimension subgroups.

[1]The talk is not (and cannot be) exhaustive compilation of his work in group rings.

In this talk[1]

The broad themes that I want to talk about are as follows:

- Augmentation ideals and dimension subgroups.
- Algebraic elements of a group algebra.

[1] The talk is not (and cannot be) exhaustive compilation of his work in group rings.

In this talk[1]

The broad themes that I want to talk about are as follows:

- Augmentation ideals and dimension subgroups.
- Algebraic elements of a group algebra.
- The Jordan decomposition property.

[1] The talk is not (and cannot be) exhaustive compilation of his work in group rings.

In this talk[1]

The broad themes that I want to talk about are as follows:

- Augmentation ideals and dimension subgroups.
- Algebraic elements of a group algebra.
- The Jordan decomposition property.
- The structure of group algebras.

[1] The talk is not (and cannot be) exhaustive compilation of his work in group rings.

In this talk[1]

The broad themes that I want to talk about are as follows:

- Augmentation ideals and dimension subgroups.
- Algebraic elements of a group algebra.
- The Jordan decomposition property.
- The structure of group algebras.
- The unit groups of group rings
  - On hyperbolic unit groups.
  - On cut groups.
  - On the central series of unit group of an integral group ring.

[1] The talk is not (and cannot be) exhaustive compilation of his work in group rings.

On augmentation ideals and dimension subgroups I

Definition

If G is a group then we have a descending central series $D_n(G)_{n\geq 1}$ of subgroups of G, called the dimension series, where

$$D_n(G) = G \cap (1 + \Delta^n),\ n \geq 1.$$

The subgroup $D_n(G)$ is called the n^{th} dimension subgroup of G.

The dimension series of any group G is closely related to its lower central series $\{\gamma_n(G)\}_{n\geq 1}$, $(\gamma_1(G) = G,\ \gamma_i(G) = [\gamma_{i-1}(G), G],\ i \geq 2)$.

- $D_n(G) \supseteq \gamma_n(G)$, for all $n \geq 1$.
- $D_n(G) = \gamma_n(G)$, for $n = 1, 2, 3$.
- E. Rips 1^{st} construted a nilpotent 2-group of class 3 having non-trivial 4^{th} dimension subgroup.

On augmentation ideals and dimension subgroups

On augmentation ideals and dimension subgroups

- Naturally, one demands the understanding of the quotient $D_n(G)/\gamma_n(G)$, called the n^{th} dimension quotient of G.

On augmentation ideals and dimension subgroups

- Naturally, one demands the understanding of the quotient $D_n(G)/\gamma_n(G)$, called the n^{th} dimension quotient of G.
- The evaluation of dimension subgroups and dimension quotients is a challenging problem in the theory of group rings, and has been a subject of investigation since 1935.

On augmentation ideals and dimension subgroups

- Naturally, one demands the understanding of the quotient $D_n(G)/\gamma_n(G)$, called the n^{th} dimension quotient of G.
- The evaluation of dimension subgroups and dimension quotients is a challenging problem in the theory of group rings, and has been a subject of investigation since 1935.

Prof. Passi has made immense contribution in this direction.

On augmentation ideals and dimension subgroups

- Naturally, one demands the understanding of the quotient $D_n(G)/\gamma_n(G)$, called the n^{th} dimension quotient of G.
- The evaluation of dimension subgroups and dimension quotients is a challenging problem in the theory of group rings, and has been a subject of investigation since 1935.

Prof. Passi has made immense contribution in this direction. For instance, he proved that for a finite p-group G, $D_4(G) = \gamma_4(G)$.

On augmentation ideals and dimension subgroups

- Naturally, one demands the understanding of the quotient $D_n(G)/\gamma_n(G)$, called the n^{th} dimension quotient of G.
- The evaluation of dimension subgroups and dimension quotients is a challenging problem in the theory of group rings, and has been a subject of investigation since 1935.

Prof. Passi has made immense contribution in this direction. For instance, he proved that for a finite p-group G, $D_4(G) = \gamma_4(G)$.
In his recent developments, he has given a new outlook to the topic, by extending its domain and generalising the concept.

- An analogy for Lie rings

Definition

For a Lie ring, L,

$$\delta_n(L) = L \cap \Delta^n(L),$$

$\Delta^n(L)$ being the augmentation ideal of the universal enveloping algebra $U(L)$ generated by L.

Theorem

For a Lie ring L, $\delta_n(L) = \gamma_n(L),\ n \leq 3$ and may differ at $n = 4$.

Theorem (Metabelian Lie ring)

For every metabelian Lie ring, L, $2\delta_n(L) \subseteq \gamma_n(L),\ n \geq 1$ and this is best possible bound for exponents of the Lie dimension quotients.

[2] With: L. Bartholdi, N.D. Gupta, C.K. Gupta, M. Hartl, R. Mikhailov, T.Sicking.

Dimension subgroups

Extended notions [3]

- Generalized dimension subgroups

Definition

$\mathcal{D}_n(G, \mathfrak{a}) := G \cap (1 + \mathfrak{a} + \mathfrak{g}^n)$ for any two sided ideal $\mathfrak{a} \in \mathbb{Z}G$, $\mathfrak{g} := \Delta(G)$.

  - Identified $\mathcal{D}_n(F, \mathfrak{a})$ for various two sided ideals $\mathfrak{a}$ in a free group ring $\mathbb{Z}F$.
  - Exhibited how some of the generalized dimension subgroups are related to derived functors.

- Relative dimension subgroups

Definition

If $N \trianglelefteq E$ and n is a positive integer, then the n^{th} dimension subgroup $D_n(E, N)$ of E relative to N is $E \cap (1 + \mathfrak{n}\mathfrak{e} + \mathfrak{e}^n)$.

  - Many theorems concerning relative dimension subgroups are proven.

[3] With: L. Bartholdi, N.D. Gupta, C.K. Gupta, M. Hartl, R. Mikhailov, T.Sicking.

On algebraic elements

Definition

Let $\mathcal{A}$ be an algebra over a commutative ring R. An element $\alpha \in \mathcal{A}$ is said to be algebraic if $f(\alpha) = 0$, for some non-zero polynomial $f(x) \in R[x]$.

Algebraic elements of interest:

- Nilpotent elements ($x^n = 0$),
- idempotents ($x^2 = x$),
- torsion units ($x^n = 1$).

Algebraic elelments in group rings

Celebrated results[4]

Passi-Passman inequality

If $\alpha \in \mathbb{C}G$ is an element satisfying a non-zero polynomial $f(x) \in \mathbb{C}[x]$, and if λ is the maximum of the absolute values of the complex roots of $f(x)$, then

$$\sum_i |\epsilon_i(\alpha)|^2 / |\kappa_i| \leq \lambda^2,$$

where $\epsilon_i : \mathbb{C}G \to \mathbb{C}$, denote the partial augmentation corresponding to the conjugacy class κ_i $(\epsilon_i(\alpha := \sum_{g\in G} \alpha_g g) = \sum_{g \in \kappa_i} \alpha_g)$.

This powerful inequality was obtained using the embedding of $\mathbb{C}G$ in a Hilbert space H and then considering the uniform closure of $\mathbb{C}G$ in $\mathbb{C}*$ algebra of bounded operators on H.

[4] With: A.W. Hales, I. S. Luthar, D. S. Passman

Theorem

Let G be a finite group. For $A = \sum_{g \in \kappa_i} A_g g \in M_n(\mathbb{C})[G] = M_n(\mathbb{C}G)$, define partial augmentation $\epsilon_i(A) = \sum_{g \in \kappa_i} Tr(A_g)$.
Let λ be the maximum of the absolute values of the roots of the minimal polynomial $m(x)$ of A over $\mathbb{C}$. Then,

$$\sum_i |\epsilon_i(\alpha)|^2 / |\kappa_i| \leq n^2 \lambda^2.$$

[5] With: A.W. Hales, I. S. Luthar, D. S. Passman

Algebraic elelments in group rings

Celebrated results[6]

As a consequence, following results on torsion units in $M_n(\mathbb{Z}G),\ n = 1,2$ were obtained.

Theorem

For a finite abelian group G, every torsion unit $A \equiv I(mod\ \Delta(G))$ in $M_2(\mathbb{Z}G)$ is similar in $M_2(\mathbb{C}G)$ to a diagonal matrix with group elements on diagonal.

Theorem

Let G be an abelian group having an abelian subgroup of index 2. Then, every torsion unit of augmentation 1 in $\mathbb{Z}G$ is rationally conjugate to an element of G, i.e, Zassenhaus conjecture holds for torsion units in $\mathbb{Z}G$, if G has an abelian group of index 2.

[6] With: A.W. Hales, I. S. Luthar, D. S. Passman

JD-Jordan decomposition

F- a field of characteristic 0, A- a finite dimensional F-algebra. For $\alpha \in A$, $\exists$ unique decomposition

$$\alpha = \alpha_s + \alpha_n,$$

where $\alpha_s, \alpha_n \in \mathbb{Q}G$, α_s semisimple (satisfies F-polynomial with no repeated roots), α_n nilpotent and $\alpha_s\alpha_n = \alpha_n\alpha_s$, which is called Jordan decomposition (JD) of α. (Generalises JD of a matrix.)

AJD-additive Jordan decomposition

R-integrally closed subring of its quotient field F, A-an R-order. If $\forall\ \alpha \in A$, $\alpha_s, \alpha_n \in A$, we say that A has AJD.

MJD-additive Jordan decomposition

If $\forall\ \alpha \in \mathcal{U}(A)$,

$$\alpha = \alpha_s \alpha_u,$$

where $\alpha_s, \alpha_u \in A$, α_s semisimple, α_u unipotent and $\alpha_s \alpha_u = \alpha_u \alpha_s$, then we say that A has MJD.

AJD $\implies$ MJD

If $\alpha \in A$ is a unit, then so is α_s, and hence $\alpha = \alpha_s \alpha_u$, where $\alpha_u = 1 + \alpha_s^{-1} \alpha_n$.

AJD $\iff$ MJD

MJD holds for $A \iff$ AJD holds in A, $\forall \alpha \in \mathcal{U}(A)$.

On the Jordan decomposition property

Cases of interest[7]

- $A = M_n(R)$, ring of $n \times n$ matrices over R.
- $A = RG$, group ring of a finite group.

Theorem (Jordan decomposition property in $M_n(R)$)

[a] *Let R be an integral domain which is integrally closed in its quotient field F of characteristic zeo, with $R \neq F$. Then,*

- *AJD holds in $M_1(R)$ and $M_2(R)$, and fails in $M_n(R)$, $n \geq 3$.*
- *MJD holds in $M_1(R)$ and $M_2(R)$, holds in $M_3(R) \iff$ units of R with 0 forms a subfield, and fails in $M_n(R)$, $n \geq 4$.*

[a] The integral Jordan decomposition of matrices was further taken up by relating it to Hochschild cohomology, in joint work with K. W. Roggenkamp and M. Soraino.

[7] With: A.W. Hales, I. S. Luthar, S. R. Arora, K. W. Roggenkamp, M. Soriano, L.E. Wilson

On the Jordan decomposition property

Cases of interest[8]

Theorem (Jordan decomposition property in RG)

Consider Wedderburn decomposition of group algebra FG,

$$FG \cong \oplus_{i=1}^{h} M_{n_i}(D_i),$$

D_i division rings. Then,

- *AJD holds in $RG \implies n_i \leq 2, \forall\ i$. Further, if $2 \nmid |G|$, then G is abelian.*
- *MJD holds in $RG \implies n_i \leq 3, \forall\ i$. Further, if neither 2 nor 3 divides $|G|$, then G is abelian.*

Moreover, if R is the ring of all algebraic integers. Then AJD holds in RG, if and only if G is abelian.

[8] With: A.W. Hales, I. S. Luthar, S. R. Arora, K. W. Roggenkamp, M. Soriano, L.E. Wilson

On the Jordan decomposition property in $\mathbb{Z}G$[9]

In series of papers, on the JD in $\mathbb{Z}G$, one gets a list of groups G for which AJD/MJD holds for $\mathbb{Z}G$ and those where it does not. We list a few of them.

Theorem (Groups G such that $\mathbb{Z}G$ has AJD-complete classification)

AJD holds in $\mathbb{Z}G$, if and only if G is one of the following:

- *an abelian group, or*
- *of the form $Q_8 \times E \times A$, $E-$elementary abelian 2-group, A abelian group, $2 \nmid |A|$, $2 \nmid o_{|A|}(2)$, or*
- *a dihedral group of order $2p$, p odd prime.*

Groups with MJD

- $\mathbb{Z}A_4$ does not have MJD.
- $\mathbb{Z}Q_{4p}$ has MJD, p an odd prime.
- $\mathbb{Z}D_{2n}$ has MJD if and only if $n = 2, 4$ or an odd prime p.

[9] With: A.W. Hales, I. S. Luthar, S. R. Arora, K. W. Roggenkamp, M. Soriano, L.E. Wilson

On the Jordan decomposition property in ZG[10]

Theorem (2- groups G such that $\mathbb{Z}G$ has MJD)

For a 2-group G, AJD holds in $\mathbb{Z}G$, if and only if G is one of the following

- *If G is abelian, then $\mathbb{Z}G$ has AJD, and hence has MJD*
- *Both non abelian groups of order 8, i.e., Q_8 and D_8 have MJD.*
- *out of 9 non abelian groups of order 16, only 5 have MJD [a].*
- *out of 44 non abelian groups of order 32, only 4 have MJD.*
- *if $|G| = 2^n,\ n \geq 5$, then $\mathbb{Z}G$ has MJD if and only if G is hamiltonian.*

[a] Some cases handled by M.M. Parmenter

[10] With: A.W. Hales, I. S. Luthar, S. R. Arora, K. W. Roggenkamp, M. Soriano, L.E. Wilson

On the Jordan decomposition property in ZG[11]

Theorem (Restriction on finite groups)

Let G be a finite group such that $\mathbb{Z}G$ has MJD. Then, one of the following holds:

- *G is either abelian or of the form $Q_8 \times E \times A$, $E-$elementary abelian 2-group, A abelian group, $2 \nmid |A|$, $2 \nmid o_{|A|}(2)$.*
- *G has order $2^a 3^b$, $a, b \in \mathbb{Z}_{\geq 0}$.*
- *$G = Q_8 \times C_p$, for some prime p, such that $2 \mid o_p(2)$.*

[11] With: A.W. Hales, I. S. Luthar, S. R. Arora, K. W. Roggenkamp, M. Soriano, L.E. Wilson

On the structure of semisimple group algebras

PCIs and Wedderburn decomposition [12]

In a series of articles, the object of investigation has been the structure of group algebras.

[12] With: G.K. Bakshi, S. Gupta, R. S. Kulkarni,

On the structure of semisimple group algebras

PCIs and Wedderburn decomposition [12]

In a series of articles, the object of investigation has been the structure of group algebras.
One of the fundamental questions in this direction is to determine a complete set of pcis of group algebras.

[12] With: G.K. Bakshi, S. Gupta, R. S. Kulkarni,

On the structure of semisimple group algebras

PCIs and Wedderburn decomposition [12]

In a series of articles, the object of investigation has been the structure of group algebras.
One of the fundamental questions in this direction is to determine a complete set of pcis of group algebras.

$\mathbb{Q}G$

For the rational group algebra $\mathbb{Q}G$, a complete set of pcis as well as explicit expression for them, is given for certain classes of groups. This leads to complete description of the structure of $\mathbb{Q}G$.

[12] With: G.K. Bakshi, S. Gupta, R. S. Kulkarni,

On the structure of semisimple group algebras

PCIs and Wedderburn decomposition [12]

In a series of articles, the object of investigation has been the structure of group algebras.
One of the fundamental questions in this direction is to determine a complete set of pcis of group algebras.

$\mathbb{Q}G$

For the rational group algebra $\mathbb{Q}G$, a complete set of pcis as well as explicit expression for them, is given for certain classes of groups. This leads to complete description of the structure of $\mathbb{Q}G$.

$\mathbb{F}_q G$

For finite semisimple group algebras $F_q G$, the complete set of pcis, the structure of $F_q G$ and the group of automorphisms have been studied for metabelian groups.

[12] With: G.K. Bakshi, S. Gupta, R. S. Kulkarni,

Let (X, d) be a metric space. For $x, y \in X$, the Gromov product of y, z with respect to x is defined to be

$$(y.z)_x = \frac{1}{2}\{d(y,x) + d(z,x) - d(y,z)\}.$$

The metric space is called δ-hyperbolic ($\delta \geq 0$) if

$$(x.y)_w \geq min\{(x.z)_w, (y.z)_w\} - \delta,$$

for all $w, x, y, z \in X$. The Cayley graph $\mathcal{G}(G, S)$ of G w.r.t. the a generating set S is the metric graph whose vertices are in one-to-one correspondence with the elements of G and which has an edge of length 1 joining g to gs, for each $g \in G$ and $s \in S$. The group G is said to be hyperbolic, if its Cayley graph is δ hyperbolic metric space for some $\delta \geq 0$.

On hyperbolic unit groups

Some results [13]

Theorem

Let G be a finite non-Hamiltonian group. Then the following are equivalent:

(i) *Exactly one Wedderburn component of $\mathbb{Q}G$ is $M_2(\mathbb{Q})$, and any other component is either $\mathbb{Q}$, or an imaginary quadratic extension of $\mathbb{Q}$ or a totally definite quaternion algebra over $\mathbb{Q}$.*

(ii) *G has a normal free complement in $\mathcal{U}_1(\mathbb{Z}G)$.*

(iii) *$\mathcal{U}_1(\mathbb{Z}G)$ is virtually free.*

(iv) *$\mathcal{U}_1(\mathbb{Z}G)$ is hyperbolic.*

Moreover, if one of the above conditions holds, then every finitely generated torsion-free subgroup of $\mathcal{U}_1(\mathbb{Z}G)$ is free. In particular, any normal torsion-free complement of G in $\mathcal{U}_1(\mathbb{Z}G)$ is free.

[13] With: S. O. Juriaans, D. Prasad, A. C. Souza Filho

On hyperbolic unit groups

Some results [14]

Theorem (Torsion groups occuring as subgroups)

If a torsion group G embeds into a hyperbolic unit group, then G must be finite and isomorphic to one of the following groups:

(i) C_5, C_8, C_{12}, *an Abelian group of exponent dividing 4 or 6;*

(ii) *a Hamiltonian 2-group;*

(iii) S_3, D_4, Q_{12}, $C_4 \rtimes C_4$.

Conversely, all of the groups listed above have hyperbolic unit groups.

[14] With: S. O. Juriaans, D. Prasad, A. C. Souza Filho

On hyperbolic unit groups

Some results [15]

Theorem (Infinite polycyclic-by-finite groups G that embed)

An infinite polycyclic-by-finite group G embeds into a group G whose unit group $\mathcal{U}_1(\mathbb{Z}G)$ is hyperbolic if and only if

(i) $T(G)$, *the set of elements of finite order in* G, *is a subgroup of* G;

(ii) $G \cong T(G) \rtimes \mathbb{Z}$;

(iii) $\mathcal{U}_1(\mathbb{Z}G) = T(G)$.

[15] With: S. O. Juriaans, D. Prasad, A. C. Souza Filho

Hyperbolicity of $\mathcal{U}_1(\mathfrak{o}_K G),\ K = \mathbb{Q}[\sqrt{d}]$.

Theorem

Let $K = \mathbb{Q}[\sqrt{d}]$, with $d (\neq 1)$, square-free integer and G a finite group. Then $\mathcal{U}_1(\mathfrak{o}_K G)$ is hyperbolic if, and only if, G is one of the groups listed below and $\mathfrak{o}_K$ is determined by the corresponding value of d:

- (i) $G \in \{C_2, C_3\}$ *and d arbitrary;*
- (ii) G *is an Abelian group of exponent dividing n for: $n = 2$ and $d < 0$, or $n = 4$ and $d = -1$, or $n = 6$ and $d = -3$.*
- (iii) $G = C_4$ *and $d < 0$.*
- (iv) $G = C_8$ *and $d = -1$.*
- (v) $G = Q_8$ *and $d < 0$ and $d \equiv 1 \ (mod\ 8)$.*

[16] With: S. O. Juriaans, D. Prasad, A. C. Souza Filho

- A group with the cut-property must necessarily be divisible by 2 or 3.

[17] With: G.K. Bakshi, S. Maheshwary

- A group with the cut-property must necessarily be divisible by 2 or 3.
- A nilpotent group with the cut-property must be a $(2,3)$-group.

[17] With: G.K. Bakshi, S. Maheshwary

On the central units of an integral group ring

Finite cut-groups: Central Units Trivial[17]

- A group with the cut-property must necessarily be divisible by 2 or 3.
- A nilpotent group with the cut-property must be a $(2, 3)$-group.
- A 3-group G has the cut-property if, and only if, for all $x \in G$, x^2 is conjugate to x^{-1}.

[17] With: G.K. Bakshi, S. Maheshwary

- A group with the cut-property must necessarily be divisible by 2 or 3.
- A nilpotent group with the cut-property must be a $(2, 3)$-group.
- A 3-group G has the cut-property if, and only if, for all $x \in G$, x^2 is conjugate to x^{-1}.
- A 2-group G has the cut-property if, and only if, for each $x \in G$, x^3 is conjugate to either x or x^{-1}.

[17] With: G.K. Bakshi, S. Maheshwary

- A group with the cut-property must necessarily be divisible by 2 or 3.
- A nilpotent group with the cut-property must be a $(2,3)$-group.
- A 3-group G has the cut-property if, and only if, for all $x \in G$, x^2 is conjugate to x^{-1}.
- A 2-group G has the cut-property if, and only if, for each $x \in G$, x^3 is conjugate to either x or x^{-1}.
- The cut-property implies the normalizer property i.e. $G.\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = N_{\mathcal{U}}(G)$. Note that all metacyclic groups have the normalizer property, only a few of them have the cut property.

[17] With: G.K. Bakshi, S. Maheshwary

On the central units of an integral group ring

Finite cut-groups: Central Units Trivial[17]

- A group with the cut-property must necessarily be divisible by 2 or 3.
- A nilpotent group with the cut-property must be a $(2,3)$-group.
- A 3-group G has the cut-property if, and only if, for all $x \in G$, x^2 is conjugate to x^{-1}.
- A 2-group G has the cut-property if, and only if, for each $x \in G$, x^3 is conjugate to either x or x^{-1}.
- The cut-property implies the normalizer property i.e. $G.\mathcal{Z}(\mathcal{U}(\mathbb{Z}[G])) = N_{\mathcal{U}}(G)$. Note that all metacyclic groups have the normalizer property, only a few of them have the cut property.
- Only finitely many metacyclic groups have the cut-property.

The various notions developed around the study of cut groups, exhibit a highly interesting interplay between group theory, representation theory, algebraic number theory and K -theory.

[17] With: G.K. Bakshi, S. Maheshwary

Metacyclic cut-groups (upto isomorphism)[18]

$\langle a,\ b \mid a^n = 1,\ b^t = 1,\ b^{-1}ab = a^{n-1}\rangle,\quad t = 2,4,6,\quad n = 3,4,6;$
$\langle a,\ b \mid a^4 = 1,\ b^t = a^2,\ b^{-1}ab = a^3\rangle,\ t = 2,4,6;$
$\langle a,\ b \mid a^6 = 1,\ b^2 = a^3,\ b^{-1}ab = a^5\rangle;$
$\langle a,\ b \mid a^n = 1,\ b^{\varphi(n)} = 1,\ b^{-1}ab = a^{\lambda_n}\rangle,\ n = 5,7,9,10,14,18;$
$\langle a,\ b \mid a^n = 1,\ b^{\frac{\varphi(n)}{j}} = 1,\ b^{-1}ab = a^{\lambda_n^2}\rangle,\quad j = 1,2,\quad n = 7,9,14,18;$
$\langle a,\ b \mid a^8 = 1,\ b^t = 1,\ b^{-1}ab = a^r\rangle,\ t = 2,4,\ r = 3,5;$
$\langle a,\ b \mid a^{12} = 1,\ b^t = 1,\ b^{-1}ab = a^5\rangle,\ t = 2,4;$
$\langle a,\ b \mid a^{12} = 1,\ b^t = a^\ell,\ b^{-1}ab = a^7\rangle,\ t = 2,6,\ \ell = t,12;$
$\langle a,\ b \mid a^{15} = 1,\ b^4 = 1,\ b^{-1}ab = a^2\rangle;$
$\langle a,\ b \mid a^{16} = 1,\ b^4 = 1,\ b^{-1}ab = a^r\rangle,\ r = 3,5;$
$\langle a,\ b \mid a^{20} = 1,\ b^4 = 1,\ b^{-1}ab = a^r\rangle,\ r = 3,13;$
$\langle a,\ b \mid a^{20} = 1,\ b^4 = a^{10},\ b^{-1}ab = a^3\rangle;$
$\langle a,\ b \mid a^{21} = 1,\ b^6 = 1,\ b^{-1}ab = a^r\rangle,\ r = 2,10;$
$\langle a,\ b \mid a^{28} = 1,\ b^6 = a^\ell,\ b^{-1}ab = a^{11}\rangle,\ \ell = 14,28;$
$\langle a,\ b \mid a^{30} = 1,\ b^4 = 1,\ b^{-1}ab = a^{17}\rangle;$
$\langle a,\ b \mid a^{36} = 1,\ b^6 = a^\ell,\ b^{-1}ab = a^7\rangle,\ \ell = 6,36;$
$\langle a,\ b \mid a^{42} = 1,\ b^6 = 1,\ b^{-1}ab = a^r\rangle,\ r = 11,19;$

[18] With G.K. Bakshi, S. Mah

Definition (RS-subgroup)

An element $x \in G$ of finite order has the RS-property (or is an RS-element) in G if

$$x^j \sim_G x^{\pm 1},\ \forall\ j \in \mathcal{U}(o(x)).$$

A subgroup of G all whose elements are RS, is called an RS-subgroup of G.

[19]With: G.K. Bakshi, S. Maheshwary

Work done in this direction-an overview

- Examined the class of cut-groups under extensions, including amalgams and HNN extensions and thus provide several interesting examples of infinite cut-groups.
- A classification of infinite metacyclic cut-groups is given.
- Classified p-groups and nilpotent groups which are cut-groups.
- Modulo the trivial units, the group of central units is isomorphic to a torsion free subgroup of $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$, consisting of symmetric central units. Consequently, it follows that all central units of $\mathbb{Z}G$ are trivial, if so are all the symmetric central units.
- The quotient $\mathcal{Z}_i(\mathcal{U})/\mathcal{Z}_i - 1(\mathcal{U})$ is of finite exponent for all $i \geq 2$, provided $\mathcal{Z}(G)$ is of finite exponent or G is generated by torsion elements of bounded exponent, where $\mathcal{Z}_i(\mathcal{U})$ denotes the i^{th} term of the upper central series of $\mathcal{U}$.

[20] With: G.K. Bakshi, S. Maheshwary

- If G is a finite group, the central height of $\mathcal{U}(\mathbb{Z}G)$, i.e., the smallest integer $n \geq 0$ such that $\mathcal{Z}_n(\mathcal{U}) = \mathcal{Z}_{n+1}(\mathcal{U})$, is at most 2.
- The central height of $\mathcal{U}$ is 2 if, and only if, G is the so-called Q^* group, and the second centre is described as well.
- In all other cases, the central height must be 0 or 1. The central height 0 essentially means, G must be a cut group with trivial centre.

[21] With: S. R. Arora, A.W. Hales, S. Maheshwary

- When is $\mathcal{U}(\mathbb{Z}(G))$ nilpotent?

- When is $\mathcal{U}(\mathbb{Z}(G))$ nilpotent?
- When is $\mathcal{U}$ residually nilpotent?

For a finite group G, the residual nilpotence of $\mathcal{U}$ has been explored by Musson and Weiss (nilpotent and p-abelian).

On residual nilpotence of $\mathcal{U}(\mathbb{Z}G)$-work in progress [22]

Set

$$\mathcal{U}_n(\mathbb{Z}G) := \mathcal{U}(\mathbb{Z}G) \cap (1 + \Delta^n(G)),\ n \geq 1,$$

so that

- $\gamma_n(\mathcal{U}) \subseteq \mathcal{U}_n(G)$.
- $\mathcal{U}_w(\mathbb{Z}G) := \cap_n \mathcal{U}_n(\mathbb{Z}G) = \langle 1 \rangle \implies \gamma_w(\mathcal{U}) = \langle 1 \rangle$.

Definition

An element $g \neq 1$ is said to have **infinite p-height** in G, if for any i and j, $\exists$ elements $x \in G$ and $a \in \gamma_i(G)$ satisfying $x^{p^j} = ga$. Denote by $T(p)$, the set of elements of infinite p-height in T.

[22] with S. Maheshwary

On residual nilpotence of $\mathcal{U}(\mathbb{Z}G)$-work in progress [23]

We have the following result for the class nilpotent groups:

Theorem

Let G be a nilpotent group. Let T be torsion group of G and let $T(p)$ denote the set of elements of infinite p-height in T. Then $\mathcal{U}_\omega(\mathbb{Z}G) = \langle 1 \rangle$ if and only if G satisfies one of the following:

(i) $T = \langle 1 \rangle$, *i.e., G is a torsion free nilpotent group.*

(ii) T *is a* $(2, 3)$ *group of exponent 6.*

(iii) T *is a p-group with $T(p) = \langle 1 \rangle$, i.e., T is a p-group which has no element of infinite p-height.*

(iv) $\langle 1 \rangle \subsetneq T(p) \subsetneq T$ *and there do not exist an element $t \in T(p)$ such that* $\langle g \rangle \cap \langle t \rangle = \langle 1 \rangle$.

[23] with S. Maheshwary

Amongst others

- A major contribution of Passi, is towards the application of various homological methods to some classical problems in the theory of group rings. [24]
- Prof V. Bardakov will throw light on his recent developments on Quandle rings. [25]

[24] With: R. Mikhailov

[25] With: V. Bardakov, M. Singh

An opportunity to express

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,
discipline, humility and passion to name a few.

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,
discipline, humility and passion to name a few.

Fellowship or award or any prize,

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,
discipline, humility and passion to name a few.

Fellowship or award or any prize,
he inspires only to rise and rise and rise.

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,
discipline, humility and passion to name a few.

Fellowship or award or any prize,
he inspires only to rise and rise and rise.
He doesn't bound limits, and sets his students free;

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,
discipline, humility and passion to name a few.

Fellowship or award or any prize,
he inspires only to rise and rise and rise.
He doesn't bound limits, and sets his students free;
to think of finite, or indefinite, like a sea.

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,
discipline, humility and passion to name a few.

Fellowship or award or any prize,
he inspires only to rise and rise and rise.
He doesn't bound limits, and sets his students free;
to think of finite, or indefinite, like a sea.

To write or to edit or to review,

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,
discipline, humility and passion to name a few.

Fellowship or award or any prize,
he inspires only to rise and rise and rise.
He doesn't bound limits, and sets his students free;
to think of finite, or indefinite, like a sea.

To write or to edit or to review,
one thing is clear in his view;

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,
discipline, humility and passion to name a few.

Fellowship or award or any prize,
he inspires only to rise and rise and rise.
He doesn't bound limits, and sets his students free;
to think of finite, or indefinite, like a sea.

To write or to edit or to review,
one thing is clear in his view;
Work could be multiplicative or trivial,

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,
discipline, humility and passion to name a few.

Fellowship or award or any prize,
he inspires only to rise and rise and rise.
He doesn't bound limits, and sets his students free;
to think of finite, or indefinite, like a sea.

To write or to edit or to review,
one thing is clear in his view;
Work could be multiplicative or trivial,
a note or a book or a research article,

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,
discipline, humility and passion to name a few.

Fellowship or award or any prize,
he inspires only to rise and rise and rise.
He doesn't bound limits, and sets his students free;
to think of finite, or indefinite, like a sea.

To write or to edit or to review,
one thing is clear in his view;
Work could be multiplicative or trivial,
a note or a book or a research article,
Excellence is utmost essential;

An opportunity to express

His presence has charisma and magnetism,
that teaches us lot more than isomorphism.
because teaching is not his only virtue,
discipline, humility and passion to name a few.

Fellowship or award or any prize,
he inspires only to rise and rise and rise.
He doesn't bound limits, and sets his students free;
to think of finite, or indefinite, like a sea.

To write or to edit or to review,
one thing is clear in his view;
Work could be multiplicative or trivial,
a note or a book or a research article,
Excellence is utmost essential;
what else could be, expected as ideal!!

To his work, he stays faithful,

To his work, he stays faithful,
uses primitive ideas, gets them rational;

To his work, he stays faithful,
uses primitive ideas, gets them rational;
his research then, witnesses augmentation,

To his work, he stays faithful,
uses primitive ideas, gets them rational;
his research then, witnesses augmentation,
this is how, induces a new dimension.

To his work, he stays faithful,
uses primitive ideas, gets them rational;
his research then, witnesses augmentation,
this is how, induces a new dimension.

A person who has touched the greatest heights,

To his work, he stays faithful,
uses primitive ideas, gets them rational;
his research then, witnesses augmentation,
this is how, induces a new dimension.

A person who has touched the greatest heights,
yet the tone of his voice is always light.

To his work, he stays faithful,
uses primitive ideas, gets them rational;
his research then, witnesses augmentation,
this is how, induces a new dimension.

A person who has touched the greatest heights,
yet the tone of his voice is always light.
A great person, inspirational researcher and an ideal teacher,

To his work, he stays faithful,
uses primitive ideas, gets them rational;
his research then, witnesses augmentation,
this is how, induces a new dimension.

A person who has touched the greatest heights,
yet the tone of his voice is always light.
A great person, inspirational researcher and an ideal teacher,
of course, none other than our Passi Sir.

To his work, he stays faithful,
uses primitive ideas, gets them rational;
his research then, witnesses augmentation,
this is how, induces a new dimension.

A person who has touched the greatest heights,
yet the tone of his voice is always light.
A great person, inspirational researcher and an ideal teacher,
of course, none other than our Passi Sir.

Many happy returns of the day, Sir.

$THANK\ YOU$!!!