GAP-exercises for group rings

1. a) Find non-isomorphic finite groups G and H such that $\mathbb{C}G \simeq \mathbb{C}H$? Is $\mathbb{Q}G \simeq \mathbb{Q}H$?

   b) Find non-isomorphic finite groups G_1 and H_1 such that $\mathbb{Q}G_1 \simeq \mathbb{Q}H_1$? Is $\mathbb{C}G_1 \simeq \mathbb{C}H_1$?

   Hint: The command `CharacterDegrees` might be useful for some part of the exercise. Type `?CharacterDegrees` in `GAP` to find out more.

2. Set $G = \langle\ (1,5,3),\ (1,2)(3,4,5,6)\ \rangle \leq S_6$ and

$$u = 2 \cdot (1,4,5,2)(3,6) - (1,5)(4,6),$$
$$v = (2,4)(3,5) - (1,3)(4,6) - (1,3,5)(2,6,4) + (1,5,3)(2,4,6) + (1,5)(2,6).$$

   What is the order of G and the structure of this group? Is $u, v \in \mathrm{V}(\mathbb{Z}G)$? Is $u, v \in \mathrm{V}(\mathbb{Q}G)$? Are they rationally conjugate to an element of G? (There was useful content in the lectures of Ángel del Río.)

3. Find an element of $\mathrm{U}(\mathbb{Z}S_3)$ which is not of the form g or $-g$ for $g \in S_3$, expressed as linear combination of the elements of S_3.

4. Find a group of least order, which is

   a) not strongly monomial. Is this group monomial?

   b) not normally monomial.

   c) strongly monomial but not normally monomial.

   d) normally monomial group but not metabelian.

   e) strongly monomial but not abelian by supersolvable. Is this group normally monomial?

   f) * monomial but not strongly monomial.

5. a) Verify that all Bass units (as in the defintion of `GAP` with 2 parameters) in $\mathbb{Z}D_{16}$ are central in $\mathrm{U}(\mathbb{Z}D_{16})$. (Hint: The command `ForAll` might be useful.)

   b) Find a finite group G such that $\mathbb{Z}G$ contains a non-trivial (i.e. not contained in G) non-central Bass unit.

   c) Write a `GAP` function that takes 3 parameters: a group ring element g and 2 integers k and m and returns the Bass cyclic unit as defined in the lecture of Eric Jespers (generalizing the pre-implemented version of `BassCyclicUnit` in `LAGUNA`). Make your function also check whether k and m are coprime and satisfy the required congruence. (Hint: the command `Sum` might be useful.)

Please turn over

6. a) Let $G = Q_8$, the quaternion group of order 8. Use `wedderga` to determine the Wedderburn decomposition of $\mathbb{Q}Q_8$. The documentation of `WedderburnDecompositionInfo` might be useful to determine the "unusual" looking components.

   b) Check that the matrices $z = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ and $v = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ satisfy the relations

$$z^3 = 1, \quad v^2 = 1, \quad z^v = z^2.$$

   c) Let $\zeta = \zeta_3 \in \mathbb{C}^\times$ be a primitive 3rd root of unity. Convince yourself that $\zeta + \zeta^2 = -1$ and hence $2\zeta^2 + 1 = -2\zeta - 1$ and that $(2\zeta + 1)^2 = -3$.

   d) Let $G = C_3 \rtimes C_4$, where the generator of the cyclic group of order 4 acts by inversion on the normal cyclic group of order 3 (this group has SmallGroup ID `[12,1]`). Determine the Wedderbrun decomposition of $\mathbb{Q}G$ using `wedderga`.
   Hint: One of the simple algebras appearing in the Wedderburn decomposition is the quaternion algebra $\left(\frac{-1,-3}{\mathbb{Q}}\right)$.

7. Calculate the nilpotency class of the unit group of $\mathbb{F}_2 D_{64}$.

   a) Without using the `UnitLib` package.

   b) Using the `UnitLib` package.