Complex representations of finite reductive groups and generic character tables

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Notation

G: connected reductive group $/K = \bar{\mathbb{F}}_p$

$F : G \to G$ Frobenius morphism with $F^k = F_{q^k}$

$G^F = G(q)$ finite group of Lie type

We want to know the character table of G^F (irreducible representations over $\mathbb{C}$)

Remark. Algebraic group G has no finite dimensional representation over $\mathbb{C}$.

The Lang-Steinberg theorem

From now let G be connected.

Theorem. [Lang-Steinberg] If $F : G \to G$ is a Frobenius morphism then $L : G \to G$, $g \mapsto gF(g^{-1})$, is surjective.

Corollary. Let G act transitively on a set M, and $\tilde{F} : M \to M$ with $\tilde{F}(m)F(g) = \tilde{F}(mg)$ for all $m \in M$, $g \in G$.

- There exists $m_0 \in M$ with $\tilde{F}(m_0) = m_0$. Write $M^{\tilde{F}}$ for set of $\tilde{F}$-stable elements.
  [Prf.: $m \in M$, then $\tilde{F}(m) = mg'$ for some $g' \in G$. Let $g' = gF(g^{-1})$, then $\tilde{F}(mg) = mg'F(g) = mg$.]
- Assume that $H = \mathrm{Stab}_G(m_0)$ is closed, and let $A = A(m_0) = H/H^0$, F induces an automorphism on A. Elements $a, a' \in A$ are F-conjugate if there is $b \in A$ such that $a' = b^{-1}aF(b)$. There is a bijection

$$\begin{array}{rcl} G^F\text{-orbits of } M^{\tilde{F}} & \longrightarrow & F\text{-conjugacy classes of } A \\ G^F\text{-orbit of } m_0 g \in M^{\tilde{F}} & \mapsto & F\text{-conjugacy class of } gF(g^{-1})H^0 \end{array}$$

Applications of Lang-Steinberg

Let G be connected reductive and $F : G \to G$ be a Frobenius morphism.

- All maximal tori T of G are conjugate and F maps maximal tori to maximal tori. Then there is an F-stable maximal torus $T_1 = F(T_1) \le G$. For $N = N_G(T_1)$ we have $N/N^0 = N/T_1 = W$, the Weyl group of G, and F induces a map $F : W \to W$. We have a bijection
$$G^F\text{-conjugacy classes of } \{T \le G \mid F(T) = T\} \xrightarrow{\sim} F-\text{conjugacy classes of } W$$
- All Borel subgroups in G are conjugate. F maps Borel subgroups to Borel subgroups. Hence there is an F-stable Borel subgroup $B = F(B) \le G$. Since $N_G(B) = B$ is connected, all F-stable Borel subgroups are conjugate under G^F.
- Let $g \in G$ and g^G its conjugacy class. Then $F(g^G) = g^G$ if and only if g is conjugate to an element $g_0 \in G^F$. Let $C = C_G(g_0)$. Then the G^F-conjugacy classes in $g^G \cap G^F$ are in bijection with the F-conjugacy classes of C/C^0.
  [In $G = \mathrm{GL}_n(\bar{\mathbb{F}}_q)$ always $C = C^0$. Hence, for each $g \in G$ the intersection $g^G \cap G^F$ is either empty or a single G^F-conjugacy class.]

Dual group

G, F determined by root datum with Frobenius action $(\Delta, \Delta^\vee, F_0)$ and the prime power q

Can exchange roles of X and Y:

Dual group G^*, F is determined by $(\Delta^\vee, \Delta, F_0^{tr})$ and q

Examples: $G^F = \mathrm{SL}_{l+1}(q)$ and $(G^*)^F = \mathrm{PGL}_{l+1}(q)$,
or $G^F = \mathrm{Spin}_{2l+1}(q)$ (type B_l) and $(G^*)^F = \mathrm{PCSp}_{2l}(q)$ (type C_l)

From Lang-Steinberg we had:

$W/F\text{-conj.} \xrightarrow{\sim} \{F\text{-stable maximal tori } T \leq G\}/G^F, \quad w \mapsto T_w \mod G^F$

$W/F\text{-conj.} \xrightarrow{\sim} \{F\text{-stable maximal tori } T \leq G^*\}/(G^*)^F, w \mapsto T_w^* \mod (G^*)^F$

Can identify $(T_w^*)^F$ with $\mathrm{Irr}(T_w^F)$

Deligne-Lusztig characters

$T = F(T) < B = F(B)$: F-stable maximal torus and Borel subgroup, $U \leq B$

$w \in W = N_G(T)/T$, $\dot{w} \in N$ with $\dot{w}T = w$

$Y(\dot{w}) := \{gU \in G/U \mid g^{-1}F(g) \in U\dot{w}U\}$

G^F operates on $Y(\dot{w})$ from the left

$T^{wF} := \{t \in T \mid {}^{w}F(t) = t\} \stackrel{\sim}{=} T_w^F$ operates on $Y(\dot{w})$ from the right

$\mathcal{H}_w := \sum_{i \geq 0} (-1)^i H_c^i(Y(\dot{w}))$, l-adic cohomology with compact support ($l \neq p$)

This becomes a virtual G^F-T^{wF}-bimodule

Definition. [Deligne-Lusztig] We have a map $R_{T_w}^G$ from virtual $\bar{\mathbb{Q}}_l T^{wF}$-modules to virtual $\bar{\mathbb{Q}}_l G^F$-modules, sending V to $\mathcal{H}_w \otimes_{\bar{\mathbb{Q}}_l T^{wF}} V$.

For $\theta \in \mathrm{Irr}(T^{wF})$ let $s \in (T^*)^{wF}$ be the corresponding element in the dual torus; write $R_{T_w}^G(s)$ for the character of the Deligne-Lusztig induced module of θ.

Properties of Deligne-Lusztig characters $R^G_{T_w}(s)$

- The definition does not depend on choices
- For each $\chi \in \mathrm{Irr}(G^F)$ there exist T, s with $\langle \chi, R^G_{T_w}(s) \rangle \neq 0$
- (Orthogonality) Let $s \in (T^*)^{wF}$ and $s' \in (T^*)^{w'F}$, then
$$\langle R^G_{T_w}(s), R^G_{T_{w'}}(s') \rangle = |\{x \in W \mid xwF(x^{-1}) = w' \text{ and } s' = s^x\}|$$
In particular $\pm R^G_{T_w}(s)$ is irreducible if the stabilizer of s in W is trivial
- If $R^G_{T_w}(s)$ and $R^G_{T_{w'}}(s')$ have a common irreducible constituent, then s and s' are conjugate in $(G^*)^F$
This induces a partition of $\mathrm{Irr}(G^F)$, parts labeled by semisimple conjugacy classes of $(G^*)^F$, and called Lusztig series:

$$\mathrm{Irr}(G^F) = \dot{\bigcup}_{s/(G^*)^F \text{ semisimple}} \mathcal{E}(G^F, s)$$

Jordan decomposition of characters

Characters in Lusztig series $\mathcal{E}(G^F, 1)$ for $s = 1$ are called unipotent

Fact: The centralizer of a semisimple element in G is again a reductive group (but in general not connected)

Theorem. [Lusztig, '80s]

(a) Gave combinatorial parameterization of unipotent characters in all cases (E.g., by partitions of n in case $GL_n(q)$), and their multiplicities in the Deligne-Lusztig characters
(b) Showed that parameterization of $\mathcal{E}(G^F, s)$ and multipicities in Deligne-Lusztig characters is the same as for unipotent characters in $C_{G^*}(s)$

Note that these descriptions are independend of q, only depend on the type of root system of G

Remarks on character values

- Values of Deligne-Lusztig characters are "computable" if known on unipotent elements (Green functions)
- Often the Deligne-Lusztig characters do not generate the space of class functions
- [Lusztig '80s] developed theory of character sheaves; addresses both problems: makes Green function computable in many cases and produces a basis of space of class functions
- Parameterization of conjugacy classes in G^F: first semisimple conjugacy classes in G^F and their centralizers, then unipotent classes in each centralizer.
- Parameteration of $\mathrm{Irr}(G^F)$: first semisimple conjugacy classes in $(G^*)^F$ and their centralizers, then unipotent characters in each centralizer.

Example of a character table in case $K = \mathbb{C}$

$G = A_5 = \langle (1,2,3,4,5), (3,4,5) \rangle$, alternating group on 5 points Conjugacy classes: $()^G$, $(1,2)(3,4)^G$, $(1,2,3)^G$, $(1,2,3,4,5)^G$, $(1,2,3,5,4)^G$

Character table (from GAP):

```
          1a 2a 3a 5a 5b

X.1        1  1  1  1  1
X.2        3 -1  .  A *A
X.3        3 -1  . *A  A
X.4        4  .  1 -1 -1
X.5        5  1 -1  .  .
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where A $= (1+\sqrt{5})/2$ and $*$A $= (1-\sqrt{5})/2$

Generic character table of $SL_2(q)$ with $q = 2^k$

$SL_2(q)$	C_1	C_2	$C_3(a)$	$C_4(a)$
χ_1	1	1	1	1
χ_2	q	0	1	-1
$\chi_3(n)$	$q+1$	1	$\zeta_1^{an} + \zeta_1^{-an}$	0
$\chi_4(n)$	$q-1$	-1	0	$-\xi_1^{an} - \xi_1^{-an}$

$\zeta_1 := \exp(\frac{2\pi\sqrt{-1}}{q-1})$, $\quad \xi_1 := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

Parameter ranges:

$\chi_3(n)$:	$n = 1, \ldots, q-2$	($\frac{1}{2}(q-2)$ characters)
$\chi_4(n)$:	$n = 1, \ldots, q$	($\frac{1}{2}q$ characters)
$C_3(a)$:	$a = 1, \ldots, q-2$	($\frac{1}{2}(q-2)$ classes)
$C_4(a)$:	$a = 1, \ldots, q$	($\frac{1}{2}q$ classes)

(BTW: $A_5 \cong SL_2(4)$)

Generic table for $SL_2(q)$ with odd q

$SL_2(q)$	$C_1(i)$	$C_2(i)$
χ_1	1	1
χ_2	q	0
χ_3	$\frac{1}{2}(q+1)(-1)^{\frac{1}{2}(q-1)i}$	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i}+\frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$
χ_4	$\frac{1}{2}(q+1)(-1)^{\frac{1}{2}(q-1)i}$	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i}-\frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$
χ_5	$\frac{1}{2}(q-1)(-1)^{\frac{1}{2}qi+\frac{1}{2}i}$	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i}+\frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$
χ_6	$\frac{1}{2}(q-1)(-1)^{\frac{1}{2}qi+\frac{1}{2}i}$	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i}-\frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$
$\chi_7(k)$	$(q+1)(-1)^{ik}$	$(-1)^{ik}$
$\chi_8(k)$	$(q-1)(-1)^{ik}$	$-(-1)^{ik}$

$\varepsilon_4 := \exp(\frac{2\pi\sqrt{-1}}{4}),\quad \zeta_1 := \exp(\frac{2\pi\sqrt{-1}}{q-1}),\quad \xi_1 := \exp(\frac{2\pi\sqrt{-1}}{q+1})$

Generic table for $SL_2(q)$ with odd q (cont.)

$SL_2(q)$	$C_3(i)$	$C_4(i)$	$C_5(i)$
χ_1	1	1	1
χ_2	0	1	-1
χ_3	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i} - \frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$	$(-1)^i$	0
χ_4	$\frac{1}{2}(-1)^{\frac{1}{2}(q-1)i} + \frac{1}{2}\sqrt{q}\varepsilon_4^{\frac{1}{2}(q-1)}$	$(-1)^i$	0
χ_5	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i} - \frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$	0	$-(-1)^i$
χ_6	$-\frac{1}{2}(-1)^{\frac{1}{2}(q+1)i} + \frac{1}{2}\sqrt{q}\varepsilon_4^{2i+\frac{1}{2}q-\frac{1}{2}}$	0	$-(-1)^i$
$\chi_7(k)$	$(-1)^{ik}$	$\zeta_1^{ik} + \zeta_1^{-ik}$	0
$\chi_8(k)$	$-(-1)^{ik}$	0	$-\xi_1^{ik} - \xi_1^{-ik}$

$$\varepsilon_4 := \exp(\tfrac{2\pi\sqrt{-1}}{4}), \quad \zeta_1 := \exp(\tfrac{2\pi\sqrt{-1}}{q-1}), \quad \xi_1 := \exp(\tfrac{2\pi\sqrt{-1}}{q+1})$$

Observations about generic character tables

Fix type of G by root datum with Frobenius action

- There are finitely many types of centralizers of semisimple elements
- The number of unipotent classes is finite
- The smallest non-trivial degree is a ”polynomial in q”
- Many numbers here are PORC (polynomial on residue classes)
- For $E_8(q)$ the generic character tables have > 10000 columns and rows, largest entries have $|W|$ summands with 8 parameters for classes and characters each
- Some applications only need parts of the table,
- e.g., my webpage has lists of all (generic) irreducible character degree for simple groups up to rank 9