Representations of reductive groups in defining characteristic

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Setup and Notation

$K = \bar{\mathbb{F}}_p$

G simple reductive group $/K$ with root datum $(X, \Phi, Y, \Phi^\vee)$

Assume G simply connected ($Y = \mathbb{Z}\Phi^\vee$, SL, Sp, Spin, …)

We consider rational representations $G \to \mathrm{GL}(M)$, $M = K^n$ (morphism of varieties)

$T < B < G$ maximal torus and Borel subgroup, this determines

$\Phi = \Phi^+ \dot\cup -\Phi^+$ (Φ^+ the positive roots)

$\omega_1, \ldots, \omega_l$ a $\mathbb{Z}$-basis of X, dual to simple coroots (fundamental weights)

$X^+ = \langle \omega_1, \ldots, \omega_l \rangle_{\mathbb{N}_0}$ (dominant weights)

Partial order on X: $\mu \leq \lambda$ iff $\lambda - \mu \in \mathbb{N}_0\Phi^+$

Weights

Restrict M to T (consider M as KT-module), then

$$M = \bigoplus_{\mu \in X} M_\mu, \text{ where}$$

$M_\mu = \{v \in M \mid vt = \mu(t)v \text{ for all } t \in T\}$ (weight spaces)

$\Lambda(M) := \{\mu \in X \mid M_\mu \neq 0\}$ (weights of M)

$\mathrm{ch}(M) : X \to \mathbb{N}_0, \mu \mapsto \dim M_\mu$ (character of M)

Theorem. [Chevalley '50s] If M is irreducible then there exists $\lambda \in \Lambda(M)$ with $\lambda \geq \mu$ for all $\mu \in \Lambda(M)$ (highest weight), $\lambda \in X^+$ is dominant.

For each dominant $\lambda \in X^+$ there is a unique irreducible module $L(\lambda)$ with highest weight λ.

Remark. This gives a parameterization of irreducible representations of G over K. But what are $\dim L(\lambda)$ and $\mathrm{ch} L(\lambda)$?

Restricted weights

$X_p := \{a_1\omega_1 + \ldots + a_l\omega_l \mid 0 \leq a_i < p\}$ (p-restricted weights)

Theorem. [Steinberg tensor product theorem, 60s] Let $\lambda \in X^+$ dominant and write $\lambda = \lambda_0 + p\lambda_1 + \ldots + p^k\lambda_k$ with $\lambda_i \in X_p$. Then

$$L(\lambda) = L(\lambda_0) \otimes_K L(\lambda_1)^{F_p} \otimes \cdots \otimes_K L(\lambda_k)^{F_p^k}.$$

Here, M^{F_p} means the twist with the standard Frobenius F_p on M.

(So, for fixed p, the determination of all $\mathrm{ch} L(\lambda)$ reduces to a finite problem.)

Weyl modules

$\mathcal{L}$: complex simple Lie algebra with root system Φ

$\mathcal{U}$: its enveloping algebra, its irreducible (complex) representations are also parameterized by the dominant weights

$V(\lambda)_{\mathbb{C}}$: irreducible representation of $\mathcal{U}$ for $\lambda \in X^+$

Chevalley basis of $\mathcal{L}$ leads to $\mathbb{Z}$-lattice $V(\lambda)_{\mathbb{Z}}$ (Kostant $\mathbb{Z}$-form)

$V(\lambda) := V(\lambda)_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$ becomes a KG-module (Weyl module)

$V(\lambda)$ has a unique irreducible quotient $L(\lambda)$

Theorems. [Weyl, Freudenthal] $\dim V(\lambda)$ and $\mathrm{ch} V(\lambda)$ are known and computable.

Affine Weyl group

$W_p := p\mathbb{Z}\Phi \rtimes W$ affine Weyl group

acts on X, $(w, \mu) \mapsto w.\mu$

Let $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and define

$$C_{\mathbb{Z}} := \{\lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Phi^+\}$$
$$\overline{C_{\mathbb{Z}}} := \{\lambda \in X \mid 0 \le \langle \lambda + \rho, \alpha^\vee \rangle \le p \text{ for all } \alpha \in \Phi^+\}$$

Then $\overline{C_{\mathbb{Z}}}$ is a fundamental domain for the action of W_p on X

Theorem. [Jantzen, Andersen, 80s] (Linkage principle) For $\lambda \in X^+$ dominant write uniquely

$$\mathrm{ch} L(\lambda) = \sum_{\mu \in X^+} a_\mu \mathrm{ch} V(\mu).$$

Then $a_\mu \neq 0$ implies $\mu \in W_p.\lambda$.

Character formula

Let $\lambda_0 \in C_{\mathbb{Z}}$ (exists if $p > h$, Coxeter number) and $\lambda = w.\lambda_0$ and $\mu = v.\lambda_0$ dominant.

Theorem. [Riche, Williamson, 2019] Let $p > 2h - 1$. Then $a_\mu = {}^pP_{v,w}(1)$, where ${}^pP_{v,w}$ is a p-Kazhdan-Lusztig polynomial.

For p "big enough" ${}^pP_{v,w} = P_{v,w}$, the Kazhdan-Lusztig polynomial.

Theorem. [Jantzen, Andersen, 80s] (Translation principle) Reduce computation of $\mathrm{ch}L(\lambda)$ for all $\lambda \in X_p$ to these cases.

Computing characters

Remarks. Character formula is difficult to evaluate, (p-)Kazhdan-Lusztig polynomials are difficult to compute.
There is no formula or conjecture for small $p \leq 2h - 1$.

There is a bilinear form on $V(\lambda)_{\mathbb{Z}}$ which can be evaluted using commutator relations in $\mathcal{U}$, vectors for different weights are orthogonal. Let B be the Gram matrix of this form restricted to $(V(\lambda)_{\mathbb{Z}})_\mu$ for a weight μ.

Theorem. [Wong] The dimension of $L(\lambda)_\mu$ in characteristic p is the rank of B mod p (as matrix over $\mathbb{F}_p$).

Combining this with various mathematical results and algorithmic ideas we were able to compute many explicit characters for groups of not too large rank and $V(\lambda)$ with weight spaces up to dimension about 10000.

Finite groups G^F

$F : G \to G$ Frobenius, such that $F^k = F_{q^k}$, q an integer

Theorem. [Steinberg 60's] The irreducible representations of G^F over K are the

$$\{L(\lambda)\,|_{G^F} \mid \lambda \in X_q\} \qquad (q\text{-restricted weights})$$

Remarks.

- The same G-representations are restricted to twisted and untwisted groups.
- [Brunat-L., 2014] Essentially reduce the case of G^F from arbitrary reductive G to the case above.
  (In general it is no longer true that all irreducibles of G^F are restrictions of irreducibles of G.)

An application: Representations of small degree in defining characteristic

Type E_8, bound $M = 100000$

All highest weights λ and primes p such that $L(\lambda)$ in characteristic p has degree at most M are:

deg	λ	p	deg	λ	p	deg	λ	p
1	(00000000)	all	23125	(00000002)	7	30132	(00000010)	3
248	(00000001)	all	26504	(00000010)	2	30132	(00000010)	5
3626	(10000000)	2	26999	(00000002)	31	30380	(00000010)	$\neq$2,3,5
3875	(10000000)	$\neq$2	27000	(00000002)	$\neq$7,31			

Have similar results for all types of groups up to rank 10 (and more) and also have the characters of these representations.