Gravitational-wave source modelling using analytical methods: a voice in the discussion panel

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The Future of Gravitational-Wave Astronomy

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Subjective To-Do List for the Post-Newtonian (PN) Two-Body Problem

- Computation of gravitational-wave luminosity of two-point-mass system at the 4PN order, both for circular and for generic quasi-elliptical orbits. This will enable construction of the 4.5PN-accurate templates for inspiralling compact binaries.
- Computation, within the PN framework, higher-order (at least 3PN-order) tidal corrections to dynamics of binaries containing neutron stars.
- Higher-order perturbative solutions of two-body problem are complicated, both from computational and from conceptual point of view. Therefore it is highly desired to have more than one independent derivation of any analytical result:
  - — making derivation, within the ADM Hamiltonian approach, of gravitational-wave luminosities of two-point-mass system at the 2PN, 3PN, ... order;
  - — making derivation of 4PN two-point-body equations of motion using an extended body model.
- General relativity is classical (non-quantum) theory, therefore one can expect that all regularization issues related to usage of Dirac-delta sources in the derivation of PN two-body results, should be resolved in 3-dimensional space.
  However, it seems today that the only procedure that solves all regularization problems is dimensional regularization (DR).
  One can look for some extension/modification of Schwartz distribution theory that would be suitable for 3-dimensional regularization.
  Such theory would simplify a lot higher-order PN computations.

PN Two-Body Problem (Without Spins)

There are two sub-problems, usually analyzed separately:

- problem of finding equations of motion (EOM),
- problem of computing gravitational-wave luminosity.

EOM	N	1PN	2PN	2.5PN	3PN	3.5PN	4PN	4.5PN	5PN	5.5PN	6PN	6.5PN
Luminosity	—	—	—	N	—	1PN	1.5PN	2PN	2.5PN	3PN	3.5PN	4PN

Deriving equations of motion:
after fixing the gauge (e.g. TT gauge within ADM Hamiltonian formalism or harmonic coordinates within Fokker-action or EFT approaches) one perturbatively solves (in space dimension $d = 3 - \varepsilon$) Einstein's equations usually using point masses (i.e. Dirac-delta sources);
UV divergences linked to point-particle description removed by DR;
presence IR divergences on top of the UV divergences linked to nonlocality in time (related to tail effects).

PN Two-Body EOM (Without Spins)

- 1PN ($\propto v^2/c^2$):
  Lorentz–Droste 1917, Einstein–Infeld–Hoffmann 1938.
- 2PN ($\propto v^4/c^4$):
  Ohta–Okamura–Kimura–Hiida 1974, Damour–Deruelle 1981, Damour 1982, Schäfer 1985, Kopeikin 1985.
- 2.5PN ($\propto v^5/c^5$):
  Damour–Deruelle 1981, Damour 1982, Schäfer 1985, Kopeikin 1985.
- 3PN ($\propto v^6/c^6$):
  Jaranowski–Schäfer 1998, Blanchet–Faye 2000, Damour–Jaranowski–Schäfer 2001, Itoh–Futamase 2003, Blanchet–Damour–Esposito-Farèse 2004, Foffa–Sturani 2011.
- 3.5PN ($\propto v^7/c^7$):
  Iyer–Will 1993, Jaranowski–Schäfer 1997, Pati–Will 2002, Königsdörffer–Faye–Schäfer 2003, Nissanke–Blanchet 2005, Itoh 2009.
- 4PN ($\propto v^8/c^8$; new feature—nonlocality in time):
  Damour–Jaranowski–Schäfer 2014 (Bini–Damour 2013), Bernard–Blanchet–Bohé–Faye–Marsat–Marchand 2017 (Bernard–Blanchet–Bohé–Faye–Marchand–Marsat 2017–2018), Foffa–Porto–Rothstein–Sturani 2019.

♣ Usage of δs simplifies computations.

♣ Effacement principle (Damour 1983): dimensions and internal structure of compact and nonrotating bodies enter their EOM only at the 5PN order.

♣ One can use δs to model source terms for black-hole spacetimes, e.g. the Brill-Lindquist solution of time-symmetric two-black-hole initial value problem (Jaranowski & Schäfer 1999).

♣ For two-body systems δ-sources together with DR give unique conservative EOM up to the 4PN order and gravitational-wave luminosities up to the 3.5PN order and for the 4.5PN order (for circular orbits).

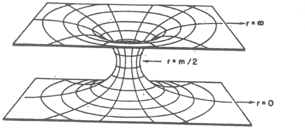

Fig. 1. A two-dimensional analog of the Schwarzschild-Kruskal manifold is shown isometrically imbedded in flat three-space. The figure shows the curvature and topology of the metric

$$ds^2=(1+m/2r)^4(dr^2+r^2d\theta^2).$$

The sheets at the top and bottom of the funnel continue to infinity and represent the asymptotically flat regions of the manifold ($r\rightarrow 0$, $r\rightarrow\infty$).

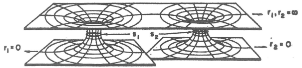

Fig. 2. A two-dimensional analog of the hypersurface of time symmetry of a manifold containing two "throats" is shown isometrically imbedded in flat three-space. The figure illustrates the curvature and topology for a system of two "particles" of equal mass m, and separation large compared to m, described by the metric

$$ds^2=(1+m/2r_1+m/2r_2)^4 ds_F^2.$$

3PN Conservative Two-Body EOM (Without Spins)

There exist three independent and compatible derivations using δ-sources and DR:

- ADM-Hamiltonian-based derivation
  (Damour, Jaranowski, & Schäfer 1998–2001),
  initial purely 3-dimensional derivation plagued by two UV-divergence-related ambiguity parameters, final non-ambiguous derivation used DR (one ambiguity parameter also fixed by requirement of Poincaré symmetry);
- harmonic-coordinate-based derivation
  (Blanchet, Damour, Esposito-Farèse, & Faye 2000–2004),
  initial purely 3-dimensional Lorentz-invariant derivation plagued by one UV-divergence-related ambiguity parameter,
  final non-ambiguous derivation used DR;
- effective-field-theory approach (Foffa & Sturani 2011),
  non-ambiguous derivation using DR.

There exist other pure 3-dimensional derivations of conservative 2PN and 3PN EOM using extended body models; they are compatible with EOM derived for the δ-sources:

- 2PN-accurate EOM by Grishchuk & Kopeikin (1985–1986);
- 2PN-accurate EOM by Pati & Will (2000–2002)
  within a direct integration of the relaxed Einstein equations;
- 3PN-accurate EOM by Itoh & Futamase (2003–2004)
  using a surface-integral approach.

4PN Conservative Two-Body EOM (Without Spins)

There exist three independent and compatible derivations using δ-sources and DR:

- ADM-Hamiltonian-based derivation (Damour–Jaranowski–Schäfer 2014)—the one IR-divergence related ambiguity parameter fixed by using results from gravitational self-force calculations (Bini–Damour 2013);
- Fokker-action-based derivation (Bernard–Blanchet–Bohé–Faye–Marsat–Marchand 2017)—the two IR-divergence ambiguity parameters fixed by using results from gravitational self-force calculations; then improved (without the use of ambiguity parameters) by rederivation of the conservative tail contribution using DR, but with a combined ("$\epsilon\eta$") regularization implemented (Bernard–Blanchet–Bohé–Faye–Marchand–Marsat 2017–2018);
- effective-field-theory approach (Foffa–Porto–Rothstein–Sturani 2019)—a pure DR calculation (where use has been made of the zero-bin subtraction method for interrelated UV and IR poles).

> *(...) the derivations in both the ADM (...) and Fokker-action (...) formalisms left room open for further improvement and clarifications, in particular, with regards to the handling of IR divergences (...) and the apparent reliance on an extra regulator beyond DR. (...) Moreover, while the renormalization procedures presented elsewhere led to the correct result, a more systematic removal of IR/UV divergences will be needed when physical logarithms in the near zone first appear at higher PN orders (due to finite-size effects). We address all of these issues (...) by providing an ambiguity-free and systematic derivation of the renormalized Lagrangian in DR, all within the confines of the PN expansion, which can be naturally extended to all orders.* [Foffa *et al.*, Phys. Rev. D **100**, 024048 (2019)]

Gravitational Waves from Inspiralling Binary on Circular Orbits

The gravitational-wave strain measured by the laser-interferometric detector and induced by gravitational waves from coalescing compact binary (made of nonspinning bodies) in circular orbits during inspiral phase:

$$\boxed{h(t) = \frac{C}{D}\left[\dot{\phi}(t)\right]^{2/3} \sin\left[2\phi(t) + \alpha\right],}$$

where $\phi(t)$ is the orbital phase of the binary [so $\dot{\phi}(t) := \mathrm{d}\phi(t)/\mathrm{d}t$ is the angular frequency],
D is the luminosity distance of the binary to the Earth, C and α are some constants.

The orbital phase $\phi(t)$ is computed from the balance equation:

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{L} \quad \Longrightarrow \quad \phi = \phi(t),}$$

which both sides have the following PN expansions:

$$E = \boxed{E_{\mathrm{N}}} + \frac{1}{c^2}\boxed{E_{1\mathrm{PN}}} + \frac{1}{c^4}\boxed{E_{2\mathrm{PN}}} + \frac{1}{c^6}\boxed{E_{3\mathrm{PN}}} + \frac{1}{c^8}\boxed{E_{4\mathrm{PN}}} + \frac{1}{c^{10}} E_{5\mathrm{PN}} + \mathcal{O}\Big((v/c)^{11}\Big),$$

$$\mathcal{L} = \boxed{\mathcal{L}_{\mathrm{N}}} + \frac{1}{c^2}\boxed{\mathcal{L}_{1\mathrm{PN}}} + \frac{1}{c^3}\boxed{\mathcal{L}_{1.5\mathrm{PN}}} + \frac{1}{c^4}\boxed{\mathcal{L}_{2\mathrm{PN}}} + \frac{1}{c^5}\boxed{\mathcal{L}_{2.5\mathrm{PN}}}$$
$$+ \frac{1}{c^6}\boxed{\mathcal{L}_{3\mathrm{PN}}} + \frac{1}{c^7}\boxed{\mathcal{L}_{3.5\mathrm{PN}}} + \frac{1}{c^8}\mathcal{L}_{4\mathrm{PN}} + \frac{1}{c^9}\mathcal{L}_{4.5\mathrm{PN}} + \frac{1}{c^{10}}\mathcal{L}_{5\mathrm{PN}} + \mathcal{O}\Big((v/c)^{11}\Big).$$

Notation

Masses of the bodies: m_1, m_2, $\quad M := m_1 + m_2$, $\quad \mu := \dfrac{m_1 m_2}{M}$, $\quad \nu := \dfrac{\mu}{M} = \dfrac{m_1 m_2}{(m_1 + m_2)^2}$, $\quad 0 \leq \nu \leq \dfrac{1}{4}$.

Dimensionless PN parameter introduced for circular orbits $\quad x := \dfrac{1}{c^2}(GM\dot{\phi})^{2/3}$.

5PN-Accurate Bounding Energy in the Center-of-Mass Frame for Circular Orbits

$$E(x;\nu) = -\frac{\mu c^2 x}{2}\Bigg(1 + e_{1PN}(\nu)\,x + e_{2PN}(\nu)\,x^2 + e_{3PN}(\nu)\,x^3 + \left(e_{4PN}(\nu) + \frac{448}{15}\nu \ln x\right)x^4$$

$$+ \left(e_{5PN}(\nu) - \left(\frac{4988}{35}\nu + \frac{656}{5}\nu^2\right)\ln x\right)x^5 + \mathcal{O}\left((v/c)^{11}\right)\Bigg),$$

$$e_{1PN}(\nu) = -\frac{3}{4} - \frac{1}{12}\nu, \qquad e_{2PN}(\nu) = -\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2,$$

$$e_{3PN}(\nu) = -\frac{675}{64} + \left(\frac{34445}{576} - \frac{205}{96}\pi^2\right)\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3,$$

$$e_{4PN}(\nu) = -\frac{3969}{128} + \left(-\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}(2\ln 2 + \gamma)\right)\nu + \left(-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right)\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4,$$

$$e_{5PN}(\nu) = -\frac{45927}{512} + \cdots + \frac{1}{512}\nu^5$$

(γ is the Euler's constant).

- At nth PN order the term $\propto \nu^n$ carries at most one power of G.
- The 5PN-order coefficient $+\frac{1}{512}\nu^5$ displayed above can be computed from the 1st post-Minkowskian (1PM) closed-form ADM Hamiltonian derived by Ledvinka, Schäfer, and Bičák (2008).
- The same 5PN-order $\propto \nu^5$ coefficient was computed by Foffa (2014) from the 1PM Lagrangian derived by him within the effective field theory approach, with the result $+\frac{3121}{32}\nu^5$.

4PN-Accurate 2-Point-Mass ADM Conservative Hamiltonian

$$H_{\leq 4\mathrm{PN}}[\mathbf{x}_a, \mathbf{p}_a] = H^{\mathrm{local}}_{\leq 4\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H^{\mathrm{nonlocal}}_{4\mathrm{PN}}[\mathbf{x}_a, \mathbf{p}_a] \quad (a = 1, 2).$$

Local-in-time 4PN-accurate Hamiltonian

$$H^{\mathrm{local}}_{\leq 4\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) = (m_1 + m_2)c^2 + H_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^2} H_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^4} H_{2\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^6} H_{3\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) + \frac{1}{c^8} H^{\mathrm{local}}_{4\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a).$$

Nonlocal-in-time 4PN Hamiltonian

(Blanchet-Damour 1988, Damour-Jaranowski-Schäfer 2014)

$$H^{\mathrm{nonlocal}}_{4\mathrm{PN}}[\mathbf{x}_a, \mathbf{p}_a] = -\frac{1}{5}\frac{G^2 M}{c^8} \dddot{I}_{ij} \times \mathrm{Pf}_{2r_{12}/c}\left(\int_{-\infty}^{+\infty} \frac{\mathrm{d}v}{|v|} \dddot{I}_{ij}(t+v)\right),$$

$\dddot{I}_{ij}$ is a 3rd time derivative of the Newtonian quadrupole moment I_{ij} of the system,

$$I_{ij} := \sum_a m_a \left(x_a^i x_a^j - \frac{1}{3}\delta^{ij}\mathbf{x}_a^2\right),$$

Pf_T denotes a Hadamard partie finie with time scale T,

$$\mathrm{Pf}_T \int_0^{+\infty} \frac{\mathrm{d}v}{v} g(v) := \int_0^T \frac{\mathrm{d}v}{v}\left(g(v) - g(0)\right) + \int_T^{+\infty} \frac{\mathrm{d}v}{v} g(v).$$

Newtonian/1PN/2PN Hamiltonians

The operation "$+(1 \leftrightarrow 2)$" used below denotes the addition for each term of another term obtained by the label permutation $1 \leftrightarrow 2$.

$$H_{\rm N}(\mathbf{x}_a,\mathbf{p}_a) = \frac{\mathbf{p}_1^2}{2m_1} - \frac{Gm_1m_2}{2r_{12}} + (1 \leftrightarrow 2),$$

$$H_{1\rm PN}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{(\mathbf{p}_1^2)^2}{8m_1^3} + \frac{Gm_1m_2}{4r_{12}}\left(-6\,\frac{\mathbf{p}_1^2}{m_1^2} + 7\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right) + \frac{G^2m_1^2m_2}{2r_{12}^2} + (1 \leftrightarrow 2),$$

$$\begin{aligned}H_{2\rm PN}(\mathbf{x}_a,\mathbf{p}_a) = {} & \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\,\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + 5\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} - 6\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right) \\ & + \frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27\,(\mathbf{p}_1\cdot\mathbf{p}_2) + 6\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right) - \frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2} + (1 \leftrightarrow 2).\end{aligned}$$

3PN Hamiltonian (Damour-Jaranowski-Schäfer 2001)

$$\begin{aligned}H_{3\rm PN}(\mathbf{x}_a,\mathbf{p}_a) = {} & -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{Gm_1m_2}{r_{12}}\Bigg(-14\,\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\,\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 4\,\mathbf{p}_1^2\,\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4m_2^2} + 6\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2} - 10\,\frac{(\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2 + \mathbf{p}_2^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4m_2^2} \\ & + 24\,\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2} + 2\,\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} + \frac{(7\,\mathbf{p}_1^2\,\mathbf{p}_2^2 - 10\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3} + \frac{(\mathbf{p}_1^2\,\mathbf{p}_2^2 - 2\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3} \\ & + 15\,\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3} - 18\,\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3} + 5\,\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\Bigg) + \frac{G^2m_1m_2}{r_{12}^2}\Bigg(\frac{1}{16}(m_1 - 27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4} \\ & - \frac{115}{16}m_1\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} + \frac{1}{48}m_2\frac{25\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 371\,\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2} + \frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3} - \frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2} \\ & - \frac{1}{8}m_1\frac{(15\,\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2) + 11\,(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2} + \frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} \\ & - \frac{1}{48}(220m_1 + 193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\Bigg) + \frac{G^3m_1m_2}{r_{12}^3}\Bigg(-\frac{1}{48}\left(425\,m_1^2 + \left(473 - \frac{3}{4}\pi^2\right)m_1m_2 + 150\,m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2} \\ & + \frac{1}{16}\left(77(m_1^2+m_2^2) + \left(143 - \frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \frac{1}{16}\left(20\,m_1^2 - \left(43 + \frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} \\ & + \frac{1}{16}\left(21(m_1^2+m_2^2) + \left(119 + \frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\Bigg) + \frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left(\left(\frac{227}{3} - \frac{21}{4}\pi^2\right)m_1 + m_2\right) + (1 \leftrightarrow 2).\end{aligned}$$

4PN Local-in-time Hamiltonian (Damour-Jaranowski-Schäfer 2014)

$$
\begin{aligned}
H^{\rm local}_{\rm 4PN}(\mathbf{x}_a,\mathbf{p}_a) = {}& \frac{7(\mathbf{p}_1^2)^5}{256m_1^9} + \frac{Gm_1m_2}{r_{12}}\bigg(\frac{45(\mathbf{p}_1^2)^4}{128m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64m_1^6m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16m_1^6m_2^2} - \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32m_1^6m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64m_1^6m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256m_1^5m_2^3} \\
&+ \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128m_1^5m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256m_1^5m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3} - \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64m_1^5m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64m_1^5m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128m_1^5m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256m_1^5m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64m_1^4m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^4m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64m_1^4m_2^4} \\
&+ \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64m_1^4m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16m_1^4m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16m_1^4m_2^4} - \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32m_1^4m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64m_1^4m_2^4} \\
&- \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32m_1^4m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128m_1^4m_2^4}\bigg) + \frac{G^2m_1m_2}{r_{12}^2}m_1\bigg(\frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128m_1^5m_2} + \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16m_1^5m_2} \\
&- \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128m_1^5m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128m_1^5m_2} + \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256m_1^5m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480m_1^4m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840m_1^4m_2^2} \\
&- \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320m_1^4m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480m_1^4m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280m_1^4m_2^2} - \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840m_1^4m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920m_1^4m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840m_1^4m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840m_1^4m_2^2} \\
&- \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8m_1^3m_2^3} + \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192m_1^3m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2^3} + \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192m_1^3m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96m_1^3m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96m_1^3m_2^3} \\
&- \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32m_1^3m_2^3} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384m_1^3m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4m_1^2m_2^4} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2m_1^2m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16m_1^2m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6m_1^2m_2^4} \\
&+ \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48m_1^2m_2^4} - \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24m_1^2m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96m_1^2m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96m_1^2m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48m_1^2m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8m_2^6}\bigg) + \frac{G^3m_1m_2}{r_{12}^3}\bigg(m_1^2\bigg(\frac{5027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152m_1^4} \\
&- \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640m_1^3m_2} + \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920m_1^3m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384m_1^3m_2} + \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800m_1^3m_2} - \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960m_1^2m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800m_1^2m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400m_1^2m_2^2} \\
&+ \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400m_1^2m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600m_1^2m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800m_1^2m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32m_2^4}\bigg) + m_1m_2\bigg(\bigg(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\bigg)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \bigg(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \bigg(\frac{375\pi^2}{8192} - \frac{23533}{1280}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\
&+ \bigg(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\bigg)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \bigg(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\bigg)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2} + \bigg(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2m_2^2} + \bigg(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2m_2^2} \\
&- \bigg(\frac{30383}{960} + \frac{36405\pi^2}{16384}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2} + \bigg(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\bigg)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2} + \bigg(\frac{2369}{60} + \frac{35655\pi^2}{16384}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2} + \bigg(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3m_2} \\
&+ \bigg(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}\bigg)\bigg) + \frac{G^4m_1m_2}{r_{12}^4}\bigg(m_1^3\bigg(\frac{64861\mathbf{p}_1^2}{4800m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8m_1m_2} + \frac{105\mathbf{p}_2^2}{32m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2m_1m_2}\bigg) + m_1^2m_2\bigg(\bigg(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\bigg)\frac{\mathbf{p}_1^2}{m_1^2} \\
&+ \bigg(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\bigg)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2} + \bigg(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\bigg)\frac{\mathbf{p}_2^2}{m_2^2} + \bigg(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} + \bigg(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2} + \bigg(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2}\bigg)\bigg) \\
&+ \frac{G^5m_1m_2}{r_{12}^5}\bigg(-\frac{m_1^4}{16} + \bigg(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\bigg)m_1^3m_2 + \bigg(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\bigg)m_1^2m_2^2\bigg) + (1\leftrightarrow 2).
\end{aligned}
$$

Looking for the Correct Three-dimensional Dirac-delta-related Regularization (1)

- "Good" δ-functions of Infeld and Plebański (1954–60); they satisy, besides having the properties of ordinary Dirac δ-functions, the condition

$$\frac{1}{|\mathbf{x}-\mathbf{x}_0|^k}\delta(\mathbf{x}-\mathbf{x}_0)=0, \quad k=1,\dots,p \quad \text{(for some positive integer } p\text{)}.$$

- A natural generalization of the concept of "good" δ-functions is "partie finie" value of function at its singular point $\mathbf{x}_0$ (here M is some non-negative integer):

$$f(\mathbf{x}_0+\epsilon\mathbf{n})=\sum_{m=-M}^{\infty}a_m(\mathbf{n})\epsilon^m, \quad \mathbf{n}:=\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|}, \quad f_{\mathrm{reg}}(\mathbf{x}_0):=\frac{1}{4\pi}\int \mathrm{d}\Omega\, a_0(\mathbf{n}).$$

- Infeld and Plebański assumed that the "tweedling of products" is always satisfied:

$$(f_1 f_2)_{\mathrm{reg}}(\mathbf{x}_0)=f_{1\mathrm{reg}}(\mathbf{x}_0)f_{2\mathrm{reg}}(\mathbf{x}_0),$$

  but this is generally wrong for arbitrary singular functions f_1 and f_2. Problems with fulfilling this property begin at the 3PN order.

- All contact integrals are evaluated as

$$\int \mathrm{d}^3x\, f(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0):=f_{\mathrm{reg}}(\mathbf{x}_0).$$

Looking for the Correct Three-dimensional Dirac-delta-related Regularization (2)

- Another consequence of employing Dirac-delta sources is necessity to differentiate singular and homogeneous functions using a distributional derivative.
- Let f be a function defined in a neighbourhood of the origin of $\mathbb{R}^3$, it is said to be a positively homogeneous function of degree λ, if for any number $a > 0$

$$f(a\,\mathbf{x}) = a^{\lambda}\, f(\mathbf{x}).$$

  Let $k := -\lambda - 2$. If λ is an integer and if $\lambda \leq -2$ (i.e., k is a nonnegative integer), then within the standard distribution theory one derives the formula

$$\bar{\partial}_i f(\mathbf{x}) = \partial_i f(\mathbf{x}) + \frac{(-1)^k}{k!}\,\frac{\partial^k \delta(\mathbf{x})}{\partial x^{i_1}\cdots\partial x^{i_k}} \times \oint_{\Sigma} \mathrm{d}\sigma_i\, f(\mathbf{x}')\, x'^{i_1}\cdots x'^{i_k},$$

  where $\bar{\partial}_i f$ on the lhs denotes the derivative of f considered as a distribution, while $\partial_i f$ on the rhs denotes the derivative of f considered as a function (which is computed using the standard rules of differentiation), Σ is any smooth close surface surrounding the origin and $\mathrm{d}\sigma_i$ is the surface element on Σ.
- The distributional derivative does not obey the Leibniz's rule. Let us suppose that it does, then

$$\bar{\partial}_i \frac{1}{r^3} = \bar{\partial}_i \left(\frac{1}{r}\,\frac{1}{r^2}\right) = \frac{1}{r^2}\,\bar{\partial}_i \frac{1}{r} + \frac{1}{r}\,\bar{\partial}_i \frac{1}{r^2}.$$

  But the rhs can be computed using standard differential calculus (no terms with δ), whereas computing the lhs one obtains some term proportional to $\partial_i \delta$.

Looking for the Correct Three-dimensional Dirac-delta-related Regularization (3)

- The Riesz-implemented Hadamard regularization is based on the Hadamard "partie finie" and the Riesz analytic continuation procedures; it relies on multiplying the full integrand, say $i(\mathbf{x})$, of the divergent integral by two regularization factors,
$$i(\mathbf{x}) \longrightarrow i(\mathbf{x})\left(\frac{r_1}{s_1}\right)^{\epsilon_1}\left(\frac{r_2}{s_2}\right)^{\epsilon_2},$$
and studying the double limit $\epsilon_1 \to 0$, $\epsilon_2 \to 0$ within analytic continuation in the complex ϵ_1 and ϵ_2 planes (here s_1 and s_2 are arbitrary 3-dimensional UV regularization scales).
- The final result of employing the 3-dimensional regularization procedures described above is ambiguous—it depends on the way one writes integrands when transforming them using standard rules: integration by parts and Leibniz's rule.

The Extended Hadamard Regularization (EHR)

- The EHR is a specific variant of 3-dimensional Hadamard regularization devised by Blanchet & Faye and used by them at the 3PN-level computations of two-point-mass EOM in harmonic coordinates (2000–01).
- The basic idea: to associate to any function $F \in \mathcal{F}$, where the set $\mathcal{F}$ comprises functions smooth on $\mathbb{R}^3$ except for the two points (around which they admit a power-like singular expansion) a pseudo-function $\mathrm{Pf}F$, which is a linear form acting on functions from $\mathcal{F}$:

$$\langle \mathrm{Pf}F, G\rangle := \mathrm{Pf}_{s_1,s_2}\int \mathrm{d}^3x\, FG, \quad \text{for any } G \in \mathcal{F},$$

  where Pf_{s_1,s_2} means partie finie of the divergent integral (it depends on two—one per each singularity—arbitrary regularization scales s_1 and s_2).
- The Dirac δ-functions δ_a are represented by the pseudo-functions $\mathrm{Pf}\delta_a$ defined by

$$\langle \mathrm{Pf}\delta_a, G\rangle := G_{\mathrm{reg}}(\mathbf{x}_a), \quad \text{for any } G \in \mathcal{F},$$

  The product $F\delta_a$ is represented by another pseudo-function $\mathrm{Pf}(F\delta_a)$:

$$\langle \mathrm{Pf}(F\delta_a), G\rangle := (FG)_{\mathrm{reg}}(\mathbf{x}_a), \quad \text{for any } G \in \mathcal{F}.$$

  As a consequence, in general $\mathrm{Pf}(F\delta_a) \neq F_{\mathrm{reg}}(\mathbf{x}_a)\mathrm{Pf}\delta_a$.
- To ensure the possibility of integration by parts, partial derivatives of singular functions are specifically treated. This leads to a distributional derivative, which differs in general from the Schwartz derivative. E.g.,

$$\bar{\partial}_i \mathrm{Pf}\frac{1}{r} = -\mathrm{Pf}\frac{n^i}{r^2} + 2\pi\mathrm{Pf}(rn^i\delta), \quad \text{Schwartz derivative gives} \quad \bar{\partial}_i\frac{1}{r} = -\frac{n^i}{r^2}.$$

The definitions adopted by EHR disagree with the DR rules.

- In generic d dimensions one can always use

$$F^{(d)}(\mathbf{x})\delta^{(d)}(\mathbf{x}-\mathbf{x}_a) = F^{(d)}_{\mathrm{reg}}(\mathbf{x}_a)\delta^{(d)}(\mathbf{x}-\mathbf{x}_a),$$

  where $F^{(d)}$ is the d-dimensional version of 3-dimensional F. This leads to the following DR rule, which disagrees with the EHR rule:

$$\left[F(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{x}_a)\right]_{\mathrm{reg}} := \Big(\lim_{d\to 3} F^{(d)}_{\mathrm{reg}}(\mathbf{x}_a)\Big)\delta^{(3)}(\mathbf{x}-\mathbf{x}_a).$$

- The EHR differentiation when applied to smooth functions of compact support, coincides with Schwartz differentiation. However, in the 3PN-level computations it operated with other singular functions and gave the results different from the results obtained by applying Schwartz differentiation.
  The definition of Schwartz differentiation is valid in d dimensions, what supports the use of this definition also in three dimensions.
- The computation using EHR can not be combined with DR. This can be seen from DR completion of the 3PN EOM in harmonic coordinates: before applying DR it was necessary to subtract all contributions, which were direct consequences of the use of EHR.
  However, at the 3PN level the difference between the final results of EHR and DR computations of two-point-mass EOM can be described in terms of one ambiguity parameter.

Looking for the Proper Modification of the Schwartz Theory?

- Inspired by the EHR of Blanchet & Faye, mathematicians have recently developed the theory of "thick distributions": Estrada & Fulling (2007) in one dimension, and Yang & Estrada (2013) in higher dimensions. This theory is connected with the EHR, but is not equivalent to the latter and it can not be used to improve regularization issues in the PN two-body problem.

> *(...) it is not correct to say that the work of Laurent Schwartz justifies everything that physicists do with the Dirac delta function, because sometimes they do things that are clearly wrong. There is a spectrum of responses to this situation. The first (chosen by too many mathematicians) is to dismiss distributions as untrustworthy, a kind of pornography that should be kept out of the hands of engineering and science students. Another (adopted by many practitioners) is to rationalize after the fact whatever interpretation of the symbols gives the right answer in the problem at hand (...) sometimes this is done in blatant contradiction to interpretations adopted in other contexts. A safer approach is to regard the delta function as a heuristic device that leads rapidly to formulas whose correctness must then be rigorously verified (e.g., by substituting a putative solution back into a differential equation). But one cannot be satisfied just with this; if distributions are unambiguously defined as linear functionals on spaces of test functions, then their properties must be unambiguous, and the mathematician should determine which formulas and calculational rules are true and why—tightening up the definitions when necessary.*
> [R.Estrada & S.A.Fulling, Int. J. Appl. Math. Stat. **10**, 25 (2007)]