Polar perturbations of the Schwarzschild background sourced by a radial infalling particle

1. Using the Green function approach, find a numerical solution of the sourced Zerilli equation

$$\frac{d^2Z(r)}{dr_\star^2} + [\omega^2 - V(r)]Z(r) = S_{pol} \ , \tag{1}$$

for a source term S_{pol} corresponding to a test particle falling radially from infinity with zero starting velocity. In the previous equation $r_\star$ is the tortoise coordinate, $V(r)$ is the Zerilli potential

$$V(r) = \frac{2(r-2M)\left(9M^3 + 9\Lambda M^2 r + 3\Lambda^2 M r^2 + \Lambda^2(\Lambda+1)r^3\right)}{r^4(3M+\Lambda r)^2} \ , \tag{2}$$

and the source function is explicitly given by (see also the supplementary `MATHEMATICA` file):

$$S_{pol} = \frac{2\sqrt{5}me^{i\omega T(r)}}{r\omega(3M+\Lambda r)^2}\left[\sqrt{2}(2M-r)\big(\omega(3M+\Lambda r)T'(r) + 2i\Lambda\big)\right. \tag{3}$$

$$\left. - 6\omega\sqrt{M^3 r} - 2\Lambda\omega\sqrt{Mr^3}\right] , \tag{4}$$

being $\Lambda = l(l+1)/2 - 1$. The function $T(r)$ and its derivative, for this orbital configuration, are given by:

$$\frac{T(r)}{M} = -\frac{\sqrt{2}}{3}\left(\frac{r}{M}\right)^{3/2} - 2\sqrt{2}\sqrt{\frac{r}{M}} + 2\log\left(\frac{\frac{\sqrt{\frac{r}{M}}}{\sqrt{2}}+1}{\frac{\sqrt{\frac{r}{M}}}{\sqrt{2}}-1}\right) \ , \tag{5}$$

and

$$T'(r) = -\frac{1}{\sqrt{2}\left(1-\frac{2M}{r}\right)\sqrt{\frac{M}{r}}} \ . \tag{6}$$

For the sake of simplicity just consider the $l = 2$ contribution. In the frequency domain, the solution will read:

$$Z(\omega, r) = Z_\infty \int_{-\infty}^{\infty} dr_\star \frac{S_{pol}Z_h}{W} \simeq e^{i\omega r_\star}\int_{-\infty}^{\infty} dr_\star \frac{S_{pol}Z_h}{W}$$
$$= e^{i\omega r_\star} Z(\omega) \ . \tag{7}$$

where Z_∞ and Z_h are the solutions of the homogeneous problem, satisfying the correct boundary conditions at infinity and at the horizon, respectively, and $W = Z'_\infty Z_h - Z_\infty Z'_h$ is their Wronskian[1].

[1] For our master equation the Wronskian has to be constant! Try it.

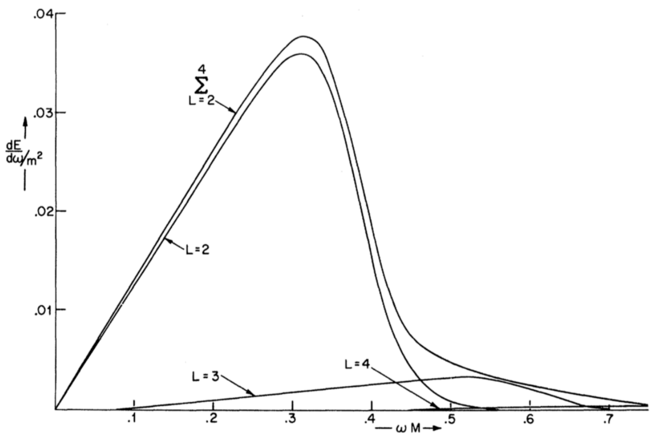

Figure 1: GW energy spectrum $dE/d\omega$ for the $\ell = 2$ polar mode and a radial infalling particle starting from rest at infinity. Taken from [1].

Having numerically integrated $Z(\omega)$ for a specific ω, try to compute the gravitational wave energy spectrum, i.e. the energy emitted for unit frequency:

$$\frac{dE}{d\omega} = \sum_l \frac{1}{32\pi} \frac{(l+2)!}{(l-2)!} \omega^2 |Z_{lm}(\omega)|^2 \quad \text{(sum over } l = 2 \text{ only)} , \tag{8}$$

and to reproduce the results of Davis et al. [1] shown in Fig. 1 (the data points of the original curve are also given as supplementary material).

2. Compute the time dependent gravitational waveform by inverting the Fourier domain solution found in the previous point, i.e.

$$\begin{aligned} Z(u) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z(\omega, r) e^{-i\omega t} d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z(\omega) e^{i\omega r_\star} e^{-i\omega t} d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Z(\omega) e^{-i\omega u} d\omega \\ &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \Re[Z(\omega) e^{-i\omega u}] d\omega , \end{aligned} \tag{9}$$

where $u = t - r_\star$ is the retarded time, and the last equality comes from the fact that $Z(t)$ is real, i.e. $Z(\omega) = Z^\star(-\omega)$[2].

[2] Test it with your code!

Bibliography

[1] M. Davis, R. RUFFINI, W. H. Press, and R. H. Price. Gravitational Radiation from a Particle Falling Radially into a Schwarzschild Black Hole. *Physical Review Letters*, 27(21):1466–1469, Nov. 1971.