

Spherical harmonic decomposition + first encounter with scattering equation

1. The following equation describes perturbations induced by a massless scalar field on the Schwarzschild background:

$$\frac{d^2\psi}{dr_\star^2} + \left[\omega^2 - f \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} \right) \right] \psi = S , \quad (1)$$

where $r_\star = r + 2M \ln(r/2M - 1)$ is the tortoise coordinate and $f = 1 - 2M/r$. In this case S is given by a gaussian packet, i.e.

$$S = f e^{-\frac{(r_\star - r_0)^2}{\sigma^2}} , \quad (r_0 = 8, \sigma = 5) . \quad (2)$$

Integrate the previous sourced equation for the parameters given by the problem, and find the power at infinity for the $l = 1$ mode as a function of the frequency ω . In particular:

- (a) Consider the homogeneous problem associated to eq. (1). At the horizon and at infinity the solution of the master equation can be written as a power series of the form:

$$\psi_h = e^{-i\omega r_\star} [a + b(r - 2M)] + \mathcal{O}(r - 2M)^2 , \quad (3)$$

and

$$\psi_\infty = e^{i\omega r_\star} (c + d/r) + \mathcal{O}(1/r^2) . \quad (4)$$

Find the coefficient (a, b, c, d) .

- (b) Find two solutions $\psi_{1,2}$ of the associated homogeneous problem, which can be obtained: (i) one starting from the horizon and integrating outward with boundary conditions given by ψ_h ; (ii) one starting from infinity integrating inward with boundary conditions given by ψ_∞ .
- (c) Compute the general solution at infinity, i.e.

$$\psi(\omega, r) = e^{i\omega r_\star} \int_{-\infty}^{\infty} \frac{\psi_1 S}{W} dr_\star , \quad (5)$$

where W is the wronskian of the two solutions $\psi_{1,2}$.

- (d) Compute the power $P = \omega^2 |\psi(\omega, r)|^2$ as a function of the frequency ω , and plot it in the interval $[0.1, 0.5]$.

2. The stress-energy tensor of a test particle orbiting around a BH can be expanded in a complete set of tensor harmonics, as:

$$\begin{aligned} \mathbf{T} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} & \left[\mathcal{A}_{\ell m}^{(0)} \mathbf{a}_{\ell m}^{(0)} + \mathcal{A}_{\ell m}^{(1)} \mathbf{a}_{\ell m}^{(1)} + \mathcal{A}_{\ell m} \mathbf{a}_{\ell m} + \mathcal{B}_{\ell m}^{(0)} \mathbf{b}_{\ell m}^{(0)} + \mathcal{B}_{\ell m} \mathbf{b}_{\ell m} \right. \\ & \left. + \mathcal{Q}_{\ell m}^{(0)} \mathbf{c}_{\ell m}^{(0)} + \mathcal{Q}_{\ell m} \mathbf{c}_{\ell m} + \mathcal{D}_{\ell m} \mathbf{d}_{\ell m} + \mathcal{G}_{\ell m} \mathbf{g}_{\ell m} + \mathcal{F}_{\ell m} \mathbf{f}_{\ell m} \right] . \end{aligned} \quad (6)$$

where (ℓ, m) are the multipole numbers. The *axial* ($\mathbf{c}_{\ell m}^{(0)}, \mathbf{c}_{\ell m}, \mathbf{d}_{\ell m},$) and *polar* ($\mathbf{a}_{\ell m}^{(0)}, \mathbf{a}_{\ell m}^{(1)}, \mathbf{a}_{\ell m}, \mathbf{b}_{\ell m}^{(0)}, \mathbf{b}_{\ell m}, \mathbf{g}_{\ell m}, \mathbf{f}_{\ell m}$) tensor harmonics basis are expressed in terms of the usual spherical harmonics $Y_{\ell m}(\theta, \phi)$ and their derivatives. For example we have:

$$\mathbf{a}_{\ell m} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_{\ell m}(\theta, \phi) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{a}_{\ell m}^{(0)} = \begin{pmatrix} Y_{\ell m}(\theta, \phi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (7)$$

$$\mathbf{a}_{\ell m}^{(1)} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & Y_{\ell m}(\theta, \phi) & 0 & 0 \\ Y_{\ell m}(\theta, \phi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (8)$$

$$\mathbf{b}_{\ell m}^{(0)} = \frac{ir}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & (\partial/\partial\theta)Y_{\ell m}(\theta, \phi) & 0 \\ 0 & 0 & 0 & 0 \\ (\partial/\partial\theta)Y_{\ell m}(\theta, \phi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (9)$$

while the expansion coefficients $(\mathcal{A}_{\ell m}^{(0)}, \mathcal{A}_{\ell m}^{(1)}, \dots, \mathcal{F}_{\ell m})$ can be computed exploiting the orthonormality properties of the tensor harmonics:

$$(A, B) = \int \int \eta^{\mu\rho} \eta^{\nu\sigma} A_{\mu\nu}^* B_{\rho\sigma} d\Omega, \quad (10)$$

where the superscript \star denotes complex conjugation and the inverse metric $\eta^{\mu\nu}$ is given by $\text{diag}(-1, 1, \frac{1}{r^2}, \frac{1}{r^2 \sin^2 \theta})$. For example, $\mathcal{A}^{(1)} = (\mathbf{a}^{(1)}, \mathbf{T})$.

We now consider the stress-energy tensor of a point particle of mass m_p , moving on a geodesics $x^\mu(\tau)$ of the Schwarzschild metric, being τ the proper time:

$$\begin{aligned} T^{\mu\nu} &= m_p \int u^\mu u^\nu \frac{\delta^{(4)}(x^\beta - y_p^\beta)}{\sqrt{-g}} d\tau \\ &= m_p \frac{dt}{d\tau} \frac{v^\mu v^\nu}{r^2 |\sin \theta_p|} \delta(r - r_p) \delta(\theta - \theta_p) \delta(\phi - \phi_p) \\ &= m_p \gamma \frac{v^\mu v^\nu}{r^2} \delta(r - r_p) \delta(\cos \theta - \cos \theta_p) \delta(\phi - \phi_p) \\ &= m_p \gamma \frac{v^\mu v^\nu}{r^2} \delta(r - r_p) \delta^{(2)}(\Omega - \Omega_p) \end{aligned} \quad (11)$$

where $v^\mu = dy_p^\mu/dt$ is the test particle 4-velocity, and γ the relativistic boost factor.

Compute the first four coefficients the expansion (6) for the stress-energy tensor (11), i.e. $(\mathcal{A}_{\ell m}, \mathcal{A}_{\ell m}^{(0)}, \mathcal{A}_{\ell m}^{(1)}, \mathcal{B}_{\ell m}^{(0)})$ [You can compare your

results with Table I of [1]. In order to get the exact same functional form of the reference, note that $v^\phi = d\phi/dt = \frac{i}{m} \frac{dY_{\ell m}}{dt} / Y_{\ell m}^*$. Remember also that the coefficients have to be multiplied by -1 if the norm of the corresponding tensor harmonics is negative. For example $(\mathbf{a}^{(1)}, \mathbf{a}^{(1)}) = -1]$

Bibliography

- [1] N. Sago, H. Nakano, and M. Sasaki. Gauge problem in the gravitational self-force: Harmonic gauge approach in the Schwarzschild background. *Physical Review D*, 67(10):3457–14, May 2003.