Numerical computation of Schwarzschild Quasi Normal Modes

1. Integrate the Regge-Wheeler equation

$$\frac{d^2R}{dr_\star^2} + [\omega^2 - V(r)]R = 0 \tag{1}$$

where $r_\star = r + 2M\log[\frac{r}{2M} - 1]$ is the tortoise coordinate, and the potential $V(r)$ is given by:

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right) \ . \tag{2}$$

using a direct integration method. Compute the real and imaginary part of the QNMs for different values of the multipole number $\ell \geq 2$. Find the QNMs by also employing a second numerical method, like the continued fraction method, i.e. by solving the following equation

$$0 = \beta_0 - \frac{\alpha_0\gamma_1}{\beta_1 -}\frac{\alpha_1\gamma_2}{\beta_2 -}\dots \ , \tag{3}$$

where the coefficients $(\alpha_n, \beta_n, \gamma_n)$ read:

$$\alpha_n = n(2 - 4iM\omega) - 4iM\omega + n^2 + 1 \ , \tag{4a}$$

$$\beta_n = -l(l+1) + 32M^2\omega^2 - n(2 - 16iM\omega) + 8iM\omega - 2n^2 + 3 \ , \tag{4b}$$

$$\gamma_n = -16M^2\omega^2 - 8iMn\omega + n^2 - 4 \ . \tag{4c}$$