# Tensor manipulations and slowly rotating BH solutions

1. Build a `MATHEMATICA` routine which computes, for a generic metric $g_{\alpha\beta}$, and a set of coordinate $x^\mu = (x^0, x^1, x^2, x^3)$, the following geometric quantities:
$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\rho}[g_{\rho\alpha,\beta} + g_{\rho\beta,\alpha} - g_{\alpha\beta,\rho}] \ , \quad (1)$$
$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} - \Gamma^\alpha{}_{k\nu}\Gamma^k{}_{\beta\mu} + \Gamma^\alpha{}_{k\mu}\Gamma^k{}_{\beta\nu} \ , \quad (2)$$
$$R_{\alpha\beta} = R^\rho{}_{\alpha\rho\beta} \quad , \quad R = R^\alpha{}_\alpha = g^{\alpha\beta} R_{\alpha\beta} \ . \quad (3)$$
where commas refer to simple derivatives, i.e. $A_{,\mu} = \partial A/\partial x^\mu$.

2. Given the Einstein field's equations
$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \ , \quad (4)$$
and the general ansatz for a static, stationary, spherically symmetric spacetime:
$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \quad x^\mu = (t, r, \theta, \phi) \ . \quad (5)$$
integrate equations (4) in the vacuum case, in order to find the values of $A(r)$ and $B(r)$ which correspond to the Schwarzschild solution, $A(r) = B(r) = 1 - 2M/r$ [hint: use the $G_{tt}$ and $G_{rr}$ components]. Compute the Ricci scalar, and the non vanishing components of the Riemann tensor for this case.

3. Using the previous solution, compute the Kretschmann invariant $\mathcal{K} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. How does it behave?

4. In most of the non-static BHs configurations alternative to General Relativity, or for Exotic Compact Objects different from the usual Kerr family, an exact analytic solution for the metric does not exist when the spin is taken into account. However, rotation can be added to the static metric given by eq. (5) using a perturbative approach, in which the spin terms are described as small corrections of the spherically symmetric background [1, 2]. The general ansatz for the perturbed metric reads:
$$ds^2 = -A(r)(1+2h)dt^2 + \frac{(1+2m)}{B(r)}dr^2 + r^2(1+2k)[d\theta^2 + \sin^2\theta(d\phi - \hat{\omega}dt)^2)] \ , \quad (6)$$

where $A(r), B(r)$ are the non-spinning background components, and $(h, m, k, \hat{\omega})$ depends of $(r, \theta)$. The angular dependence can be expanded in terms of the Legendre polynomials according to their symmetry properties, while the radial components are expressed as a series in powers of the angular momentum $J$. Note that $h, m, k$ ($\tilde{\omega}$) contain even (odd) powers of $J$ only. At the third order in the spin we have:

$$h(r,\theta) = [h_0(r)P_0(\cos\theta) + h_2(r)P_2(\cos\theta)]\chi^2 + \mathcal{O}(\chi^4) \ , \tag{7a}$$

$$m(r,\theta) = [m_0(r)P_0(\cos\theta) + m_2(r)P_2(\cos\theta)]\chi^2 + \mathcal{O}(\chi^4) \ , \tag{7b}$$

$$k(r,\theta) = [k_0(r)P_0(\cos\theta) + k_2(r)P_2(\cos\theta)]\chi^2 + \mathcal{O}(\chi^4) \ , \tag{7c}$$

and

$$\hat{\omega} = \hat{\omega}_1(r)S_1(\theta)\chi + \mathcal{O}(\chi^3) \ , \tag{8}$$

where $P_\ell$ are the legendre polynomials and $S_\ell = -\frac{1}{\sin\theta}\frac{dP_\ell}{d\theta}$. Inserting the previous ansatz into the vacuum field's equations (4) we can solve for the unknown metric functions $(h, m, k, \hat{\omega})$ order by order in the BH spin parameter $\chi = J/M^2$. Note also that the metric is invariant for rescaling $r \to f(r)$, so we can fix function $k_0(r)$ to zero without loss of generality.

Given the 6 unknown metric functions, compute the the first order correction in the spin, i.e. $\omega_1(r)$.

[**hint 1** To simplify the expressions, work with mixed components of the Einstein tensor $G^\alpha{}_\beta$. Moreover, at the linear order in the spin, there is only one non-vanishing component of the field's equations.]

[**hint 2** The solution of each unknown metric component will depend on some arbitrary constants of integrations, which can be fixed by asking: (i) asymptotic flatness, (ii) that the perturbation are regular at the horizon, (iii) and that the solution matches that one of an isolated stationary source in the far field limit (in this limit $g_{t\phi} = -\frac{2J}{r}$)].

Extra point. Can you also find the values of $(h_0, h_2, m_0, m_2, k_2)$, in order to get the following expressions:

$$m_0 = -\frac{J^2}{(r-2M)r^3} \ , \tag{9}$$

$$m_2 = -\frac{J^2(r-5M)}{Mr^4} \ , \tag{10}$$

$$h_0 = -\frac{J^2}{(r-2M)r^3} \ , \tag{11}$$

$$h_2 = \frac{J^2(M+r)}{Mr^4} \ , \tag{12}$$

$$k_2 = \frac{J^2(2M+r)}{Mr^4} \ . \tag{13}$$

5. The horizon and the ergosphere, are specified by the largest root of the equations $g_{\phi\phi}g_{tt} - g_{t\phi}^2 = 0$ and $g_{tt} = 0$, respectively. Compute them as a function of the spin BH parameter $\chi$. Also, find the *angular velocity* of the horizon, defined as $\Omega_h = -\lim_{r\to r_h} \frac{g_{t\phi}}{g_{\phi\phi}}$. The latter allows to derive the BH moment of inertia, $I = J/\Omega_h$.

6. For a stationary-axisymmetric spacetime, the ISCO of massive particles corresponds to the radius at which the second derivative of the effective potential

$$V(r) = \frac{1}{g_{rr}}\left(\frac{E^2 g_{\phi\phi} + 2ELg_{t\phi} + L^2 g_{tt}}{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}\right) , \tag{14}$$

where $E$ and $L$ are the specific energy and angular momentum of the test particle

$$E = -\frac{g_{tt} + g_{t\phi}\omega_\phi}{\sqrt{-g_{tt} - 2g_{t\phi}\omega_\phi - g_{\phi\phi}\omega_\phi^2}} \ , \ L = \frac{g_{t\phi} + g_{\phi\phi}\omega_\phi}{\sqrt{-g_{tt} - 2g_{t\phi}\omega_\phi - g_{\phi\phi}\omega_\phi^2}} \ , \tag{15}$$

and $\omega_\phi$ is the azimuthal angular velocity

$$\omega_\phi = \frac{-g_{t\phi,r} + \sqrt{g_{t\phi,r}^2 - g_{tt,r}g_{\phi\phi,r}}}{g_{\phi\phi,r}} . \tag{16}$$

By solving $V''(r) = 0$, find the spin corrections (at the second order) to the ISCO radius. Assume that the motion takes place in the equatorial plane, i.e. fix $\theta = \pi/2$.

# Bibliography

[1] J. B. Hartle. Slowly Rotating Relativistic Stars. I. Equations of Structure. *Astrophys. J.*, 150:1005, Dec. 1967.

[2] J. B. Hartle and K. S. Thorne. Slowly Rotating Relativistic Stars. II. Models for Neutron Stars and Supermassive Stars. *Astrophys. J.*, 153:807, 1968.