Emergent behavior in self-organized dynamics: from consensus to hydrodynamic flocking

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ICTS Colloquium, TIFR Bangalore Jan. 18, 2018

Collaborators:

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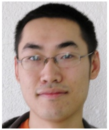

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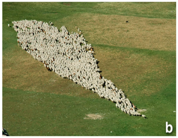
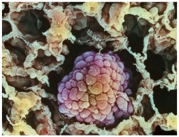
Examples of self-organized dynamics

- "Living agents": flocks of birds; schools of fish; colonies of ants, bacteria, cells





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  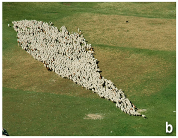

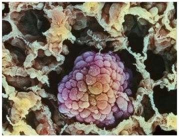

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   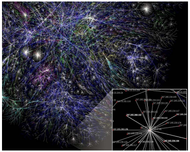

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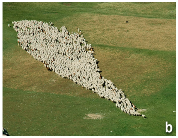

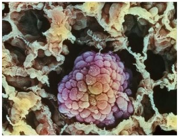

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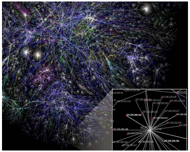
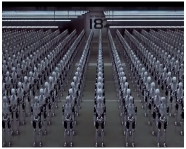
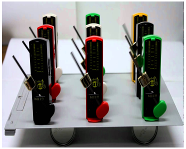
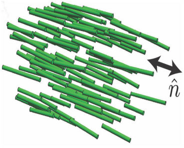


- "Non-living agents": sensor-based robots, UAVs, micro-motors, nematic fluids, ...



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- Key issue: the notion of 'local neighborhood'

Outline

1. Rules of engagement: alignment
   - Krause model for opinion dynamics
   - Sensor-based motion – the rendezvous problem
   - Vicsek model for flocking; phase transition
   - Cucker-Smale models for flocking — near and far from equilibrium
2. $t \to \infty$: The emergence of consensus, parties, leaders, ...
   - Large time behavior — consensus, flocking, . . .
   - Synchronization — Kuramoto model
   - Taking tendency into account — emergence of leaders
   - A general perspective
3. $N \to \infty$: Social hydrodynamics
   - Kinetic description
   - From kinetic to hydrodynamic description of flocking
   - Hydrodynamic alignment — smooth solutions must flock
   - Critical thresholds in flocking hydrodynamics

A basic paradigm in collective dynamics — alignment

- A general class of $N \gg 1$ agents identified w/ "traits" $\{\mathbf{p}_i(t)\}_{i=1}^N$:
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- Neighborhood $\mathcal{N}_i$ — geometric neighborhood dictated by ϕ

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- State space — vectors of "opinions": $\{\mathbf{x}_i(t)\}_{i=1}^{N}$

[2] U. Krause (1997,2000), R. Hegselmann & U.Krause: Opinion dynamics and bounded confidence models, analysis and simulation (2002,2004)

Example#1: Krause model[2] for opinion dynamics

- State space — vectors of "opinions": $\{\mathbf{x}_i(t)\}_{i=1}^N$
- Krause-Hegselmann model (1997) — interaction through **local** averaging:

$$\mathbf{x}_i(t+\Delta t) = \frac{1}{N_i} \sum_{|\mathbf{x}_i - \mathbf{x}_j| \leqslant R} \mathbf{x}_j(t), \qquad N_i := \#\{\mathbf{x}_j : |\mathbf{x}_j - \mathbf{x}_i| < R\}$$

- "Environmental averaging": $\mathbf{x}_i(t+\Delta t) = \sum_{j \in \mathcal{N}_i} a_{ij}\mathbf{x}_j(t)$ $\quad \sum_j a_{ij} = 1$
- Act on difference of opinions: $a_{ij} = \dfrac{\phi(|\mathbf{x}_i - \mathbf{x}_j|)}{\deg_i}$ $\quad \phi(r) = \mathbb{1}_{[0,R)}(r)$
- $\deg_i = \sum_{k \in \mathcal{N}_i} \phi(|\mathbf{x}_i - \mathbf{x}_k|) \rightsquigarrow N_i$ — the degree of influence on agent$_i$

⋆ Typical question in collective dynamics: does "averaging" lead to "consensus"?

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Example#2: Sensor-based networks

- State space — vectors of positions: $\{\mathbf{p}_i(t)\}_{i=1}^N \rightsquigarrow \{\mathbf{x}_i(t)\}_{i=1}^N$
- Mean-shift gradient flow[3] (non parametric clustering):

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- Local means are shifted in direction of maximal increase of density
- Robotic agents: the "rendezvous problem"[3b–3d] $\mathbf{x}_i(t)-\mathbf{x}_j(t) \xrightarrow{t\to\infty} 0$?

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[3b] On limited visibility — Ji & Egerstedt (2007); Bellaiche & Bruckstein (2015);
[3c] Time delay — Olfati-Saber & Murray (2004), Somarakis & Baras (2013-);
[3d] Connectivity of mobile networks — Zalvanos, Pappas, et. al (2007-2011)

Example #3: Vicsek model[4] — alignment of orientations

- Fix a speed s. Averaging of orientations $\{\mathbf{p}_i(t)\}_{i=1}^N \rightsquigarrow \{\boldsymbol{\omega}_i(t)\}_{i=1}^N \in \mathbb{S}^{d-1}$.

$$\boldsymbol{\omega}_i(t+\Delta t) = \Big(s \sum_{j\in\mathcal{N}_i} a_{ij}\,\boldsymbol{\omega}_j(t) + \text{noise}\Big) \times \frac{1}{|s\sum_j a_{ij}\,\boldsymbol{\omega}_j(t) + \text{noise}|}$$

- 2D additive Noise= uniform in angle in $[-\tau, \tau]$
- $a_{ij} = \dfrac{\phi(|\mathbf{x}_i - \mathbf{x}_j|)}{\deg_i} \quad \rightsquigarrow \quad \mathbf{v}_i(t) := \dfrac{s}{\deg_i} \displaystyle\sum_{j:|\mathbf{x}_i - \mathbf{x}_j|<R} \phi(|\mathbf{x}_i - \mathbf{x}_j|)\boldsymbol{\omega}_j(t)$

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Vicsek model as an alignment dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \alpha\sum_{j\in\mathcal{N}_i} a_{ij}\left(\mathbf{v}_j - \frac{\langle \mathbf{v}_i, \mathbf{v}_j\rangle}{|\mathbf{v}_i|^2}\mathbf{v}_i\right)$$

- Beyond environmental averaging:
  Projection $\frac{\langle \mathbf{v}_i, \mathbf{v}_j\rangle}{|\mathbf{v}_i|^2}$ enforces fixed speed, $|\mathbf{v}_i(t)| = s$, and noise (τ)

[4] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, O. Shochet (PRL 1995)

Example#4: Cucker-Smale model[5]— velocity alignment

- State space — vectors of velocities: $\{\mathbf{p}_i(t)\}_{i=1}^N \rightsquigarrow \{\mathbf{v}_i(t)\}_{i=1}^N$

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Short-range interactions involve 'nearby' neighbors $\deg_i \rightsquigarrow N_i$

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Example#5: far from equilibrium

- (with S. Motsch[6]): $a_{ij}(\mathbf{x}(t)) = \frac{1}{\deg_i}\phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|)$

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[6] A new model for self-organized dynamics and its flocking behavior (2011)

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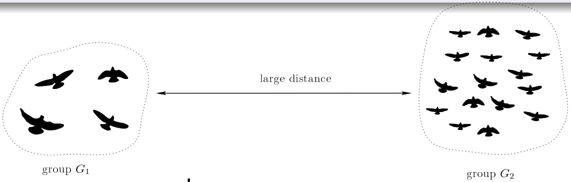

$\star$ $\deg_i(i \in G_1) \approx N_1\phi_0 \quad \rightsquigarrow \quad \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \dfrac{\alpha}{N_1\phi_0}\sum_{G_1}\phi_{ij}\left(\mathbf{v}_j(t) - \mathbf{v}_i(t)\right)$

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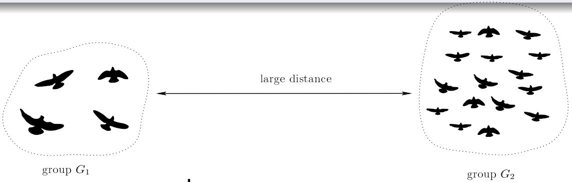

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- $A_\phi = \{a_{ij} = \dfrac{\phi_{ij}}{\deg_i}\}$ is not symmetric ...

but involves the symmetric graph Laplacian $L_A := I - D^{1/2}A_\phi D^{-1/2}$

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Outline

1. Rules of engagement: alignment
   - Krause model for opinion dynamics
   - Sensor-based motion – the rendezvous problem
   - Vicsek model for flocking; phase transition
   - Cucker-Smale models for flocking — near and far from equilibrium
2. $t \to \infty$: The emergence of consensus, parties, leaders, ...
   - Large time behavior — consensus, flocking, . . .
   - Synchronization — Kuramoto model
   - Taking tendency into account — emergence of leaders
   - A general perspective
3. $N \to \infty$: Social hydrodynamics
   - Kinetic description
   - From kinetic to hydrodynamic description of flocking
   - Hydrodynamic alignment — smooth solutions must flock
   - Critical thresholds in flocking hydrodynamics

Emerging behavior as $t \to \infty$ — local models (symmetric)

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$\star$ Interplay between dynamics on graph and graph driven by the dynamics

Cucker-Smale vs. Motsch-Tadmor model (global influence)

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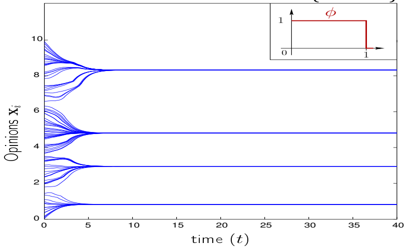
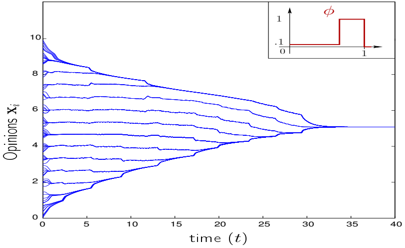
How "rules of engagement" influence the emergence of consensus?

- 100 uniformly distributed opinions: $\phi(r) = a\mathbf{1}_{\{r \leqslant \frac{1}{\sqrt{2}}\}} + b\mathbf{1}_{\{\frac{1}{\sqrt{2}} \leqslant r < 1\}}$

$a = b = 1: \ \phi(r) = \mathbf{1}_{\{0<r<1\}}$



$(a, b) = (0.1, 1)$



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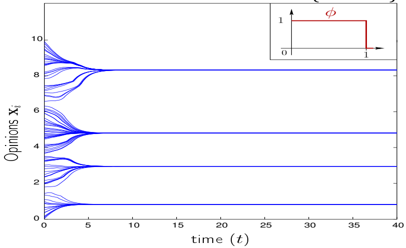

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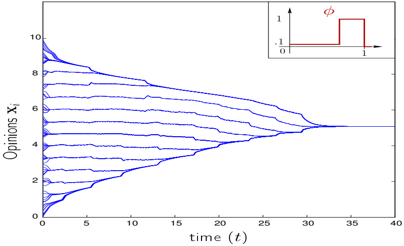

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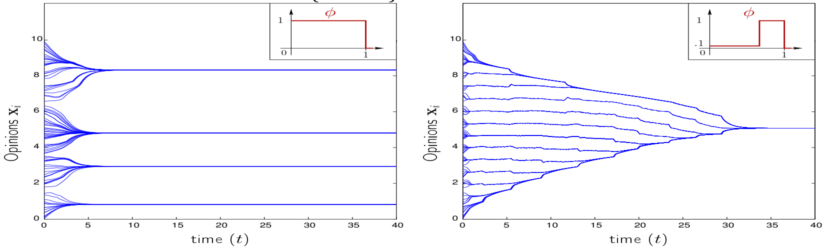

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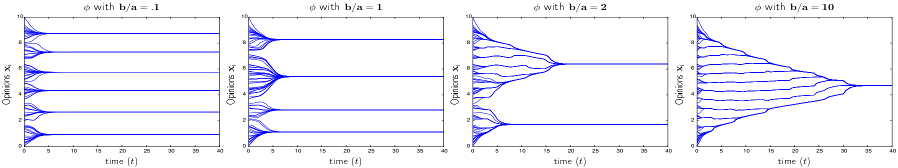

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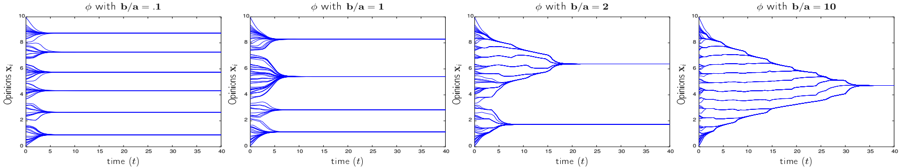

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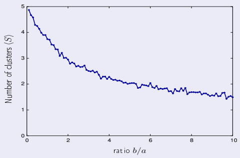
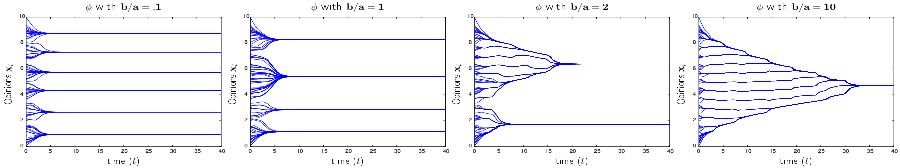
- Heterophilious dynamics enhances connectivity[6]:
  The # of clusters, $K\,(b/a)$, is decreasing $\rightsquigarrow$



[6] Motsch & ET., Heterophilious dynamics enhances consensus, SIAM Review 2014

How “rules of engagement” influence the emergence of consensus?

- 100 uniformly distributed opinions: $\phi(r) = a\mathbf{1}_{\{r\leqslant\frac{1}{\sqrt{2}}\}} + b\mathbf{1}_{\{\frac{1}{\sqrt{2}}\leqslant r<1\}}$

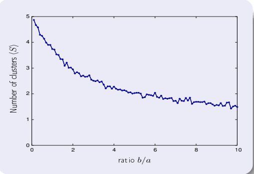


- Homophilious dynamics: align with those that think alike $(a \gg b)$ vs.
- Heterophilious dynamics: "bonding with the different" $(a \ll b)$

- Heterophilious dynamics enhances connectivity[6]: The $\#$ of clusters, $K(b/a)$, is decreasing $\rightsquigarrow$

$\star$ Heterophilious dynamics enhances consensus —

$\phi(\cdot)_{|0<r<R}$ is increasing $\rightsquigarrow$ $\lambda_2(L_{A(\mathbf{x}(t))}) > \eta > 0$?



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"Regarding networks and homophily, birds of a feather flock together, but people are influenced by those they like. Both these processes result in the same outcome (similar people together in groups), but there is no standard accepted way of separating these two processes".

Introduction to Mathematical Sociology, P. Bonacich & P. Lu

Fluctuations and mean distance of interacting clusters

[7] Garnier, Papanicolaou, Yang, Consensus convergence with stochastic effects (2017)

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- Empirical distribution $\rho^N(t,\mathbf{x}) := \dfrac{1}{N}\sum_i \delta_{\mathbf{x}-\mathbf{x}_i(t)} \;\cong\; \overline{\rho} + \rho_1(t,\mathbf{x})$

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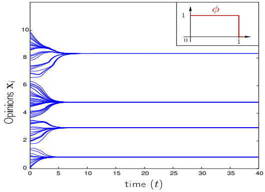

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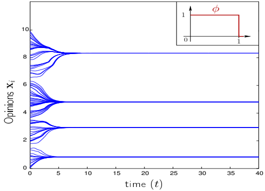

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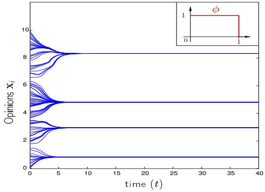

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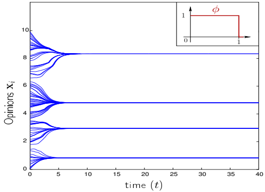

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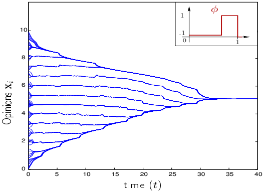

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Other aspects in collective dynamics (beyond alignment)

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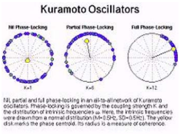
- An example — emergence of leaders ...

Example #6: Synchronization

- Kuramoto model[8] $\{\mathbf{p}_i\} \rightsquigarrow$ phases $\{\theta_i\}$ or frequencies $\{\omega_i = \dot{\theta}_i\}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_i(t) = \Omega_i + \frac{K}{N}\sum_j \sin(\theta_j - \theta_i) \rightsquigarrow \Omega_i + \frac{K}{\deg_i}\sum_j a_{ij}(\theta_j - \theta_i), \; a_{ij} = \frac{\sin(\theta_j - \theta_i)}{\theta_j - \theta_i}$$

- $|\Omega_i| < \alpha r$: steady states; more oscillators recruited into synchronized clusters ($r \approx 1$) as $\alpha > \alpha_c$ increases
- $|\Omega_i| > \alpha r$: no synchronization ($r \approx 0$) is possible for $\alpha < \alpha_c$

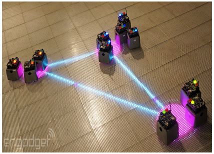


Stewart Heitmann

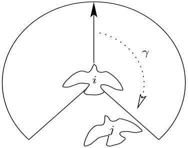
[8] Kuramoto, Lecture Notes Phys. (1975, 1984)...Acebron et. al. RevModPhys (2005)

[8b] Ha et. al (2010 –), Gerard-Varet, Dietert, Fernandez, Giacomin (2016)

Example#7: tendency to follow in "thinking agents"



Robots $\mapsto$ rely on laser



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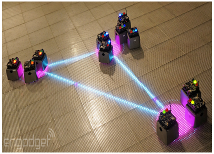

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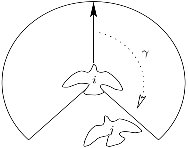

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- Influenced by the **projection** of those moving ahead
  tracing using pheromones rather than vision . . .
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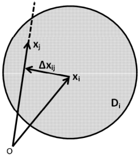

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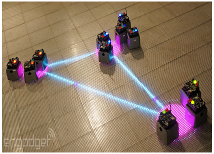

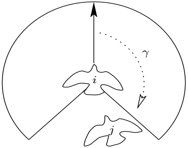

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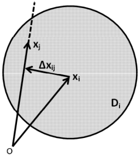

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}_i(t) \quad = \quad \frac{1}{\deg_i}\sum_{j\in\mathcal{N}_i}\phi(|\mathbf{x}_i-\mathbf{x}_j|)\left(\frac{\langle\mathbf{p}_i,\mathbf{p}_j\rangle}{|\mathbf{p}_j|^2}\mathbf{p}_j-\mathbf{p}_i\right),$$

Tendency and emergence of leaders – 1st-order model

$$\frac{d}{dt}\mathbf{x}_i(t) = \sum_{j\in\mathcal{N}_i} a_{ij}\mathbf{x}_j - \mathbf{x}_i, \quad a_{ij} = \frac{1}{\deg_i}\phi(|\mathbf{x}-\mathbf{x}_j|)\frac{\langle \mathbf{x}_i, \mathbf{x}_j\rangle}{|\mathbf{x}_j|^2}$$

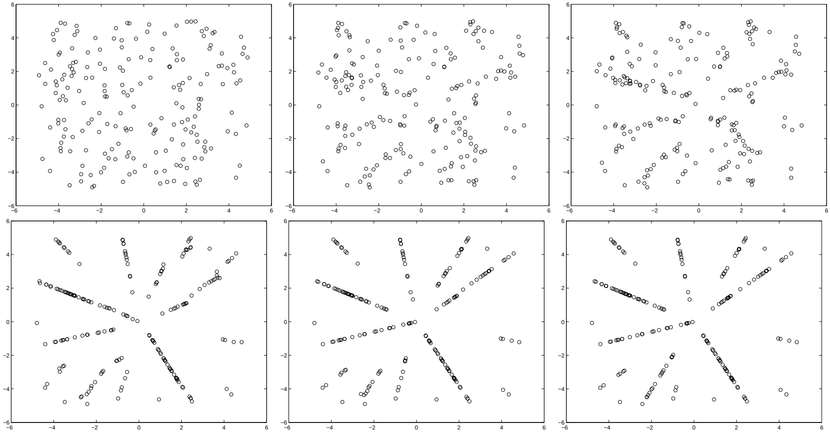

Figure: Random initial conditions, $\phi = \mathbb{1}_{[0,1]}$. Snapshots: t = 0, 0.3, 0.5, 2, 5, 70.

Self-organized dynamics

Self-organized dynamics — different questions/tools arise in different fields

- **Biology** — The role of empirical data

  Flocks, swarms, colonies, ... — how are they formed?

  Since there is no Newton's law — what are the rules of engagement?

  $\star$ Agents <u>are</u> different – are these observed patterns observed patterns system specific?

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  Models are different — but deep analogies in patterns of equilibrium
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- **Mathematics** — Agent-based models; non-local PDEs
  Agent-based ⇝ kinetic models ⇝ macroscopic models
  ⋆ Numerical and analytical studies of 'social hydrodynamics'

Outline

1. Rules of engagement: alignment
   - Krause model for opinion dynamics
   - Sensor-based motion – the rendezvous problem
   - Vicsek model for flocking; phase transition
   - Cucker-Smale models for flocking — near and far from equilibrium
2. $t \to \infty$: The emergence of consensus, parties, leaders, ...
   - Large time behavior — consensus, flocking, ...
   - Synchronization — Kuramoto model
   - Taking tendency into account — emergence of leaders
   - A general perspective
3. $N \to \infty$: Social hydrodynamics
   - Kinetic description
   - From kinetic to hydrodynamic description of flocking
   - Hydrodynamic alignment — smooth solutions must flock
   - Critical thresholds in flocking hydrodynamics

Second limit — behavior of large crowds[9] $N \to \infty$

- Empirical distribution $f^N := \dfrac{1}{N}\sum_j \delta_{\mathbf{x}-\mathbf{x}_j(t)} \otimes \delta_{\mathbf{v}-\mathbf{v}_j(t)} \longrightarrow f(t,\mathbf{x},\mathbf{v})$

Vlasov eq. for distribution $f(t,\mathbf{x},\mathbf{v})$

$$f_t + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \alpha\nabla_{\mathbf{v}}\cdot Q(f,f) = 0$$

[9] Ha & ET (2008); Canizo, Carrillo, Rosado (2009); Carrillo-Fornasier-Rosado-Toscani (2009), Motsch-ET (2014)

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- $Q(f,f)$ assembles binary interactions: alignment, repulsion, noise, ...

$$Q(f,f) = \frac{1}{\deg(t,\mathbf{x})}\int_{\mathbb{R}^{2d}} \phi(|\mathbf{x}-\mathbf{y}|)(\mathbf{w}-\mathbf{v})f(t,\mathbf{x},\mathbf{v})f(t,\mathbf{y},\mathbf{w})d\mathbf{y}d\mathbf{w} + \ldots$$

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- $Q(f,f)$ assembles binary interactions: alignment, repulsion, noise, ...

$$Q(f,f) = \frac{1}{\deg(t,\mathbf{x})}\int_{\mathbb{R}^{2d}} \phi(|\mathbf{x}-\mathbf{y}|)(\mathbf{w}-\mathbf{v})f(t,\mathbf{x},\mathbf{v})f(t,\mathbf{y},\mathbf{w})d\mathbf{y}d\mathbf{w} + \ldots$$

- Flocking ($K=1$) $\quad f \rightsquigarrow \rho(t,\mathbf{x})\delta(\mathbf{v}-\mathbf{u}(t,\mathbf{x}))$....

[9] Ha & ET (2008); Canizo, Carrillo, Rosado (2009); Carrillo-Fornasier-Rosado-Toscani (2009), Motsch-ET (2014)

Second limit — behavior of large crowds[9] $N \to \infty$

- Empirical distribution $f^N := \frac{1}{N}\sum_j \delta_{\mathbf{x}-\mathbf{x}_j(t)} \otimes \delta_{\mathbf{v}-\mathbf{v}_j(t)} \longrightarrow f(t,\mathbf{x},\mathbf{v})$

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.... recovered in terms of moments —

$$\begin{bmatrix} \text{density} \ldots\ldots\ldots\ \rho \\ \text{momentum} \ldots\ \rho\mathbf{u} \end{bmatrix} = \int \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} f(t,\mathbf{x},\mathbf{v})d\mathbf{v}$$

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Flocking hydrodynamics

$$\begin{cases} \text{mass}: & \partial_t\rho + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}) = 0 \\ \text{momentum}: & \partial_t(\rho\mathbf{u}) + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u} + P(f)) = \rho\,\mathcal{A}_\rho(\mathbf{u}) \end{cases}$$

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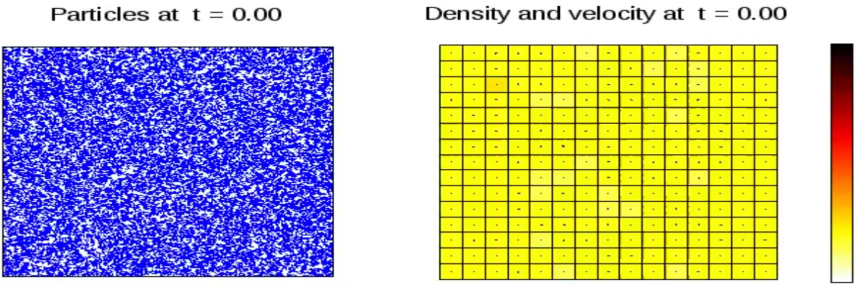
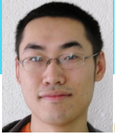
[10] R. Shvydkoy & ET, Eulerian dynamics with commutator forcing (2016)
[10b] Ha & ET(2008); Carrillo et. al.(2012); Karper, Mellet, Trivisa (2013)

Hydrodynamic vs. agent-base description

Vicsek model: agent-base model vs. hydrodynamic description



Flocking behavior – bounded kernels




- Classical solutions must flock

11

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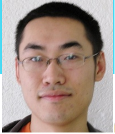


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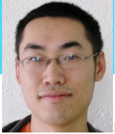


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Theorem[11]. Set diameter $\left[\mathbf{u}(t)\right]_\infty := \sup_{\mathbf{x},\mathbf{y}\in \text{Supp}\,\rho(t,\cdot)} |\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{y})|$

If $\mathbf{u} \in C^1 \rightsquigarrow \dfrac{d}{dt}\left[\mathbf{u}(t)\right] \leqslant -\alpha\, \mu(t)\, \left[\mathbf{u}(t)\right]$:

$\mu_\infty(t)$ is coefficient of ergodicity $\geqslant \min_{\mathbf{x},\mathbf{y}\in \text{Supp}\,\rho(t,\cdot)} \phi(|\mathbf{x}-\mathbf{y}|)$

Then global interaction $\int^\infty \phi(s)ds = \infty \rightsquigarrow$ unconditional flocking[11b]:

$$\text{unconditional flocking}\ldots \begin{cases} \mathbf{u}(t,\cdot) \xrightarrow{t\to\infty} \overline{\mathbf{u}} \\ \rho(t,\mathbf{x}) - \rho_\infty(\mathbf{x} - t\overline{\mathbf{u}}) \xrightarrow{t\to\infty} 0 \end{cases}$$

[11] ET & C. Tan, Proc. Roy. Soc. A (2014);
[11b] Independent of the closure relation! S.-Y. Ha & ET, KRM (2008)

Flocking behavior – bounded kernels

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- When does $\mathbf{u}(t,\cdot) \in C^1$-solution exist? What about local interaction?

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- "Local action" —
  Limiting case $\beta = 2$ — Navier-Stokes eqs. $L = \Delta$

$$\rightsquigarrow \qquad (\rho\mathbf{u})_t + \nabla(\rho\mathbf{u}\otimes\mathbf{u}) = \nabla(\rho^2 D\mathbf{u}), \quad D\mathbf{u} = \{\partial_i u_j\}$$

On the question of global regularity – global solutions exist

- For bounded ϕ's —
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12

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⋆ In contrast to blow-up in fractional Burgers: regularity iff[12d] $\beta \geqslant 1$

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- The role of (i) no vacuum and (ii) the spectral gap in 2D dynamics

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[12b] R. Shvydkoy & ET, Eulerian dynamics with commutator forcing (2016);
[12c] Do, Kiselev, Ryzhik & Tan, Global regularity for fractional Euler alignment (2017)
[12d] Caffarelli-Vasseur, Kiselev-Nazarov, Constantin-Vicol

Progress — mostly one- and two-dimensional models


- Critical threshold – 1D flocking hydrodynamics[13]

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[13b] Shvydkoy & ET Eulerian dynamics with commutator forcing. I, II and III (2017)
[13c] S. He & ET, Global regularity of 2D flocking hydrodynamics (2017)

Critical threshold – 1D flocking hydrodynamics[14]

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[14] Y.-P. Choi, J. Carrillo, E.T., C. Tan (2015)

Critical threshold – 1D flocking hydrodynamics[14]


- 1D alignment:

$$\rho_t + (\rho u)_x = 0$$

$$u_t + uu_x = \int \phi(|x-y|)(u(t,y) - u(t,x)\rho(t,y)\mathrm{d}y$$

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- Riccati balanced by alignment: $(\mathrm{d} + \phi * \rho)' = -\mathrm{d}(\mathrm{d} + \phi * \rho)$
- Global smooth solution under a critical threshold condition —

iff u_0' "is not too negative": $u_0'(x) + \phi * \rho_0(x) \geqslant 0$

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- Why $\rho_+(t) = \max_x \rho(x,t) < \infty$ matters?

Regularization by fractional dissipation

$\rho_t + u\rho_x + u_x\rho = 0$ 'implies' $u_x > -\infty$ (at least along particle path)

[15] Constantin-Vicol (2012);

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and likewise for momentum — $m_t + um_x + em = \rho\Lambda^\beta m$ ($\beta = 1$ critical)

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- Global smooth solution $0 < \beta < 2$ with no CT; Singularity helps!

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Spectral gap in 2D flocking hydrodynamics[16]

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Spectral gap in 2D flocking hydrodynamics[16]

- C-S hydrodynamics: $\mathbf{u}_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} = \phi * (\rho \mathbf{u}) - \mathbf{u} \phi * \rho$

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[16] S. He & ET, Global regularity of 2D flocking hydrodynamics (2017)

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- Global regularity (+ flocking) provided: $\begin{cases} \mathrm{div}_{\mathbf{x}}(\mathbf{u}_0)(x) \geqslant -\phi * \rho_0(x), \\ \max_x |\eta(\mathcal{S}_0(x))| \leqslant \frac{1}{2} M_0 \phi_\infty \end{cases}$

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Ongoing study

R. Shvydkoy

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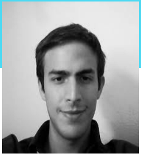

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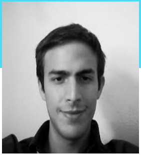

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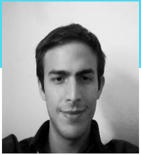

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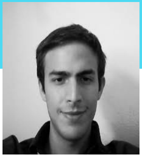

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- How different 'rules of engagement' dictate large time behavior?

KI-Net: Kinetic description of emerging challenges in multiscale problems of natural sciences

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THANK YOU

Acknowledgment. Work was supported by NSF, Ki-Net and ONR.