MODULI OF p-DIVISIBLE GROUPS (AFTER FARGUES, FONTAINE, SCHOLZE AND WEINSTEIN)

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Moduli of p-divisible groups have been studied intensively since the 1960's by Lubin, Tate, Dieudonné, Manin, Grothendieck, Messing and many others. Starting in the 1990's, Rapoport and Zink and their school, and Fontaine and his school have developed the theory much further, in relation with the study of Shimura varieties and p-adic Hodge theory, respectively.

The last decade has seen new perspectives emerging, characterized by (a) the prominence of perfectoid spaces in p-adic geometry (b) the geometrization of p-adic Hodge theory.

The goal of this mini-course is to give a modern introduction to the subject and survey some of the recent breakthroughs. To avoid technical complications, perfectoids will be rarely mentioned and we shall work over $C = \mathbb{C}_p$ most of the time. The Lubin-Tate and Drinfeld moduli spaces will serve as our main examples.

(1) Lecture 1 will begin with a review of *finite flat group schemes* and *p-divisible groups.* We shall then construct the Lubin-Tate moduli space and its coverings (the *Lubin-Tate tower*).
(2) Lecture 2 will be devoted to two "crystalline" constructions: the *universal covering* $\widetilde{G}$ of a p-divisible group G, and the *Grothendieck-Messing crystal* MG. Using the first notion, we shall present a simple description, found recently by Weinstein, of the Lubin-Tate tower at the infinite level. We shall also discuss Dieudonné modules and F-isocrystals.
(3) Lecture 3 will be devoted to the two *period maps.* We shall start by studying a big commutative diagram relating the two crystalline constructions from Lecture 2. This diagram also contains the maps leading to the construction of the Grothendieck-Messing period map π_{GM} and the Hodge-Tate period map π_{HT}. Although the Lubin-Tate example will still accompany us, we shall work in the more general framework of Rapoport-Zink spaces and period domains. Toward the end of the lecture we shall add PEL structure and introduce the Drinfeld moduli problem.
(4) Lecture 4 will be devoted to the *Fargues-Fontaine curve* and the *geometrization* of the concepts studied in the first three lectures. Following a review of some of Fontaine's rings we shall construct the (schematic) Fargues-Fontaine curve, emphasizing the analogy with the complex projective line (and where it breaks down). We shall study vector bundles on "the curve" and the concept of "modification". We shall apply this geometric picture to the classification of p-divisible groups, first over $\mathcal{O}_C/p$, and then over $\mathcal{O}_C$. The "big commutative diagram" from lecture 3 will reappear in a natural geometric context.
(5) Lecture 5 will contain *applications* to *Galois representations* and *duality* of Rapoport Zink spaces. We shall explain how geometrization of p-adic Hodge theory leads to easy conceptual proofs of some very deep theorems: the theorem of Colmez and Fontaine that "weakly

admissible = admissible" and Faltings' theorem on the "isomorphism between the Lubin-Tate tower and the Drinfeld tower at the infinite level". Of course, all the difficulties are now hidden in the construction and basic properties of "the curve", yet the new proofs are amazingly transparent. Due to the ignorance of the speaker, and time limitations, we shall say nothing about the spectecular program of geometrizing the Local Langlands Correspondence.