Long time behaviour of reduced density matrices after 2D quantum quench
(thermalization in 2D CFT and hs BH)

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Chandrasekhar lectures discussion meeting
ICTS, Bangalore
December 10, 2014

GM, T. Morita (1302.0859, 1107.4048, 1103.1558), P. Caupta, GM, R. Sinha (1306.4974), GM, R.Sinha, N. Sorokhaibam (1405.6695, 1412.xxxx)
+ related

Motivation

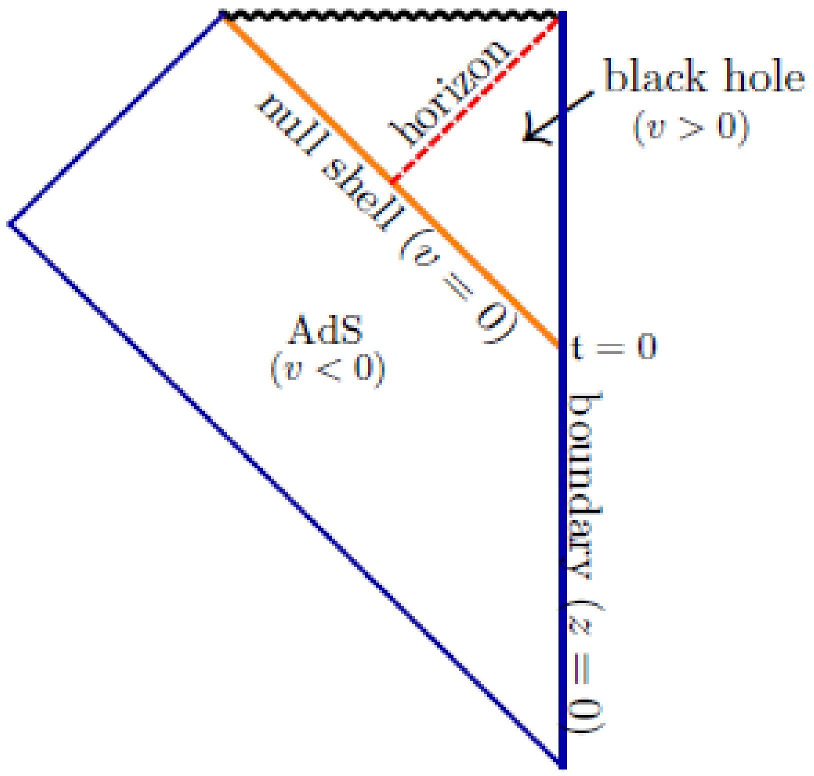
Universality in non-equilibrium phenomena (cf. Polkhovnikov-Sengupta-Silva-Vengalattore review 1007.5331)

Can time-dependent behaviour be described in terms of a few, robust parameters?

Are these related to equilibrium properties (assuming that an equilibrium exists)?

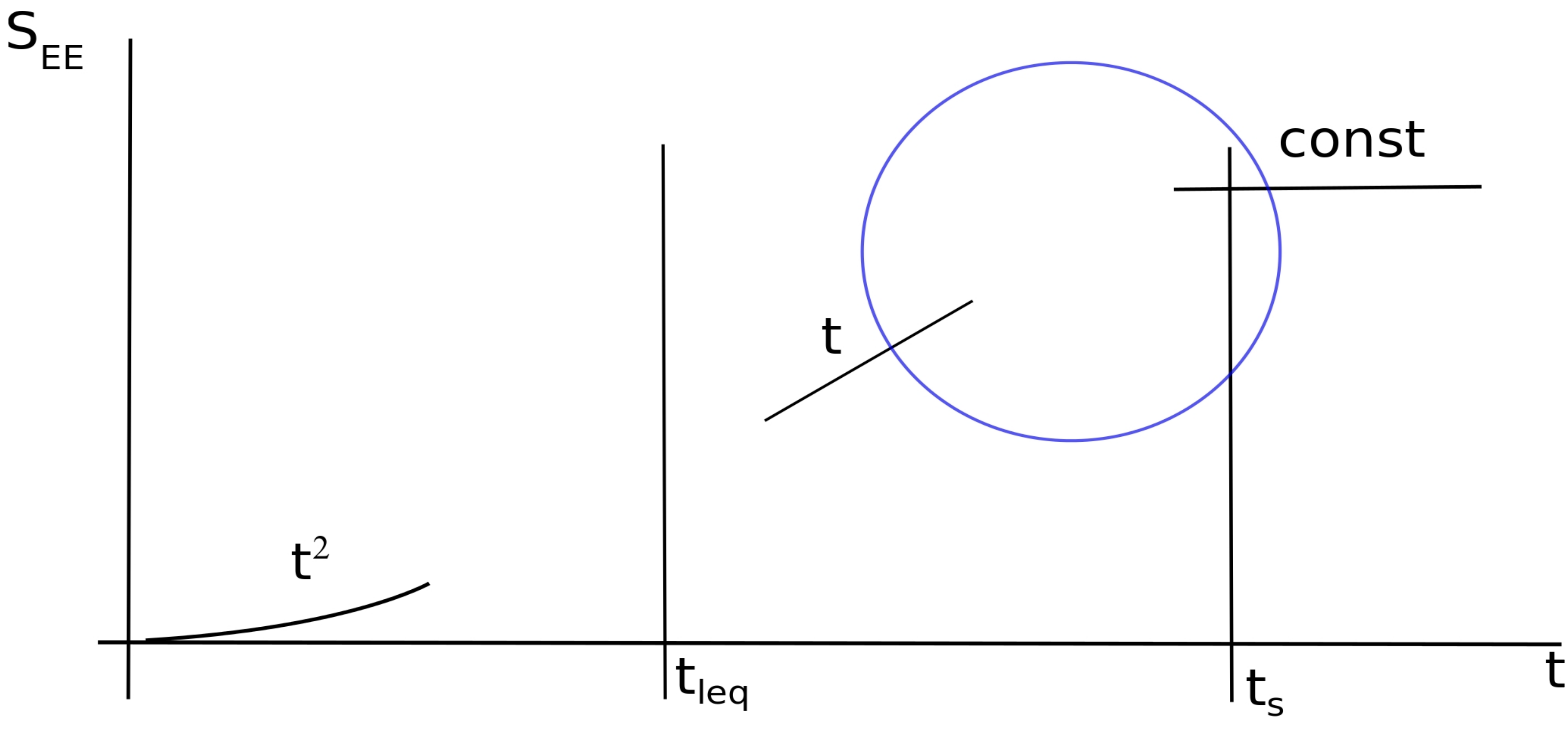
From AdS/CFT

Thermalization= black hole formation; hence to study approach to thermalization, study gravitational collapse geometries.

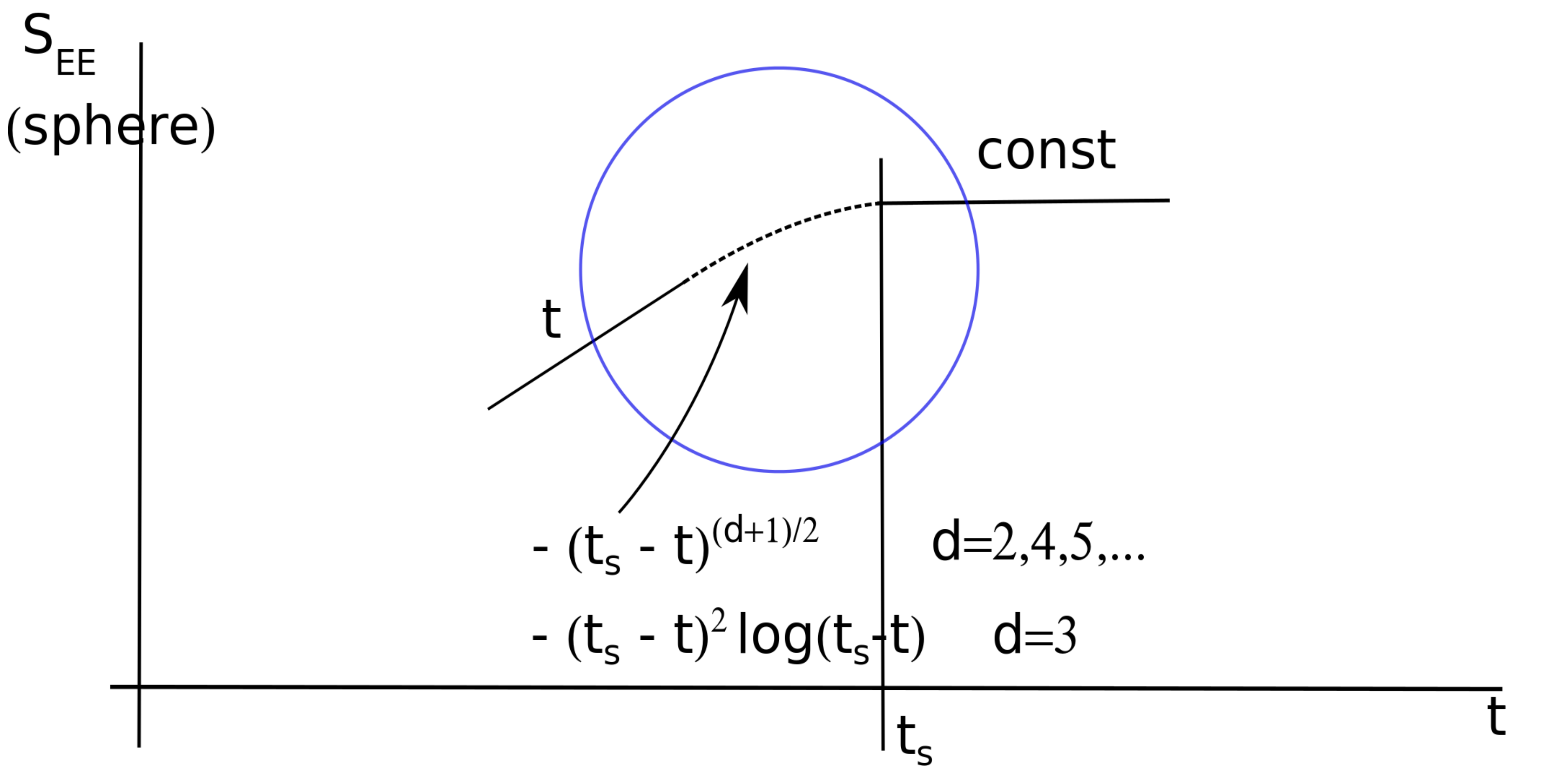


Vaidya metric

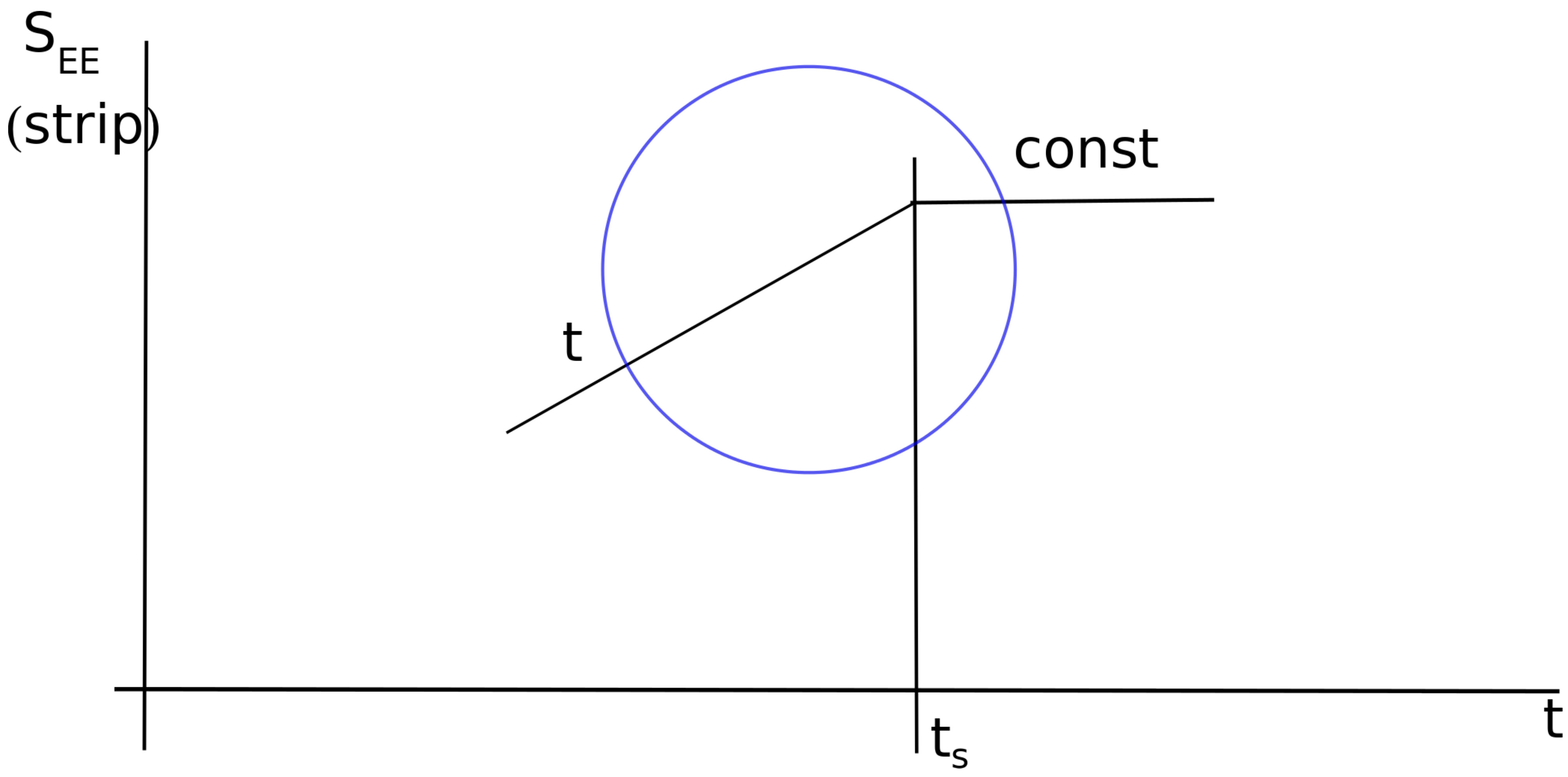
Predictions from holography (Vaidya metric) Liu-Suh 1305.7244, 1311.1200, Hubeny-Rangamani-Tonni 1302.0853) for dynamical entanglement entropy (also Takayanagi PCTS 2012 review)



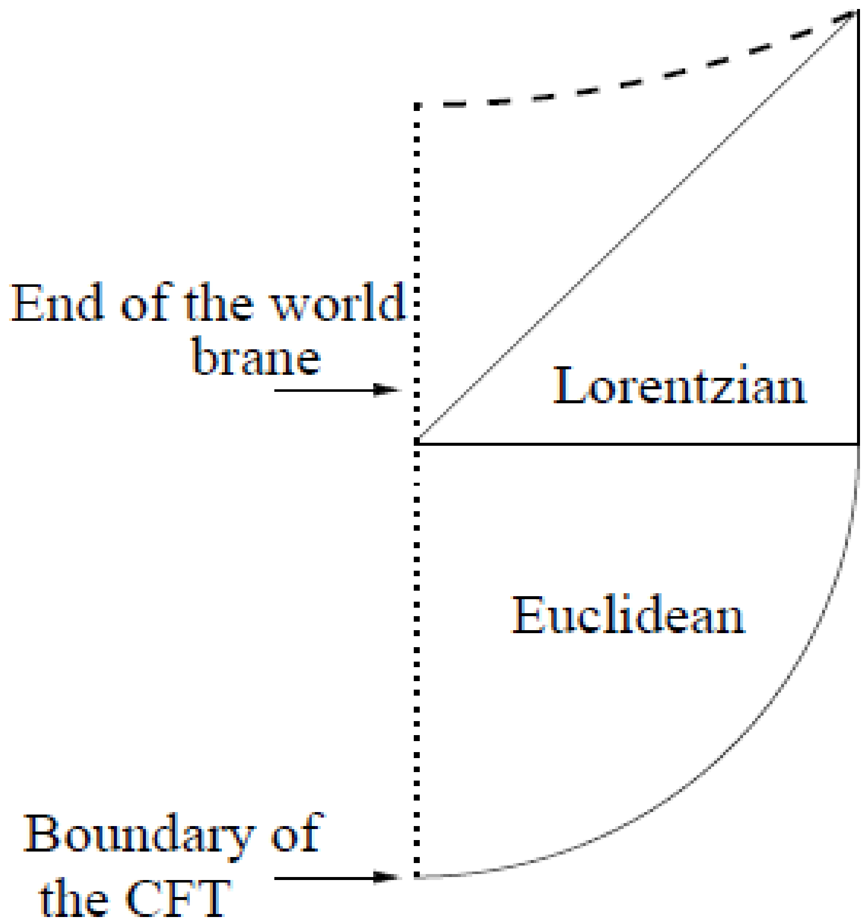
More details on thermalization
(for dynamical entanglement of a spherical region)



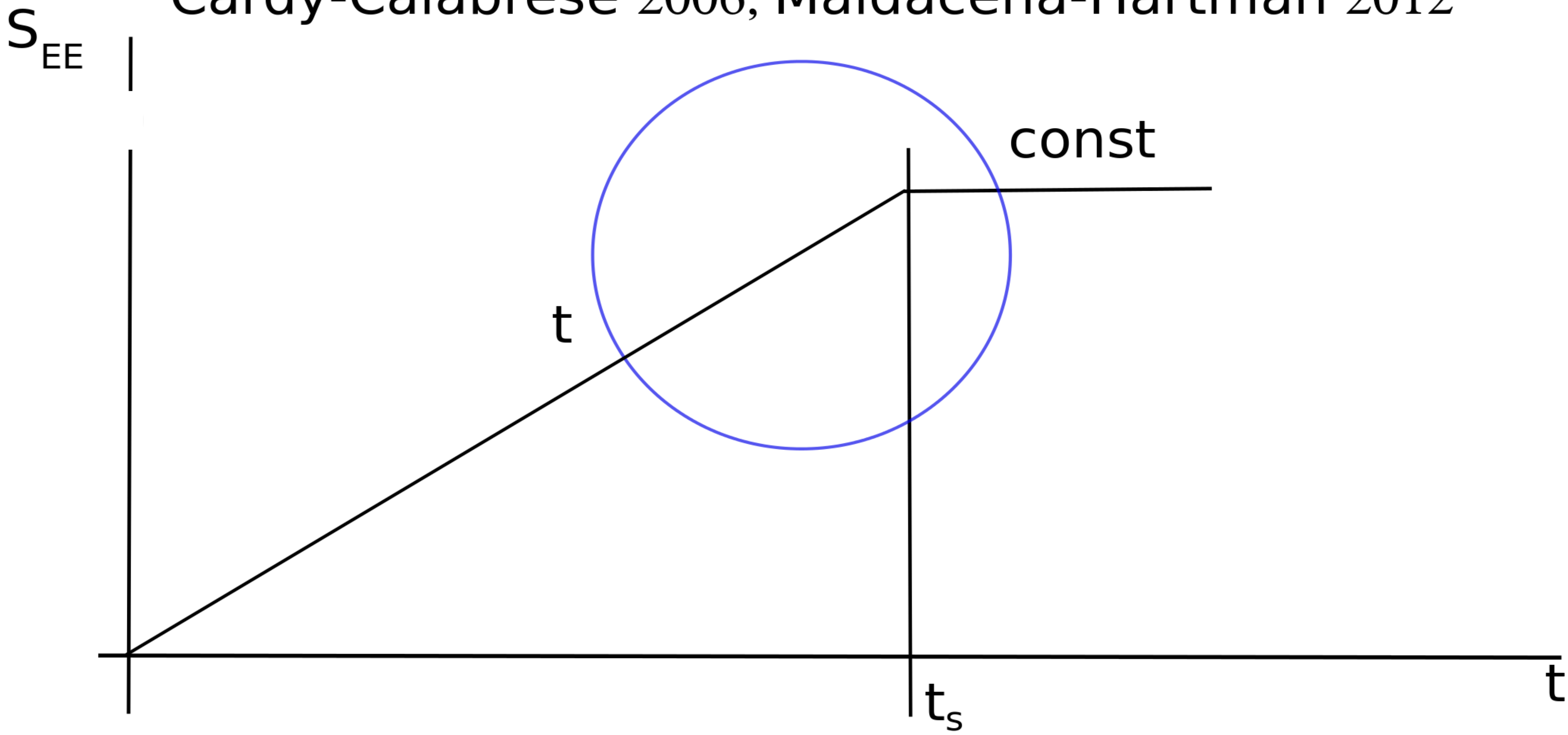
Thermalization for strip geometry



Hartle-Hawking quench

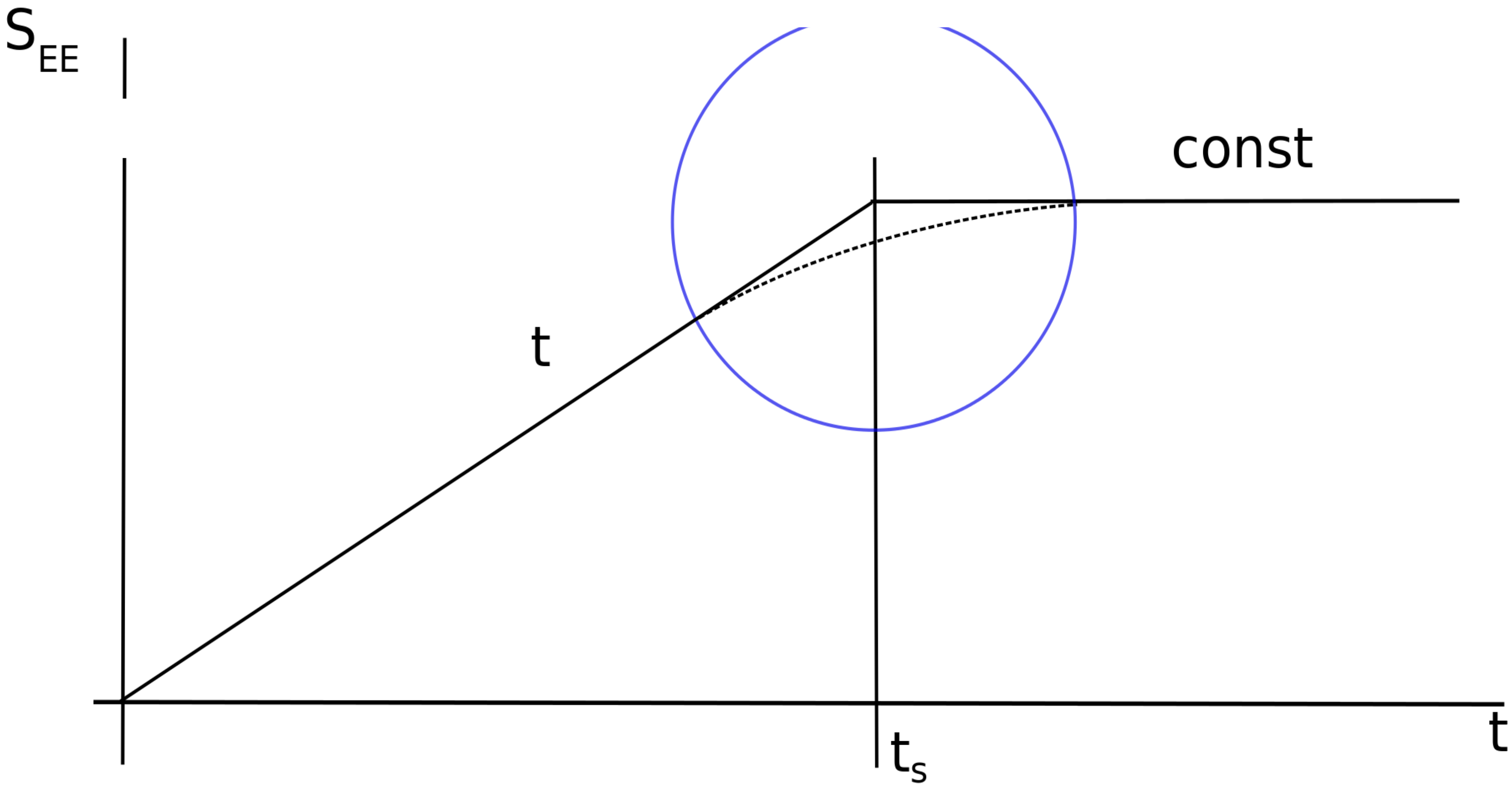


Cardy-Calabrese 2006, Maldacena-Hartman 2012



Instantaneous quench

$$S(t) = 2ts_{\rm eq}, \qquad s_{\rm eq} = \frac{\pi c}{6\beta}, \qquad t < t_s = l/2$$
$$= ls_{\rm eq} = \frac{\pi c l}{6\beta}, \qquad t > t_s$$



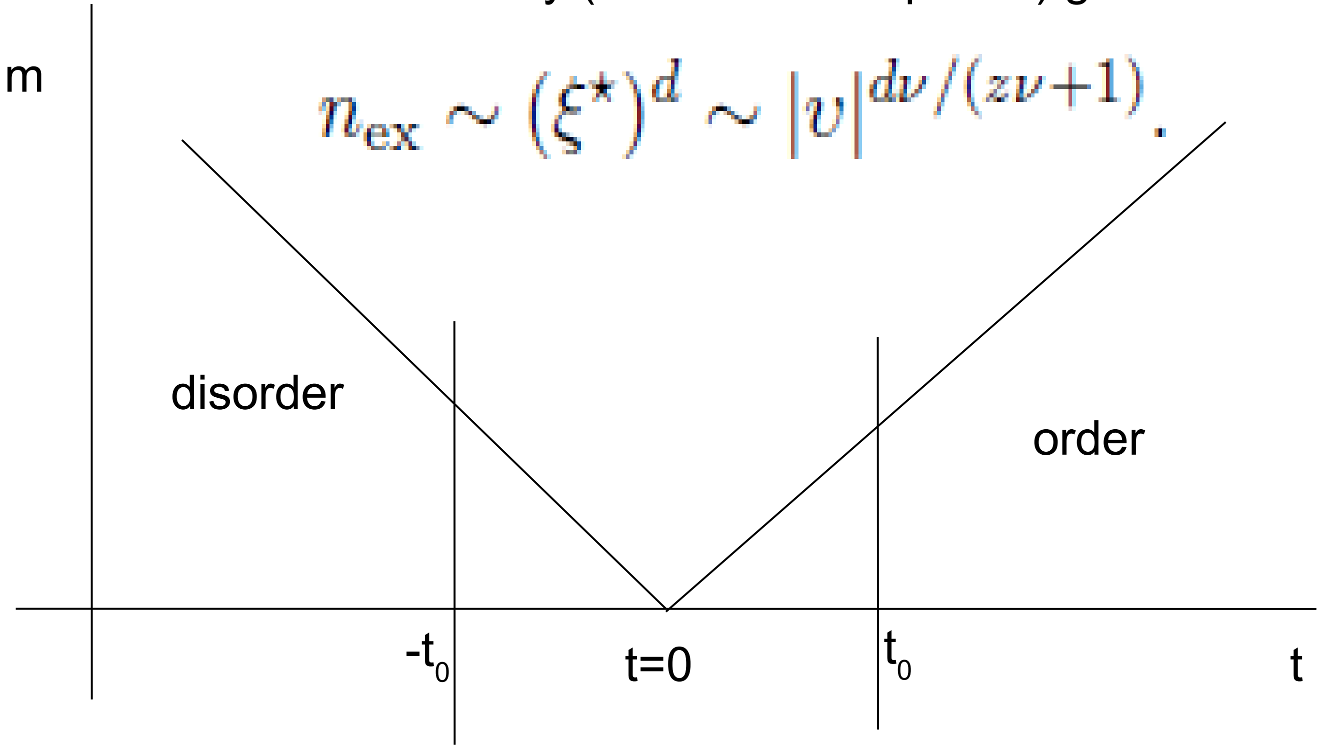
There is a scaling region where the neck is smoothened; these carry non-trivial information about the most relevant operator. These also have interpretation as the quasinormal modes. **This will be one of our regions of interest.**

Quench at a finite speed:

Kibble Zurek: parameteric transition through a critical point. Universal features remember about the critical theory and the speed of quench

Detail: disordered---> ordered: let the gap m(t) pass through a critical point m(0)=0 at finite speed v. Initially, in the far past, the Schrodinger evolution is adiabatic. After t_0 such that $t_0 = 1/m(t_0)$, it is not adiabatic, similar to a sudden quench to an excited state in the gapless theory. At late times the defect density (in the ordered phase) goes as

$$n_{\text{ex}} \sim (\xi^{\star})^d \sim |v|^{d\nu/(z\nu+1)}.$$



Where ν is the correlation function exponent. Z is the Lifshitz exponent.

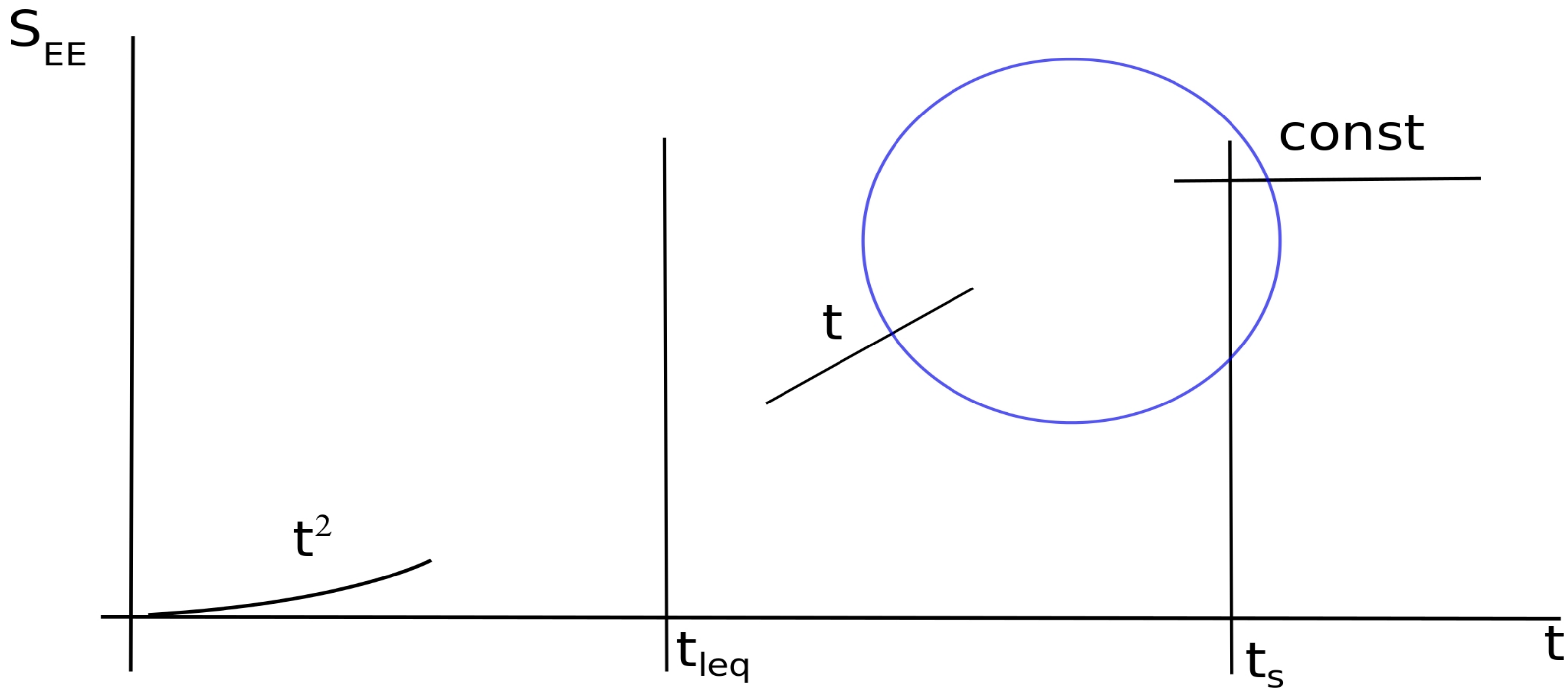
The limit to instantaneous quench: (Das-Myers et al 2013-14)

If the quenching parameter is taken to be the coefficient of a relevant operator $\mathcal{O}_\Delta$ with a total change $\delta\lambda$ over a time interval δt then the asymptotic one-point function of the operator is given by

$$\langle \mathcal{O}_\Delta \rangle_{ren} \sim \delta\lambda \, (\delta t)^{d-2\Delta}$$

The limit $\delta t \to 0$ to instantaneous quench is subtle [Das-Galante-Myers 1411.7710]

Back to entanglement entropy



How universal are the above features? Is this true also for other observables?

How does one determine the time scales, and the rates in the various time segments?

Is an asymptotic behaviour guaranteed? Under what circumstances? Is the asymptotic behaviour thermal?

Plan of the talk

1. Instantaneous (global) quenches in a 2D CFT; asymptotic time development of

--------- thermalization function

---------- entanglement entropy

---------- equal time correlation functions

The new result will be the calculation of thermalization rates as a function of conserved charges of the initial state (alternatively, as a function of chemical potentials of the final ensemble)

2. Comparison of the above with “formation” of higher spin black holes

--------- equilibrium properties

--------- quasinormal frequencies

1. Instantaneous (global) quenches in a 2D CFT.

Since the 2D CFT does not have a scale associated with it (apart from the standard UV cutoff defining the theory), scales can arise from

(a) the initial state,

and/or

(b) the choice of the observable whose time-development is being studied.

We will first discuss a class of initial states studied extensively by Cardy, Calabrese et al.

$$|\psi\rangle = e^{-\kappa H}|\mathrm{Bd}\rangle$$

The idea is to essentially start with a conformal "boundary" state at the t=0 boundary. E.g. for c=1 free scalar field theory, a Dirichlet boundary state is

$$|\mathrm{Bd}\rangle = e^{\sum_n a_n^\dagger \tilde{a}_n^\dagger}|0\rangle$$

Receives contribution from all modes, including infinite momenta; non-normalizable. This requires introduction of the exponential cutoff. Here n/L = momentum.

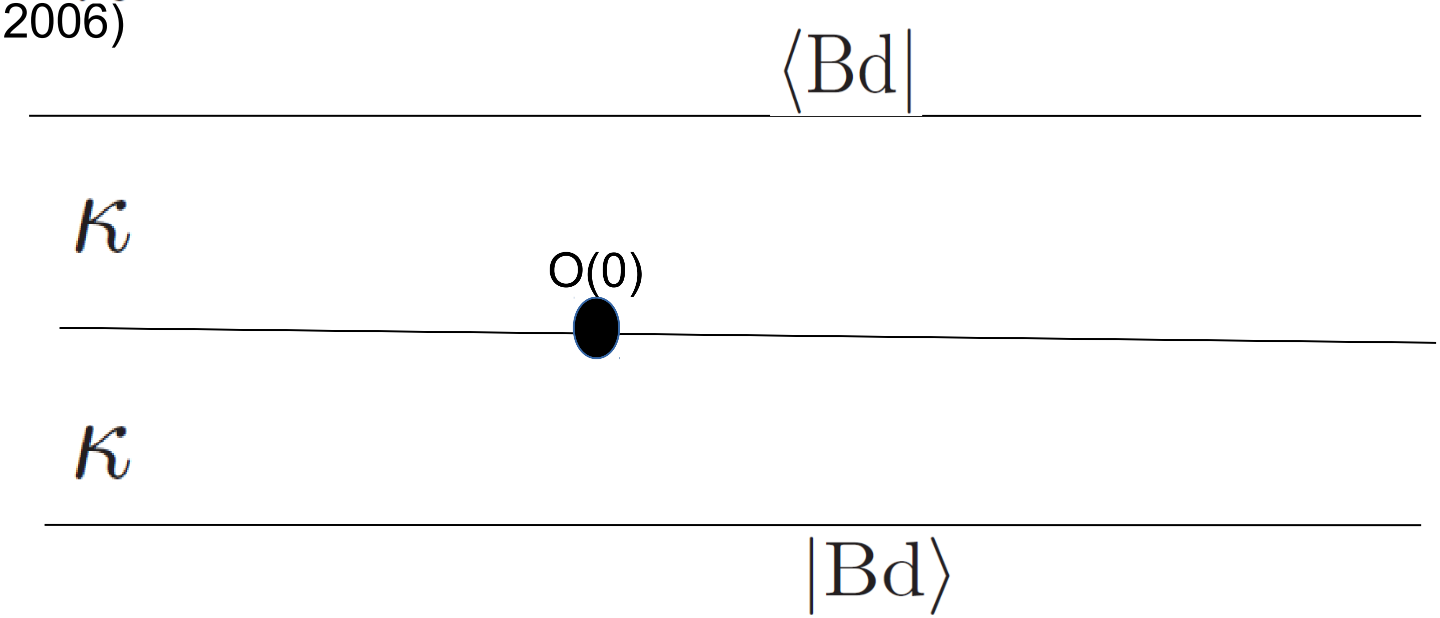
The cutoff κ introduces a scale in the intial wavefunction. 2 ways to understand:

(i) $\exp[-\kappa n N_n / L]$ This implies the existence of a momentum cutoff in the wavefunction

$$p_{max} = n_{max}/L = 1/\kappa$$

It follows that the energy density $\mathcal{E} \propto 1/\kappa^2$

(ii) κ can be understood as an imginary time evolution (Cardy-Calabrese 2006)



Euclidean CFT representationn of $\langle \psi_0 | O(0) | \psi_0 \rangle$

This allows representation of real time correlation functions in the quenched state as EFT objects with complex time.

Consequence: (1) system attains equilibrium which is thermal, with

$$\kappa = \beta/4$$

(2) large time limit of one-point functions is given by

$$\langle \phi_k(0,t) \rangle = \langle \phi_k(0,t) \rangle_{\mu=0} \times a(\mu) e^{-\gamma_k t}$$

$$\gamma_k = 4\pi h_k/\beta$$

(3) The thermalization rate of the reduced density matrix is given by

$$I(t) = 1 - \alpha e^{-2\gamma t}, \quad \gamma = 4\pi h_m/\beta$$

For more general cutoffs, involving other operators than just the Hamiltonian, the above trick does not work. E.g. consider

$$|\psi\rangle = e^{-\kappa_1 H - \kappa_2 W_3}|\mathrm{Bd}\rangle$$

The first line of reasoning appears to still work for the free scalar model:

$$\exp[-(\kappa_1 n + \kappa_2 n^2) N_n / L]$$

1

We get

$$L p_{max} = n_{max} = \frac{\sqrt{\kappa_1^2 + 4\kappa_2} - \kappa_1}{2\kappa_2}$$

For small κ_2

$$n_{max} = \frac{1}{\kappa_1}\left(1 - \frac{1}{4}\kappa_2/\kappa_1^2\right) \sim \frac{1}{\beta}(1 - \mu/\beta^2)$$

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Thermalization with chemical potentials, GGE and higher spin black holes

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Abstract

We study long time behaviour of local observables following a quantum quench in 1+1 dimensional conformal field theories. These theories, which include integrable models, are characterized by additional conserved charges besides the energy. We compute the time-dependent one-point functions of arbitrary quasiprimary fields as a function of the chemical potentials conjugate to the conserved charges. This allows us to show that the reduced density matrix for an interval, computed in a large class of initial states, asymptotically approaches that in an equilibrium ensemble. We compute the thermalization rate γ in a systematic perturbation in the chemical potentials, using the short interval expansion and a new technique to sum over an infinite number of Feynman diagrams. In a holographic situation the final equilibrium ensemble, which is a generalized Gibbs ensemble (GGE), corresponds to a higher spin black hole. We also show in a specific example that the thermalization rate γ computed in the field theory agrees with the quasinormal frequency of a scalar field in the dual higher spin black hole.

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Contents

1 Introduction and Summary

The study of thermalization in closed interacting quantum systems has a long history (see, e.g. [1] for a review). It has been known ever since the celebrated work of Fermi, Pasta and Ulam (FPU) that interacting classical systems need not necessarily equilibrate. The question of finding sufficient conditions for thermalization in quantum systems is also an open one. Recently, the advent of holography has linked the issue of thermalization in strongly coupled quantum field theories to another important, classical, problem of black hole formation (see, e.g. [2, 3] and references therein). In the latter setting too, the issue of gravitational collapse of a given matter distribution is rather nontrivial; indeed there is an interesting debate in the current literature regarding the fate of perturbations in anti-de-Sitter spacetimes. In this paper, we will concern ourselves with a proof of thermalization in two-dimensional conformal field theories (CFTs) after a quantum quench and will calculate the rate of approach to equilibrium. We will also discuss the gravity dual of the equilibrium state in terms of certain (higher spin) black holes and compare their quasinormal frequencies with the CFT results.

To begin, consider a closed system (represented by a quantum field theory in d spacetime dimensions) that is in its ground state for time $t < 0$ and is instantaneously excited to a non-stationary state $|\psi_0\rangle$ which is treated as the initial state for further time evolution in $t > 0$.[4] We will use the following definition of thermalization. Let us consider a local operator (or a string of local operators) O.[5] We will say that the observable O "thermalizes" if [6]

$$\langle O(t)\rangle \xrightarrow{t \gg t_0} \langle O\rangle_{eqm} = \mathrm{Tr}(\rho_{eqm} O) \tag{1}$$

where the equilibrium ensemble has the form

$$\rho_{eqm} = \frac{1}{Z}\exp[-\beta H - \sum_i \mu_i W_i], \quad Z = \mathrm{Tr}\exp[-\beta H - \sum_i \mu_i W_i] \tag{2}$$

We assume here that the system posssesses additional conserved charges W_i besides the hamiltonian H. Here β, μ_i are the temperature and chemical potentials chosen such that the equilibrium values of the energy and the other conserved charges coincide with those of the initial state. Eqn. (1) may be understood as a grand canonical version of the quantum ergodic hypothesis (QEH) [1]. In case of an integrable model, we have an infinite number of commuting conserved charges W_i. In this case the grand canonical ensemble is called the Generalized Gibbs ensemble (GGE). The possible asymptotic approach (1) in such a case has been discussed in [7] (see also [8]).

In general, the question of whether or not (1) is actually satisfied is complicated, and depends on the choice of the system, the initial state and the observable. To delink from the dependence on specific choice of observables, we will find it useful, following [9], to use the notion of a thermalization function which depends only on the initial state and the choice of a local region A (see footnote 5).

The thermalization function: Let us first define, for a spatial region A, the "dynamical reduced density matrix" $\rho_{dyn,A}(t)$ as

$$\rho_{dyn,A}(t) = \mathrm{Tr}_B\ \rho_{dyn}(t),\ \rho_{dyn}(t) \equiv (\exp[-iHt]|\psi_0\rangle\langle\psi_0|\exp[iHt]) \tag{3}$$

and the "equilibrium reduced density matrix" $\rho_{eqm,A}(\beta,\mu)$ as

$$\rho_{eqm,A}(\beta,\mu_i) = \mathrm{Tr}_B\ \rho_{eqm} = \mathrm{Tr}_B\ \exp[-\beta H - \mu_i W_i] \tag{4}$$

[4]This is in the class of time evolutions grouped together as a quantum quench [4, 5].

[5]The consideration of local operators can be motivated by equilibration models like the Caldeira-Leggett model [6]. The idea is to regard operators in a local region A as the "system" which is coupled to the rest which is treated as the "bath" (this division is captured naturally by a reduced density matrix, as we will see below). Note that couplings between localized observables exist even in "free" theories which are typically characterized by decoupled modes which are non-local, e.g. momentum modes.

[6]Here $O(t) = e^{iHt} O e^{-iHt}$ is a Heisenberg operator.

B is the (infinite) spatial region which forms the complement of A.[7] The entire Hilbert space is assumed to be of the form $\mathsf{H}_A \otimes \mathsf{H}_B$. Tr_B implies tracing over H_B. The motivation for considering a reduced density matrix in the context of thermalization is described in footnote 5.

We define the "thermalization function for the region A" as the overlap $I(t)$ between the dynamical reduced density matrix and the equilibrium reduced density matrix [9] [8]

$$I(t) = \mathrm{Tr}(\hat{\rho}_{dyn,A}(t)\hat{\rho}_{eqm,A}(\beta,\mu_i)) = \frac{\mathrm{Tr}(\rho_{dyn,A}(t)\rho_{eqm,A}(\beta,\mu_i))}{[\mathrm{Tr}(\rho_{dyn,A}(t)^2)\mathrm{Tr}(\rho_{eqm,A}(\beta,\mu_i)^2)]^{1/2}} \tag{5}$$

Here $\hat{\rho} = \rho/\sqrt{\mathrm{Tr}\rho^2}$ denotes a 'square-normalized' density matrix.[9]

Armed with these definitions, we will now briefly state our results.

Results: In this paper, we will restrict ourselves to 1+1 dimensional CFT, parametrized by coordinates $\sigma, t \in \mathbb{R}^2$. We take the entangling region A to be a single interval of length l: $\sigma \in (-l/2, l/2)$. We will define our initial quenched state $|\psi_0\rangle$ as follows [10]

$$|\psi_0\rangle = \exp[(-\beta H - \mu_i W_i)/4]|Bd\rangle \tag{6}$$

Here $|Bd\rangle$ is a conformal boundary state; the exponential factor cuts off the UV modes to make the state normalizable. The cut-off parameters are denoted by β, μ_i because they turn out to coincide with the β, μ_i of the equilibrium ensemble, in the sense mentioned below (2).

A. The main result of our paper is that the thermalization function has the following form

$$I(t) = 1 - \alpha\, e^{-2\gamma t} + ..., \tag{7}$$

where

$$\gamma = \frac{2\pi}{\beta}\left[\Delta + \sum_n \tilde{\mu}_n \mathsf{Q}_n + O(\mu^2)\right], \quad \mathsf{Q}_n = \frac{1}{(2\pi)^{n-2}}\left(i^n \mathsf{q}_n + (-i)^n \bar{\mathsf{q}}_n\right) \tag{8}$$

We will define the dimensionless quantities

$$\tilde{l} \equiv l/\beta, \tilde{\mu}_n \equiv \frac{\mu_n}{\beta^{n-1}}, \tag{9}$$

The result (7) is valid in the limit $t \gg \beta$ [11], the coefficient α is expressed in a double expansion in $\tilde{\mu}, \tilde{l}$. Δ, Q_n refer, respectively, to the scaling dimension ($= 2h$) and the

[7] Throughout this paper, we will consider field theories with an infinite spatial extent.

[8] Here and below we will write Tr_A as Tr when it is obvious from the context.

[9] Note that operators in a Hilbert space H can themselves be regarded as vectors in $\mathsf{H} \times \mathsf{H}^*$; under this interpretation $\mathrm{Tr}(A\ B)$ defines a positive definite scalar product. With this understanding, we will regard the hatted density matrices as unit vectors.

[10] Here we can take, for W_i, generic conserved charges besides the hamiltonian; however, for the purposes of this paper, we will identify them with W_i-charges of 2D CFT (much of what we say will go through independent of this specific choice as long as these charges mutually commute and are defined from charge densities which are quasiprimary fields of the conformal algebra).

[11] More strictly, for $t - l/2 \gg \beta$, but the distinction is not visible in the short interval expansion employed here which is valid for $\tilde{l} = l/\beta \ll 1$.

(shifted) W_n-charges (58) of the specific primary field ϕ which corresponds to the 'slowest transient' (i.e. it has the minimum Δ among all quasiprimary fields which have a non-zero expectation value in the initial state $|\psi_0\rangle$).[12] The $\mu = 0$ result corresponds to the result quoted for $I(t)$ in [9]. [13]

B. The main ingredient used to derive the result (7) is the calculation of the one-point function of a primary field ϕ_k in the state $|\psi_0\rangle$ (6) as a function of the chemical potentials $\tilde{\mu}_n$. The one-point function, at large times t, turns out to behave as (see, e.g. (81))and (81) for details)

$$\langle \phi_k(0,t) \rangle = \langle \phi_k(0,t) \rangle_{\mu=0} \times a(\mu) e^{-\gamma_k t} \tag{10}$$

where we have ignored faster transients. The time-independent quantity $a(\mu)$ is of the form $a(\mu) = 1 + o(\mu)$. The important point to note that the thermalization rate that appears here is the *same as in (8)*, where Δ, Q_n refer to properties of the field ϕ_k. The one-point function with zero chemical potential is given by (97).

Consequences: Let us work out some immediate consequences of (7).

1. Thermalization: Eq. (7) implies that

$$I(t) \xrightarrow{t\to\infty} 1, \tag{11}$$

Since $I(t)$, as defined in (5), satisfies a Cauchy-Schwarz inequality [14], the above implies that

$$\begin{aligned} &(i) \quad \hat{\rho}_{dyn,A}(t) \xrightarrow{t\to\infty} \hat{\rho}_{eqm,A} \\ &(ii) \quad \langle \psi_0 | O(\sigma_1, t) O(\sigma_2, t) ... | \psi_0 \rangle = \mathrm{Tr}(\hat{\rho}_{dyn,A}(t) O(\sigma_1) O(\sigma_2) ...) \\ &\qquad \xrightarrow{t\to\infty} \mathrm{Tr}(\hat{\rho}_{eqm,A} O(\sigma_1) O(\sigma_2) ...), \quad \sigma_1, \sigma_2, ... \in A \end{aligned} \tag{12}$$

The first equality in (ii) follows from the fact that expectation values of local operators in the region A are given by their expectation values in the reduced density matrix.

This proves thermalization (1) for our choice of initial states.

2. Eq. (7) implies the following asymptotic behaviour of the dynamical reduced density matrix

$$\hat{\rho}_{dyn,A}(t) = \hat{\rho}_{eqm,A}(\beta, \mu_i) \left(1 - \alpha\, e^{-2\gamma t} + ...\right) + \hat{Q} \left(\sqrt{2\alpha}\, e^{-\gamma t} + ...\right) \tag{13}$$

As before, the omitted terms represent higher transients. To prove this, we decompose the unit vector $\hat{\rho}_{dyn,A}(t)$ along the unit vector $\hat{\rho}_{eqm,A}(\beta, \mu_i)$ and a (possibly time-dependent) orthogonal unit vector $\hat{Q}$ (here we use the notion of unit vectors and scalar

[12] We will assume here that the spectrum of such Δ's is bounded below by a finite positive number. In case of a free scalar field theory, we can achieve this by considering a compactified target space.

[13] Our result differs from [9] by a factor of 2.

[14] This can be proved by using the scalar product of footnote 9.

product defined in footnote 9). The component along the first vector follows from (7) and the orthogonal component is determined by the total normalization. The operator $\hat{Q}$ thus satisfies

$$\mathrm{Tr}(\hat{Q}^2) = 1,\ \mathrm{Tr}(\hat{Q}\ \hat{\rho}_{eqm,A}(\beta, \mu_i)) = 0 \tag{14}$$

This does not uniquely determine $\hat{Q}$. However, we will nevertheless be able to use (13) to relate our results to the known time-dependence of entanglement entropy [10, 4], and of one-point function of primary operators. In particular, we will find below that the exponential time-dependence in (13) corresponds to a rounding off of the dynamical entanglement entropy curve around $t = l/2$.

3. Quasinormal modes: In a holographic model, (12) describes an approach to a higher spin black hole (HSBH) geometry, as the equilibrium ensemble in the CFT has been interpreted in [11, 12] in terms of a HSBH. Therefore, (13) represents a perturbation of the bulk fields from a background HSBH geometry. The exponential decay thus corresponds to the known exponential decay of quasinormal modes in the bulk. We will compare our exponent $\gamma(\tilde{\mu})$ below with quasinormal frequencies of HSBH [13] (more details will be provided in a forthcoming publication [14]). *** Refer to QNM section.

The plan of the paper is as follows. In Section

2 Calculation of $I(t)$ for $\mu_i = 0$

Let us write (7) in the form

$$\begin{aligned}
& I(t) = Z_{sc}/\sqrt{Z_{ss}Z_{cc}} = \hat{Z}_{sc}/\sqrt{\hat{Z}_{ss}\hat{Z}_{cc}}, \\
& Z_{sc} \equiv \mathrm{Tr}(\rho_{dyn,A}(t)\rho_{eqm,A}(\beta,\mu)),\ \hat{Z}_{sc} = Z_{sc}/(Z_s Z_c) \\
& Z_{ss} \equiv \mathrm{Tr}(\rho_{dyn,A}(t)\rho_{dyn,A}(t)),\ \hat{Z}_{ss} = Z_{ss}/Z_s^2, \\
& Z_{cc} \equiv \mathrm{Tr}(\rho_{eqm,A}(\beta,\mu)\rho_{eqm,A}(\beta,\mu)),\ \hat{Z}_{cc} = Z_{cc}/Z_c^2, \\
& Z_s = \mathrm{Tr}(\rho_{dyn}(t)) = \langle\psi_0|\psi_0\rangle,\ Z_c = \mathrm{Tr}(\rho_{\beta,\mu})
\end{aligned} \tag{15}$$

For the rest of the section, we will put $\mu_i = 0$. Some of the results of this section are briefly described in [9]. We will come back to non-zero chemical potentials in the next section. Here the subscripts s and c refer to a 'strip' and a 'cylinder' geometry as described above Eq. (19).

For convenience we will first compute these quantities in Euclidean time $\tau = it$ and later analytically continue back to Lorentzian time. With this, each of the expressions Z_{sc}, Z_{ss}, Z_{cc} is of the form

$$\mathrm{Tr}(\rho_{A,1}\rho_{A,2}) = \int_{\text{geometry 1}} \mathbf{D}\varphi_1 \int_{\text{geometry 2}} \mathbf{D}\varphi_2\ \delta(F[\varphi_1,\varphi_2]) \exp\left(-S[\varphi_1] - S[\varphi_2]\right) \tag{16}$$

where $S[\varphi]$ represents the action for the CFT (with fields φ) and the delta-functional in the measure represents a gluing condition between a geometry '1' and a geometry '2' along a 'cut' which is the location, at a particular time τ, of the spatial interval $A : \sigma \in (-l/2, l/2)$ [15] For Z_{ss}, both geometries are that of a strip of the Euclidean plane described by complex coordinates $(w, \bar{w}) = \sigma \pm i\tau$ defined by boundaries at $\tau = \pm\beta/4$ with boundary conditions determined by the boundary state $|Bd\rangle$ introduced in (6). For Z_{cc}, both geometries are that of a cylinder cut of the Euclidean plane with identified boundaries at $\tau = -\beta/4, 3\beta/4$. The geometries for both Z_{ss} and Z_{cc} are familiar from calculations of Entanglement Renyi entropy (of order 2) and can be calculated from appropriate correlation functions of twist fields [15] which exchange two identical geometries. For Z_{sc}, the two glued geometries are different (that of a strip and a cylinder), hence the method of twist operators do not apply in a straightforward fashion. (See Figure 1). In this paper, we will therefore, employ the method of the short interval expansion.

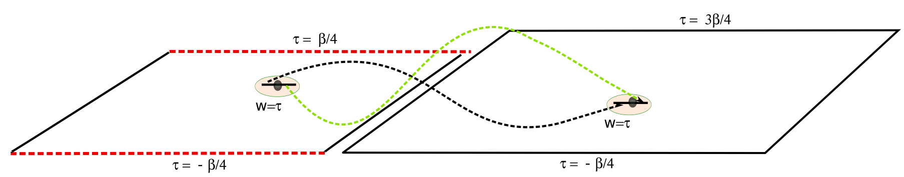

Figure 1: The glued strip and cylinder along the cut.

2.1 Short interval expansion

The idea of the short interval expansion [16] is as follows. To begin, we express the functional integral (16) as an overlap of two wavefunctions in $\mathsf{H}_1 \otimes \mathsf{H}_2$, as follows

$$Z_{12} = \mathrm{Tr}(\rho_{A,1}\rho_{A,2}) = \langle \psi_{out}|\psi_{in}\rangle = \int_{w_1 \in \mathcal{D}_1} \mathbf{D}\overline{\varphi}_1(w_1) \int_{w_2 \in \mathcal{D}_2} \mathbf{D}\overline{\varphi}_2(w_2) \;\; \psi_{in}[\overline{\varphi_1}, \overline{\varphi_2}] \; \psi^*_{out}[\overline{\varphi_1}, \overline{\varphi_2}]$$
$$\psi_{in}[\overline{\varphi_1}, \overline{\varphi_2}] \equiv \int_{w_1 \in \mathcal{D}_1} \mathbf{D}\varphi_1(w_1) \int_{w_2 \in \mathcal{D}_2} \mathbf{D}\varphi_2(w_2) \delta(\varphi_1|_{\partial\mathcal{D}_1} - \overline{\varphi_1}) \delta(\varphi_2|_{\partial\mathcal{D}_2} - \overline{\varphi_2}) \delta(F[\varphi_1, \varphi_2]) \exp\left(-S[\varphi_1] - S[\varphi_2]\right)$$
$$\psi_{out}[\overline{\varphi_1}, \overline{\varphi_2}] \equiv \int_{w_1 \notin \mathcal{D}_1} \mathbf{D}\varphi_1(w_1) \int_{w_2 \notin \mathcal{D}_2} \mathbf{D}\varphi_2(w_2) \delta(\varphi_1|_{\partial\mathcal{D}_1} - \overline{\varphi_1}) \delta(\varphi_2|_{\partial\mathcal{D}_2} - \overline{\varphi_2}) \exp\left(-S[\varphi_1] - S[\varphi_2]\right) \quad (17)$$

Here $\mathcal{D}_1$ (respectively, $\mathcal{D}_2$) is a small disc drawn around the cut in geometry 1 (respectively, geometry 2).

Note that only $|\psi_{in}\rangle$ depends on the gluing condition since the delta functional in the measure does not affect $|\psi_{out}\rangle$. The basic point of the short interval is that in the limit when the length l of the cut is small compared with the characterizing length scale of the geometries (in our case, when $l \ll \beta$), the wavefunction $\psi_{in}[\varphi_1, \varphi_2]$ becomes jointly localized at the centre $(w_1, \bar{w}_1)$ of the disc $\mathcal{D}_1$ *and* at the centre $(w_2, \bar{w}_2)$ of the disc $\mathcal{D}_2$ [16], and hence

[15] To be precise, $\delta[F] = \delta(\varphi_1(A_<) - \varphi_2(A_>)) \, \delta(\varphi_1(A_>) - \varphi_2(A_<))$, where $A_<$ ($A_>$) represents the limiting value from below (above) the cut.

[16] We will take the centre of the disc in each geometry to coincide with the centre of the cut, which has coordinates $w = i\tau, \bar{w} = -i\tau$.

can be expanded in terms of local operators, as follows

$$|\psi_{in}\rangle = \sum_{k_1,k_2} C_{k_1,k_2}\ \phi_{k_1}(w_1,\bar{w}_1)\,\phi_{k_2}(w_2,\bar{w}_2)|0\rangle_1 \otimes |0\rangle_2 \tag{18}$$

Here k_1, k_2 label a complete basis of quasiprimary operators of the CFT Hilbert space. Each term in the sum represents a factorized wavefunction (between geometries 1 and 2), which, therefore, gives [17]

$$\begin{aligned}
\hat{Z}_{sc} &= \sum_{k_1,k_2} C_{k_1,k_2}\langle\phi_{k_1}(w_1,\bar{w}_1)\rangle_{str}\langle\phi_{k_2}(w_2,\bar{w}_2)\rangle_{cyl},\\
\hat{Z}_{ss} &= \sum_{k_1,k_2} C_{k_1,k_2}\langle\phi_{k_1}(w_1,\bar{w}_1)\rangle_{str}\langle\phi_{k_2}(w_2,\bar{w}_2)\rangle_{str},\\
\hat{Z}_{cc} &= \sum_{k_1,k_2} C_{k_1,k_2}\langle\phi_{k_1}(w_1,\bar{w}_1)\rangle_{cyl}\langle\phi_{k_2}(w_2,\bar{w}_2)\rangle_{cyl}
\end{aligned} \tag{19}$$

Here the subscripts *str* and *cyl* refer to "strip", and "cylinder" respectively. The one-point functions are evaluated on the respective geometries without any cut (see Section B for more details). The glued functional integral (16), (17) is recovered by summing over k_1, k_2 with the coefficients C_{k_1,k_2}; , as clear from (19) these are determined by the gluing condition and depend on the size of the cut [16] (see Section A for more details).

2.1.1 Conformal map

We will find it convenient to compute correlators on the cylinder by mapping the operators to the infinite plane, using the conformal map

$$z = i\exp[(2\pi/b)w] \tag{20}$$

The same map also maps the strip to the upper half plane (UHP). With this, the mid-point of the cut gets mapped to

$$z = i\exp[(2\pi\tau/b)] = i\exp[-2\pi t/\beta] \tag{21}$$

where the last expression uses the Lorenzian time $t = -i\tau$. The 'image' point for correlators on the strip gets mapped to

$$z' = \bar{z} = -i\exp[(2\pi\tau/b)] = -i\exp[2\pi t/\beta] \tag{22}$$

Note that for large times $z \to 0$ where $z' \to -i\infty$.

2.2 Proof of thermalization

Armed with the ingredients above, we will now prove (11).

[17] In case geometries 1 and 2 are identical, the superscripts in $w_i, \bar{w}_i, i = 1, 2$ indicate which sheet we are considering.

Proof:
Note that the only one-point functions on the strip that survive in the limit $t \to \infty$ are ones for which

$$\langle \phi_k \rangle_{cyl} = \langle \phi_k \rangle_{str}$$

All other one-point functions for which the value on the strip differs from the one on the cylinder are those which have a time-dependence given by $(z-\bar{z})^{-2h_k} \propto \exp[-4\pi t h_k/\beta]$ (see Section B). Hence in the large time limit all such one-point functions vanish.

In this limit, therefore, $Z_{sc} = Z_{ss} = Z_{cc}$, hence

$$I(t \to \infty) = 1$$

which proves (11).

2.3 Thermalization rate γ

To evaluate $I(t)$ at finite times, we organize the terms in $\hat{Z}_{sc}, \hat{Z}_{ss}, \hat{Z}_{cc}$ as follows

$$\hat{Z}_{sc} = C_{0,0}(1+S_1^{sc}),\ S_1^{sc} = \sum_a \hat{C}_{a,0}(\langle\phi_a\rangle_{str} + \langle\phi_a\rangle_{cyl}) + \sum_{ab} \hat{C}_{a,b}\langle\phi_a\rangle_{str}\langle\phi_b\rangle_{cyl}$$
$$\hat{Z}_{ss} = C_{0,0}(1+S_1^{ss}+S_2^{ss}),\ S_1^{ss} = 2\sum_a \hat{C}_{a,0}\langle\phi_a\rangle_{str} + \sum_{ab} \hat{C}_{a,b}\langle\phi_a\rangle_{str}\langle\phi_b\rangle_{str},\ S_2^{ss} = \sum_k \hat{C}_{k,k}(\langle\phi_k\rangle_{str})^2$$
$$\hat{Z}_{cc} = C_{0,0}(1+S_1^{cc}),\ S_1^{cc} = 2\sum_a \hat{C}_{a,0}\langle\phi_a\rangle_{cyl} + \sum_{ab} \hat{C}_{a,b}\langle\phi_a\rangle_{cyl}\langle\phi_b\rangle_{cyl} \tag{23}$$

where $a, b, ...$ denote descendents of the identity operator, k labels other primaries (than the identity) and their descedents. $\hat{C} \equiv C/C_{0,0}$. Using the results in Sections A and B we get

$$\begin{aligned}
S_1^{sc} &= -a_T\tilde{l}^2\left(1+O(\tilde{l})^2\right) + a_{T\bar{T}}\tilde{l}^4 e^{-8\pi\tilde{t}}\left(1+O(\tilde{l})^2\right) + O(e^{-8\pi\tilde{t}})\\
S_1^{ss} &= -a_T\tilde{l}^2\left(1+O(\tilde{l})^2\right) + 2a_{T\bar{T}}\tilde{l}^4 e^{-8\pi\tilde{t}}\left(1+O(\tilde{l})^2\right) + O(e^{-8\pi\tilde{t}})\\
S_2^{ss} &= \sum_k \left[a_k\tilde{l}^{4h_k}e^{-8\pi h_k\tilde{t}}\left(1+O(\tilde{l})^2\right) + O(e^{-12\pi h_k\tilde{t}})\right]\\
S_1^{cc} &= -a_T\tilde{l}^2\left(1+O(\tilde{l})^2\right)\\
a_T &= \frac{c\pi^2}{24},\ a_{T\bar{T}} = \frac{A_{T\bar{T}}\pi^4}{8c}\ a_k = \frac{A_k^2}{n_k}\left(\frac{\pi}{2}\right)^{4h_k}
\end{aligned} \tag{24}$$

To this order, it is easy to see that the contribution to $I(t)$ from descendents of identity, demarcated by $a_T, a_{T\bar{T}}$, vanishes. The leading contribution to $I(t)$, demarcated by a_k, occurs only in D_2 and comes from the primary field $\phi_m(z,\bar{z})$ for which h_k is the minimum ($= h_m$). With this we get

$$I(t) = 1 - \alpha\exp[-\gamma t] + ...,\ \gamma = 8\pi h_m = 4\pi\Delta_m \tag{25}$$

which is of the form (7) for $\mu = 0$, with

$$\alpha \equiv \frac{A_m^2}{n_m}\left(\frac{\pi}{2}\right)^{4h_m}(\tilde{l})^{4h_m}\left(1+O(\tilde{l})^2\right) \tag{26}$$

The discarded terms in (25) are faster transients. This proves (7) for zero chemical potential. This result has already appeared in [9]. [18]

The leading time-dependence of $I(t)$ comes from $\langle\phi_m(w,\bar{w})\rangle^2_{str}$. Note that, at large t

$$\langle\phi_k(w,\bar{w})\rangle_{str} \propto \cosh(2\pi t/\beta)^{-2h_m} \sim \exp[-2\pi\Delta_m t/\beta] \tag{27}$$

where we have used Eqs. (97) and (98), and the fact that scaling dimension is $\Delta_m = 2h_m$.

3 Comparison with calculations of EE

According to our calculations, the rate of fall-off of the time-dependent density matrix is

$$\hat{\rho}_{dyn,A}(t) = (1-\alpha\exp(-2\gamma t))\hat{\rho}_{eqm,A} + \hat{Q}(t)\sqrt{2\alpha}\exp(-\gamma t) \tag{28}$$

After large times, the expression becomes

$$\begin{aligned}\hat{\rho}_{dyn,A}(t) &\simeq \hat{\rho}_{eqm,A} + \hat{Q}(t)\sqrt{2\alpha}\exp(-\gamma t)\\ &= \hat{\rho}_{eqm,A}(1+\exp(-\gamma t)\sqrt{2\alpha}\hat{\rho}^{-1}_{eqm,A}\hat{Q}(t))\end{aligned} \tag{29}$$

We shall use the expression 29 to compute the time-dependent correction to the entanglement entropy. For doing so, we calculate the following quantity,

$$\begin{aligned}\log\hat{\rho}_{dyn,A}(t) &= \log\hat{\rho}_{eqm,A} + \log(1+\exp(-\gamma t)\sqrt{2\alpha}\hat{Q}(t)\hat{\rho}^{-1}_{eqm,A})\\ &\simeq \log\hat{\rho}_{eqm,A} + \exp(-\gamma t)\sqrt{2\alpha}\hat{Q}(t)\hat{\rho}^{-1}_{eqm,A}\end{aligned} \tag{30}$$

The dynamical entanglement entropy is then given by

$$\begin{aligned}S_{dyn,A} &= -Tr(\hat{\rho}_{dyn,A}(t)\log\hat{\rho}_{dyn,A}(t))\\ &\simeq -Tr(\hat{\rho}_{eqm,A}\log\hat{\rho}_{eqm,A}) - Tr(\hat{Q}(t))\sqrt{2\alpha}\exp(-\gamma t) - Tr(\hat{Q}(t)\log\hat{\rho}_{eqm,A})\sqrt{2\alpha}\exp(-\gamma t)\end{aligned}$$

or,

$$S_{dyn,A}(t) = S_{eqm,A} - Tr(\hat{Q}(t)(1+\log\hat{\rho}_{eqm,A}))\sqrt{2\alpha}\exp(-\gamma t) \tag{31}$$

Here, $\gamma = (4\pi h_m/\beta)$.

The above expression gives us the form for the decay of the entanglement entropy for a single interval on a strip. This generic statement can be tested in the premises of the formulation due to [10] where we shall explicitly show the analogy.

In [10], one considers 2 disjoint intervals on 2 separate field theories (a thermal CFT and its thermal double) and calculates the entanglement entropy for the same. Our case is that of a single interval on a strip (half-cylinder). To calculate the entanglement entropy for such a system using the replica trick, one must calculate the n-th order Rényi entropy.

The trace of the n-th order Rényi entropy is given by the 2 pt. function of a twist and an anti-twist operator inserted at the 2 end points of the interval on the strip.

$$I_n = \mathrm{Tr}(\hat{\rho}^n_A) = \langle\sigma_+(w_1,\bar{w}_1)\tilde{\sigma}_-(w_2,\bar{w}_2)\rangle_{str} \tag{32}$$

[18] Our exponent differs from Cardy's value by a facor of 2.

Using the method of images (on a cylinder), one can write the above 2 pt. function as a four point function on the cylinder,

$$\langle \sigma_+(w_1, \bar{w}_1)\tilde{\sigma}_-(w_2, \bar{w}_2)\rangle_{str} = \langle \sigma_+(w_1)\tilde{\sigma}_-(w_2)\sigma_+(w_1')\tilde{\sigma}_-(w_2')\rangle_{cyl} \tag{33}$$

Now, however, all the twist/anti-twist operators are holomorphic alone.
In [10], the n-th order Rényi entropy is given by

$$I_{n,Mal} = \langle \sigma_+(w_1, \bar{w}_1)\sigma_-(w_2, \bar{w}_2)\sigma_+(w_3, \bar{w}_3)\sigma_-(w_4, \bar{w}_4)\rangle_{cyl} \tag{34}$$

Our calculation involves just the holomorphic part of the above. Thus,

$$J_n = \sqrt{I_{n,Mal}} = \langle \sigma_+(w_1)\sigma_-(w_2)\sigma_+(w_3)\sigma_-(w_4)\rangle_{cyl}$$

with $w_3 = w_2'$ and $w_4 = w_1'$. The twist operators are inserted at the following points:

$$w_1 = 0, \quad w_2 = l, \quad , w_3 = w_2' = l + i\frac{\beta}{2} + 2t, \quad w_4 = w_1' = +i\frac{\beta}{2} + 2t \tag{35}$$

The map from the cylinder to the UHP is $z = \exp\left(\frac{2\pi w}{\beta}\right)$. The n-th order Rényi entropy is then given by,

$$J_n = \left(\frac{\beta}{2\pi}\right)^{-4h_n}\left(2\sinh\frac{\pi l}{\beta}\right)^{-4h_n} x^{2h_n} G_n(x) \tag{36}$$

with

$$x = \frac{(z_1 - z_2)(z_2' - z_1')}{(z_1 - z_2')(z_2 - z_1')} = \frac{2\sinh^2(\frac{\pi l}{\beta})}{\cosh(\frac{2\pi l}{\beta}) + \cosh(\frac{4\pi t}{\beta})} \tag{37}$$

and

$$G_n(x,) = \langle \sigma_+(0)\sigma_-(x)\sigma_+(1)\sigma_-(\infty)\rangle_{\mathbb{C}} \tag{38}$$

The function in 38 refers to twist operators that have been inserted on a single plane. In [10], they consider the limit,

$$\frac{t}{\beta}, \frac{l}{2\beta} \gg 1 \tag{39}$$

where l is the size of the interval. In this limit, the cross ratio becomes

$$x \sim \frac{\exp\left(\frac{2\pi l}{\beta}\right)}{\exp\left(\frac{2\pi l}{\beta}\right) + \exp\left(\frac{4\pi t}{\beta}\right)} \tag{40}$$

To calculate the time independent part of the EE, they consider the large time limit,

$$\left(t - \frac{l}{2}\right) \gg \beta \tag{41}$$

In this limit, the cross ratio $x \sim 0$ and this is the dominating OPE channel. We are, however, interested in calculating the asymptotic fall-off behaviour of the EE. Thus, we shall consider the limit,

$$\left(t - \frac{l}{2}\right) > \beta \tag{42}$$

In this limit, the cross-ratio $x \sim \exp(-\frac{4\pi}{\beta}(t - l/2))$. The dominating OPE channel is still $x \sim 0$.
Calculating 38, in the $x \sim 0$ OPE channel, while taking into account the contribution from the operator with the lowest dimension, we get

$$G_n(x) \simeq \left(\frac{1}{x^{2h_n}} + C^2_{\sigma_+\sigma_- O_m} \frac{1}{x^{2h_n - h_m}} \right) \tag{43}$$

Replacing this into expression 36, we get

$$J_n = \left(\frac{\beta}{\pi} \sinh \frac{\pi l}{\beta} \right)^{-4h_n} \left(1 + C^2_{\sigma_+\sigma_- O_m}(n) x^{h_m} \right) \tag{44}$$

Thus, the entanglement entropy is given by

$$\begin{aligned} S_{A(dyn)} &= S_A^{(n)}|_{n=1} = -\partial_n J_n|_{n=1} \\ &= \frac{c}{3} \log \left(\frac{\beta}{\pi} \sinh \frac{\pi l}{\beta} \right) f_1(x) + (\partial_n \log f_n(x))|_{n=1}[f_1(x)] \end{aligned} \tag{45}$$

where

$$f_n(x) = \left(1 + C^2_{\sigma_+\sigma_- O_m}(n) x^{h_m} \right) \tag{46}$$

Since we are interested in finding only the large time fall-off behaviour of $S_{A,dyn}(t)$, we can choose to ignore the 2nd term in the above expression because it would contribute as a higher order transient.
Therefore, in the large time limit (with $x \sim \exp(-\frac{4\pi}{\beta}(t - l/2))$),

$$\begin{aligned} S_{A(dyn)} &\simeq \left[\frac{c}{3} \log \left(\frac{\beta}{2\pi\epsilon} \exp \frac{\pi l}{\beta} \right) \right] \left(1 + C^2_{\sigma_+\sigma_- O_m}(n=1) x^{h_m} \right) \\ &\simeq \left(\frac{\pi c}{3} \frac{l}{\beta} + 4 S_{div} \right) \left(1 + C^2_{\sigma_+\sigma_- O_m}(1) \exp(-\frac{4\pi h_m}{\beta}(t - l/2)) \right) \end{aligned} \tag{47}$$

Here,

$$S_{div} = \frac{c}{12} \log \left(\frac{\beta}{2\pi\epsilon} \right) \tag{48}$$

Thus, the dynamical entanglement entropy settles down to the equilibrium entanglement entropy in the following fashion,

$$S_{A(dyn)} \simeq S_{A(eqm)} (1 + C^2_{\sigma_+\sigma_- O_m} \exp(-\frac{4\pi h_m}{\beta}(t - l/2)) \tag{49}$$

Here, we should also briefly mention the result at early times. The limits for calculating corrections in this case would be,

$$\frac{t}{\beta}, \frac{l}{2\beta} \gg \qquad \left(\frac{l}{2} - t \right) > \beta \tag{50}$$

In these limits, the cross ratio is $x \sim (1-\exp(-\frac{4\pi}{\beta}(t-l/2)))$ and the $x \sim 1$ channel dominates. The form of the $S_{A(dyn)}$ would, however, still remain the same with $(t-l/2)$ now replaced by $(l/2-t)$.
Comparing the result 49 to the one we got in (31), it is easy to see that a short interval expansion of the above result would give us a match, to leading order, in the asymptotic fall-off dynamics of the entanglement entropy. The important hint that this result provides is the possibility that the short-interval expansion may not be all that important in proving thermalization.

4 Effect of turning on a single chemical potential μ_n

For simplicity we will first turn on only a single μ_n for a specific n, and put all other chemical potentials to zero. We will define the following notations

$$
\begin{aligned}
&|\psi(t)\rangle = \exp[-\mu_n W_n/4]\ |\psi^{(0)}(t)\rangle, \qquad |\psi^{(0)}(t)\rangle = \exp[-\beta H/4 - iHt]|Bd\rangle,\\
&\rho_{dyn}(t) = \exp[-\mu_n W_n/4]\ \rho^{(0)}_{dyn}(t)\ \exp[-\mu_n W_n/4],\ \rho^{(0)}_{dyn}(t) = |\psi^{(0)}(t)\rangle\langle\psi^{(0)}(t)|\\
&\rho_{eqm}(\beta,\mu) = \exp[-\mu_n W_n]\ \rho_{eqm}(\beta,0)
\end{aligned}
\tag{51}
$$

Here $|\psi_0\rangle$, $\rho_{dyn}(t)$, $\rho_{eqm}(\beta,\mu)$ are as defined in (6),(3) and (2), and $|\psi^0(0)\rangle \equiv \exp[-\beta H/4]|Bd\rangle$.

In the presence of μ_n, $I(t)$ is again given by the equation (15), which, in turn, is given by (19), with the following changes in the one-point functions:

$$
\begin{aligned}
\langle\phi_k(w,\bar{w})\rangle_{str} &\to \langle\phi_k(w,\bar{w})\rangle^{\mu}_{str} \equiv \frac{\langle e^{-\frac{\mu_n}{4}W_n}\phi_{k1}(w,\bar{w})e^{-\frac{\mu_n}{4}W_n}\rangle_{str}}{\langle e^{-\frac{\mu_n}{2}W_n}\rangle_{str}}\\
\langle\phi_{k1}(w,\bar{w})\rangle_{cyl} &\to \langle\phi_{k1}(w,\bar{w})\rangle^{\mu}_{cyl} \equiv \frac{\langle e^{-\mu_n W_n}\phi_{k1}(w,\bar{w})\rangle_{cyl}}{\langle e^{-\mu_n W_n}\rangle_{cyl}}
\end{aligned}
\tag{52}
$$

To see the above, note that the effect of the μ-deformation (51), in terms of the discussions in (16)-(19), is to (a) insert $e^{-\mu_n W_n/4}$ before and the after the cut in the strip, and (b) insert $e^{-\mu_n W_n}$ before or after the cut in the cylinder. The denorminators in the above equation ensure the proper normalization $\langle 1\rangle^{\mu}_{s} = \langle 1\rangle^{\mu}_{c} = 1$ (these follow from the use of the normalized quantities $\hat{Z}$ in (19).

4.1 One-point function on the strip with μ-deformation

In the following we will compute $\langle\phi_k(w,\bar{w})\rangle^{\mu}_{s}$.

4.1.1 $O(\mu_n)$ calculation

We note that the charge W_n can be written as a contour integral over a conserved holomporhic current $\mathcal{W}(w_1)$ and its antiholomporphic counterpart $\bar{\mathcal{W}}(\bar{w}_1)$: [19]

$$
\begin{aligned}
W_n &= \frac{1}{2\pi}\int_\Gamma W_{\tau\tau\ldots\tau} d\sigma = \frac{1}{2\pi}\int_\Gamma \left(i^n dw_1\, \mathcal{W}_n(w_1) + (-i)^n d\bar{w}_1\, \bar{\mathcal{W}}_n(\bar{w}_1)\right) \\
&= \frac{1}{2\pi}\left(\frac{2\pi}{\beta}\right)^{n-1}\left[i^n \int_\Gamma dz_1 \left(z_1^{n-1}\mathcal{W}_n(z_1) + \sum_{m=1}^{\lfloor n/2\rfloor} a_{n,n-2m} z_1^{n-2m-1}\mathcal{W}_{n-2m}(z_1)\right) + (\text{antihol})\right]
\end{aligned} \tag{53}
$$

In the first line we consider Γ to be along $\tau =$ constant line, along which $dw_1 = d\bar{w}_1 = d\sigma$. In the second line we have transformed both the currents and the integration variable to z-coordinates using the conformal map (20); the lower order currents appear, with known numerical coefficients $a_{n,2-2m}$, in the conformal transformation of $\mathcal{W}_n$ currents which are not primary. [20]

Using this representation of the W_n-charge, we can easily show that

$$
\begin{aligned}
&\delta_\mu \langle \phi_k(w,\bar{w})\rangle_{str} = \langle \phi_k(w,\bar{w})\rangle^\mu_{str} - \langle \phi_k(w,\bar{w})\rangle_{str} \\
&= -\frac{\mu_n}{4}\frac{1}{2\pi}\int_{\Gamma+\Gamma'} dw_1\, i^n \langle \mathcal{W}_n(w_1)\phi_k(w,\bar{w})\rangle_{str,conn} + \text{antihol} \\
&= -A_k \frac{\mu_n}{8\pi}(2\pi/\beta)^{n-1} i^n \int_{\Gamma+\Gamma'} \left[dz_1 \left(z_1^{n-1}(zz')^{2h_k}\langle \mathcal{W}_n(z_1)\phi_k(z)\phi_k(z')\rangle_{conn} + \ldots\right) + \text{antihol}\right]
\end{aligned} \tag{54}
$$

The notation A_k is as in (98). The ellipses denote the lower order currents indicated in (53) — the calculation of these is a straightforward generalization of the steps below and will not be written explicitly. The connected three-point function $\langle \mathcal{W}_n \phi\phi\rangle$ is given by

$$
\langle \mathcal{W}_n(z_1)\phi_k(z)\phi_k(z')\rangle_{conn} = q_n (z_1 - z)^{-n}(z_1 - z')^{-n}(z - z')^n \langle \phi_k(z)\phi_k(z')\rangle_{conn} \tag{55}
$$

where q_n is the $\mathcal{W}_n$-charge of the field ϕ_k. We now note the result for following indefinite integral

$$
\begin{aligned}
I_n(z_1, z, z') &= \int dz_1 z_1^{n-1}(z_1 - z)^{-n}(z_1 - z')^{-n}(z - z')^n \\
&= R_n\!\left(\frac{z}{z'}\right)\ [\ln(z_1 - z) - \ln(z_1 - z')] + \ldots,\ R_n(x) = 1 + o(x),
\end{aligned} \tag{56}
$$

In the definite integral, all terms coming from the upper limit $z_1 = \infty$ all cancel out amongst themselves; the contribution from the lower limit $z_1 = 0$ gives

$$
-\ln(-z) + \ln(-z') = 4\pi t/\beta
$$

[19] In the following, we will denote the integration variables along contours by $w_1, w_2, ..$ in the w-coordinates, and $z_1, z_2, ..$ in the z-coordinates.

[20] Note, e.g. that $T(z_0)\mathcal{W}_4(z_1)$ OPE has a $T(z_1)$ term, besides the usual $\mathcal{W}_4(z_1)$ and $\partial\mathcal{W}_4(z_1)$ terms. This means that under a conformal map, e.g. (20), the $\mathcal{W}_4$ current mixes with T.

where we have used the vaues of (z, z') as in (21) and (22). The omitted terms in (56) are subleading at large t (see Appendix C). Multiplying the above by a factor of 2 (to take into account the two contours), combining with the antiholomorphic contribution, and taking into account the lower order currents denoted by ... in (54), we get

$$\delta_\mu \langle \phi_k(w, \bar{w}) \rangle_{str} = \langle \phi_k(w, \bar{w}) \rangle_{str} (1 - \tilde{\mu}_n (\mathsf{Q}_n 2\pi t/\beta + r_1) + O(\mu^2))$$
$$\mathsf{Q}_n = \frac{1}{(2\pi)^{n-2}} \left(i^n \mathsf{q}_n + (-i)^n \bar{\mathsf{q}}_n \right) \tag{57}$$

where we have used the shifted charges

$$\mathsf{q}_n = q_n + \sum_{m=1}^{\lfloor n/2-1 \rfloor} a_{n,n-2m} q_{n-2m} \tag{58}$$

and their anitholomorphic counterparts. In (57) r_1 is some time-independent term which is not important at large t, compared to the linear t piece.

4.1.2 Higher orders in μ_n

Before proceeding with the higher order calculation, let us redo the $O(\mu)$ calculation in a simpler fashion. The result (56) is actually more general:

$$J_n(z_1, z) \equiv \int dz_1 g(z_1)(z_1 - z)^{-n} = R_n(z) \ln(z_1 - z) + \text{meromorphic in } (z_1 - z) \text{ where}$$
$$R_n(z) = \text{Residue}_{z_1 = z}[g(z_1)(z_1 - z)^{-n}] = \frac{1}{2\pi i} \oint_z dz_1 \left[g(z_1)(z_1 - z)^{-n} \right] \tag{59}$$

The result can be obtained by an explicit calculation by the substitution $z_1 = z + y$, $g(z_1) = g(z) + y g'(z) + ... + y^{n-1}/(n-1)! g^{(n-1)}(z) + ...$ Alternatively it can be understood by deforming the contour of integration in J_n so that the upper limit z_1 is moved around z by an angle 2π, $y \to y e^{2\pi i}$, so that $R_n(z) \ln(y) \to R_n(z)(\ln(y) + 2\pi i)$; this identifies the coefficient of the log as the residue. In terms of the figure 2, the coefficient of $\log(z_1 - z)$ in the integral $\int_\Gamma$ (or $\int_{\Gamma'}$) is the same as $1/(2\pi i) \oint_{\Gamma - \Gamma'}$.

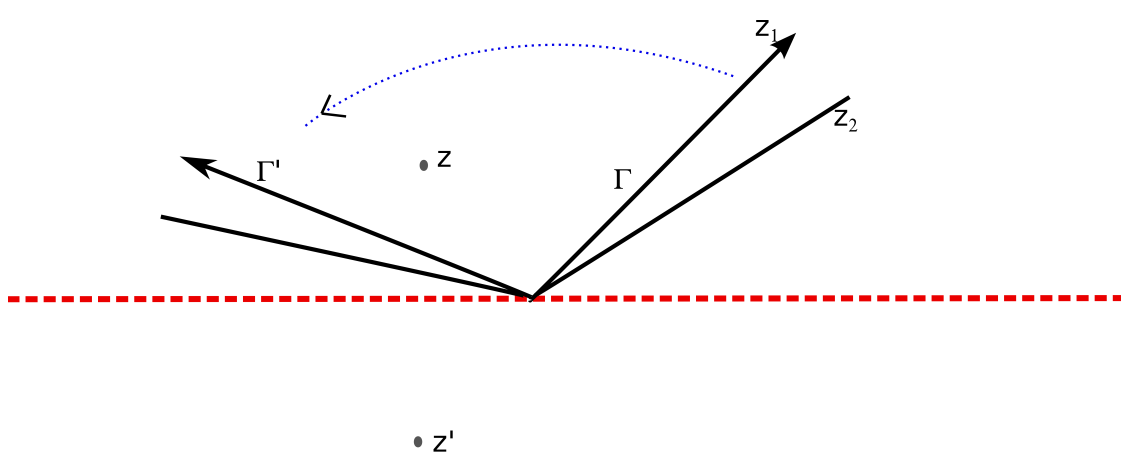

Figure 2: The contour deformation argument.

A similar argument also shows that the coefficient of $\log(z_1 - z')$ in the same integral is minus the above residue (as is seen explicitly in (56)); this follows because the contour $\oint_{\Gamma-\Gamma'}$ can be deformed to an *clockwise contour* around z'.

Interestingly, note that although we began with a calculation of the anticommutator $\{W_n, \phi\}$, the coefficient of the log term is given rather by the commutator $[W_n, \phi]$ (recall the relation between $\oint_{\Gamma-\Gamma'}$ and the commutator.

The deformation to $\oint_{\Gamma-\Gamma'}$ implies that we can shrink the closed contour to an arbitrary small size, hence the dominant contribution to the coefficient of $\log(z_1-z)$ in $\langle \mathcal{W}_n(z_1)\phi_k(z)\phi_k(z')\rangle_{conn}$ comes from the leading OPE:

$$\mathcal{W}_n(z_1)\phi_k(z) = \frac{q_n\phi_k(z)}{(z_1 - z)^n} \tag{60}$$

This, and the above remark about the coefficient of $\log(z_1 - z')$ (that it has an extra minus sign) gives us the Eq. (56) entirely.

The extension to $O(\mu_n^2)$ and higher orders is now easy. By similar arguments as above, we can show that

$$\begin{aligned}&\int_{\Gamma_1} dz_1 z_1^{n-1} \int_{\Gamma_2} dz_2 z_2^{n-1} \langle \mathcal{W}_n(z_1)\mathcal{W}_n(z_2)\phi_k(z)\phi_k(z)\rangle_{conn}/\langle\phi_k(z)\phi_k(z)\rangle_{conn}\\ &= (\log(z_1 - z) - \log(z_1 - z'))^2 q_n^2 + \text{linear log} + ...\end{aligned} \tag{61}$$

Essentially this is shown by deforming the contour Γ_1 and Γ_2 successively, and repeatedly using the leading OPE (60).

Thus, to $O(\mu^2)$, we have

$$\delta_\mu\langle\phi_k(w,\bar{w})\rangle_{str} = \langle\phi_k(w,\bar{w})\rangle_{str}(1 - \tilde{\mu}_n(\mathsf{Q}_n 2\pi t/\beta + r_1) + \frac{1}{2}\mu_n^2(\mathsf{Q}_n 2\pi t/\beta)^2(1 + O(1/t)) + O(\mu^3)) \tag{62}$$

where we have included the contribution of lower order currents which mix with $\mathcal{W}_n$ under conformal transformations (see (53)).

By extending to arbitrary orders, we can show that (see Section 5 for a general proof)

$$\langle\phi_k(w,\bar{w})\rangle^\mu_{str} = \langle\phi_k(w,\bar{w})\rangle_{str} \times \alpha(\mu)\exp[-\mu_n\mathsf{Q}_n 2\pi t/\beta + O(\mu^2))] \tag{63}$$

4.2 One-point function on the cylinder in the presence of μ_n

Expanding (52) for the cylinder case, we get

$$\langle e^{-\mu_n W_n}\phi_{k1}(w,\bar{w})\rangle_{cyl}/\langle e^{-\mu_n W_n}\rangle_{cyl} = \langle\phi_{k1}(w,\bar{w})\rangle - \mu(\langle W_n\phi_{k1}(w,\bar{w})\rangle - \langle W_n\rangle\langle\phi_{k1}(w,\bar{w})\rangle + ...$$

By arguments similar to the case of the one-point function on the strip, the $O(\mu_n)$ term gets related to the connected two point function $\langle\mathcal{W}_n(w_1)\phi_{k1}(w,\bar{w})\rangle$, For primary fields ϕ_k, or quasiprimaries which are not descendents of the identity operator. the one-point function vanishes because of translational symmetry. For the descendents and quasiprimary operators, the z,z' dependent parts vanish by the same argument. And the z, z' independent part, that arises from the conformal transformation(20), is basically the disconnected correlation function. So all the time independent terms cancelled, just as in the $\mu = 0$ case.

4.3 Proof of thermalization in the presence of μ_n

The statement of thermalization in the absence of any higher spin chemical potentials has been proven. We would like to show that thermalization happens even in the presence of any higher spin chemical potential. The equivalent mathematical statement would be the following

$$I^{\mu}(t \to \infty) \to 1 \tag{64}$$

The structure of $I^{\mu}(t)$ is,

$$I^{\mu}(t) = \frac{\hat{Z}^{\mu}_{sc}}{\sqrt{\hat{Z}^{\mu}_{ss}}\sqrt{\hat{Z}^{\mu}_{cc}}} \tag{65}$$

Here,

$$\begin{aligned}
\hat{Z}^{\mu}_{sc} &= \sum_{k_1,k_2} C_{k1,k2}\langle\phi_{k1}(w_1,\bar{w}_1)\rangle^{\mu}_{str}\langle\phi_{k2}(w_2,\bar{w}_2)\rangle^{\mu}_{cyl} \\
\hat{Z}^{\mu}_{ss} &= \sum_{k1,k2} C_{k1,k2}\langle\phi_{k1}(w_1,\bar{w}_1)\rangle^{\mu}_{str}\langle\phi_{k2}(w_2,\bar{w}_2)\rangle^{\mu}_{str} \\
\hat{Z}^{\mu}_{cc} &= \sum_{k1,k2} C_{k1,k2}\langle\phi_{k1}(w_1,\bar{w}_1)\rangle^{\mu}_{cyl}\langle\phi_{k2}(w_2,\bar{w}_2)\rangle^{\mu}_{cyl}
\end{aligned}$$

The way we proved thermalization in the absence of the higher spin chemical potentials was by showing that in the large time limit, only the purely holomorphic/anti-holomorphic operators in the numerator and the denominator survived. In such a case, $Z_{sc} = Z_{ss} = Z_{cc}$ and the thermalization function $I(t) \to 1$. In the presence of the higher spin chemical potentials, however, it is not obvious how $\langle\phi_k(w,\bar{w})\rangle^{\mu}_{str} = \langle\phi_k(w,\bar{w})\rangle^{\mu}_{cyl}$ in the large time limit. Also, there maybe linear t terms present even in the one pt. function of the purely holomorphic/anti-holomorphic operators.
For example, the following is the form of the one-pt. function on the strip in the presence of a μ_3 chemical potential:

$$\begin{aligned}
\langle\phi_k(w,\bar{w})\rangle^{\mu}_{str} &= \frac{\langle e^{-\mu_3 W^3/4}\phi_k(w,\bar{w})e^{-\mu_3 W^3/4}\rangle_{str}}{\langle e^{-\mu_3 W^3/2}\rangle_{str}} \\
&= \langle\phi_k(w,\bar{w})\rangle^{(0)}_{str} - \frac{\mu_3}{2}(\langle W^3\phi_k(w,\bar{w})\rangle - \langle W^3\rangle\langle\phi_k(w,\bar{w})\rangle + ...
\end{aligned}$$

where linear t terms may arise from the $\langle W\phi\rangle$ terms.
To understand how thermalization happens, we will spilt the functions in 66 into different parts according to the different classes of operators they receive contributions from.
Thus,

$$\begin{aligned}
\hat{Z}_{sc} &= 1 + \sum_a C_a\langle\phi_a\rangle^{\mu}_{str}\langle\phi_a\rangle^{\mu}_{cyl} + \sum_{a,b} C_{a;b}\langle\phi_a\rangle^{\mu}_{str}\langle\phi_b\rangle^{\mu}_{cyl} \\
\hat{Z}_{ss} &= 1 + \sum_a C_a\langle\phi_a\rangle^{\mu}_{str}\langle\phi_a\rangle^{\mu}_{str} + \sum_{a,b} C_{a;b}\langle\phi_a\rangle^{\mu}_{str}\langle\phi_b\rangle^{\mu}_{str} \\
\hat{Z}_{cc} &= 1 + \sum_a C_a\langle\phi_a\rangle^{\mu}_{cyl}\langle\phi_a\rangle^{\mu}_{cyl} + \sum_{a,b} C_{a;b}\langle\phi_a\rangle^{\mu}_{cyl}\langle\phi_b\rangle^{\mu}_{cyl}
\end{aligned} \tag{66}$$

Here, 'a' refers to purely holomorphic/anti-holomorphic operators; and (a,b) refer to operators which have both the holomorphic and the anti-holomorphic part.

$\underline{a = (h,0)}$ or $\underline{a = (0,\bar{h})}$:

For these kind of operators (both in and outside the conformal block of identity), the one pt. function on the cylinder and the strip (with $\mu = 0$) are the same,i.e.,

$$\langle \phi_h(w) \rangle_{str} = \langle \phi_h(w) \rangle_{cyl} \tag{67}$$

The $\langle W_0^{(3)} \phi_h(w) \rangle_{str}^{\mu}$ term comes with a coefficient of $(\mu/2)$ as against the analogous term on the cylinder which comes with a coefficient of μ. However, on the strip, the anti-holomorphic current term $\langle \bar{\mathcal{W}}(\bar{w}_1)\phi_h(w) \rangle_{str}$ is also non-zero and is in fact equal to its holomorphic counterpart. This supplies the factor of two which compensates for the half in the coefficient. The same can be shown happening at higher orders in μ as well. Thus,

$$\langle \phi_h(w) \rangle_{str}^{\mu} = \langle \phi_h(w) \rangle_{cyl}^{\mu} \tag{68}$$

to all orders in μ. Now, calculating the second term with a single insertion of $\mathcal{W}^3$ current,

$$\langle W_{(0)}^3 \phi_h(w) \rangle_{cyl} = z^h \int_{\Gamma_1} dz_1 \langle \mathcal{W}^3(z_1) \phi_h(z) \rangle_{cyl} z_1^3 \tag{69}$$

However, one must note that,

$$\langle \phi_h(w) \rangle_{cyl}^{\mu} \propto \langle \phi_h(w) \rangle_{cyl}^{(0)} \tag{70}$$

For primary operators, $\langle \phi_h(w) \rangle_{cyl}^{(0)} = 0$. Hence, $\langle \phi_h(w) \rangle_{cyl}^{\mu} = 0$.
For quasi-primary operators, $\langle \phi_h(w) \rangle_{cyl}^{(0)} = const$. Hence, $\langle \phi_h(w) \rangle_{cyl}^{\mu} = const$ as well.

$\underline{a \equiv (h,h)}$ and $\underline{b \equiv (h,h)}$

For primary operators of this kind, the one point function (with $\mu = 0$) on the strip has an exponential time-dependence and hence, dies off in the large time limit. However, quasi-primary operators of this kind have a time independent part that survives. This constant value is, however, equal to its time independent counterpart on the cylinder. Thus,

$$\langle \phi_{(h,h)}(w,\bar{w}) \rangle_{str} = \langle \phi_{(h,h)}(w,\bar{w}) \rangle_{cyl} \tag{71}$$

in the limit $t \to \infty$.
Let us consider the second term in (66) with the insertion of a purely holomorphic (anti-holomorphic) current $\mathcal{W}^3(w)$ $(\bar{\mathcal{W}}^3(\bar{w}))$. Such a one point function of a primary operator has a non-trivial time dependence when considered on the strip. However, in the large time limit, this goes to zero. On the cylinder, such a one point function is already zero. For a quasi-primary operator also the same story applies. They are zero on the cylinder and non-zero on the plane at finite time. However in the large time limit, they go to zero as well.

At order μ_3^2, there are 2 insertions of the $\mathcal{W}^3$ current. Each such insertion provides us with a factor of $4t^2$ on the strip as compared to $2t^2$ on the cylinder.

The net effect of considering these kind of operators on the functions (66) is

$$\hat{Z}_{sc} = 1 + C_{h,0;h,0} \tag{72}$$

5 One point function with an arbitrary number of chemical potentials

We are interested in calculating the higher order terms in the thermalization function. To quadratic order, this would require us to calculate the one point function in the presence of two $\mathcal{W}$-currents. We define the integrand as

$$G_{mn}(z_1, z_2, z, z') = z_1^{m-1} z_2^{n-1} \frac{\langle W_m(z_1) W_n(z_2) \phi(z) \phi(z') \rangle_{\mathbb{C}}}{\langle \phi(z) \phi(z') \rangle_{\mathbb{C}}} \tag{73}$$

In order to evaluate the four point function above, we need to use the following $\langle W\phi \rangle$ OPE,

$$W_m(z_1)\phi(z) = q_m \frac{\phi(z)}{(z_1 - z)^m} + \alpha_m \frac{\phi_{m,1}(z)}{(z_1 - z)^{m-1}} + \dots \tag{74}$$

where $\alpha_{m,1}$ is an operator of conformal dimension $(h+1, 0)$.
Using the above OPE, we calculate the four point function in (73) as

$$\begin{aligned}
\langle W_m(z_1) W_n(z_2) \phi(z) \phi(z') \rangle_{\mathbb{C}} &= \Bigg[\frac{q_m}{(z_1 - z)^m} \langle W_n(z_2) \phi(z) \phi(z') \rangle_{\mathbb{C}} \\
&\quad + \frac{\alpha_m}{(z_1 - z)^{m-1}} \langle W_n(z_2) \phi_{m,1}(z) \phi(z') \rangle_{\mathbb{C}} + \dots + (z \leftrightarrow z') \Bigg] \\
&= \Bigg[\frac{q_m}{(z_1 - z)^m} \Bigg[\frac{q_n}{(z_2 - z)^n} \langle \phi(z) \phi(z') \rangle_{\mathbb{C}} + \frac{\alpha_n}{(z_2 - z)^{n-1}} \langle \phi_{n,1}(z) \phi(z') \rangle_{\mathbb{C}} \\
&\quad + \dots + (z \leftrightarrow z') \Bigg] \\
&\quad + \frac{\alpha_m}{(z_1 - z)^{m-1}} \Bigg[\frac{q_{n;m,1}}{(z_2 - z)^n} \langle \phi_{m,1}(z) \phi(z') \rangle_{\mathbb{C}} \\
&\quad + \frac{\alpha_{n;m,1}}{(z_2 - z)^{n-1}} \langle \phi_{m,1;n,1}(z) \phi(z') \rangle_{\mathbb{C}} + (z \leftrightarrow z')] \Bigg] + \dots + (z \leftrightarrow z') \Bigg]
\end{aligned}$$

The integrand 73 is then

$$\begin{aligned}
G_{mn}(z_1, z_2, z, z') &= q_m q_n \Bigg[\frac{z_1^{m-1} z_2^{n-1}}{(z_1 - z)^m (z_2 - z)^n} (1 + O(z_1 - z) + O(z_2 - z)) \\
&\quad + \frac{z_1^{m-1} z_2^{n-1}}{(z_1 - z)^m (z_2 - z')^n} (1 + O(z_1 - z) + O(z_2 - z')) + (z \leftrightarrow z') \Bigg]
\end{aligned}$$

Performing an indefinite integral over the open contours Γ_1 and Γ_2, we get

$$\begin{aligned}
\int_{\Gamma_1} dz_1 \int_{\Gamma_2} dz_2 G_{mn}(z_1, z_2; z, z') &\simeq (q_m \log(z_1 - z) q_n \log(z_2 - z) + q_m \log(z_1 - z) q_n \log(z_2 - z') \\
&\quad + (z \leftrightarrow z')) + \text{ linear log terms}
\end{aligned}$$

The lower limit of the integral gives

$$\int_{\Gamma_1} dz_1 \int_{\Gamma_2} dz_2 G_{mn}(z_1, z_2; z, z') \simeq q_m q_n (\log(-z) - \log(-z'))^2 \tag{75}$$

This answer gives us the $\mu^2 t^2$ correction term to the thermalization function with appropriately squared coefficients.
In order to calculate the higher order correction terms, we shall have to calculate the one point function in the presence of an arbitrary number of higher spin charges. Thus,

$$\begin{aligned}\langle\phi(w,\bar{w})\rangle^{\mu}_{strip} &= \mathcal{N}_{\mu}\langle \exp(-\frac{\mu_m}{4}W_m \phi(w,\bar{w}) \exp(-\frac{\mu_n}{4}W_n))\rangle_{strip} \\ &= \mathcal{N}_{\mu}\langle\sum_{r=1}^{\infty}\frac{(-1/4)^r}{r!}\mu_{m_1}W_{m_1}\ldots\mu_{m_r}W_{m_r}\phi(w,\bar{w})\sum_{s=1}^{\infty}\frac{(-1/4)^s}{s!}\mu_{n_1}W_{n_1}\ldots\mu_{n_s}W_{n_s}\rangle_{strip}\end{aligned} \tag{76}$$

where $\mathcal{N}_{\mu}^{-1} = \langle \exp(-\frac{\mu_n}{4}W_n)\phi(w,\bar{w})\exp(-\frac{\mu_n}{4}W_n)\rangle_{stip}$.
The (r,s)-th term in the above expression is given by

$$\begin{aligned}&-\frac{(1/4)^{(r+s)}}{r!s!}\prod_{i=1}^{r}\mu_i\prod_{j=1}^{s}\mu_j\langle W_{m_1}\ldots W_{m_r}\phi(w,\bar{w})W_{n_1}\ldots W_{n_s}\rangle_{strip} \\ &= -\frac{(1/4)^{(r+s)}}{r!s!}\prod_{i=1}^{r}\mu_i\prod_{j=1}^{s}\mu_j \\ &2^{r+s}\langle\int_{\Gamma_{m_1}}\mathcal{W}_{m_1}(w_{m_1})\ldots\int_{\Gamma_{m_r}}\mathcal{W}_{m_r}(w_{m_r})\int_{\Gamma_{n_1}}\mathcal{W}_{n_1}(w_{n_1})\ldots\int_{\Gamma_{n_s}}\mathcal{W}_{n_s}(w_{n_s})\phi(w,\bar{w})\rangle_{strip}\end{aligned} \tag{77}$$

Just as in (73), we can define the above as an integrand

$$G_{m_1\ldots m_r n_1\ldots n_s} = \prod_{i=1}^{r} z_{m_i}^{r-1}\prod_{j=1}^{s} z_{m_j}^{s-1}\frac{\langle\mathcal{W}_{m_1}(z_{m_1})\ldots\mathcal{W}_{m_r}(z_{m_r})\mathcal{W}_{n_1}(z_{n_1})\ldots\mathcal{W}_{n_s}(z_{n_s})\phi(z)\phi(z')\rangle_{\mathbb{C}}}{\langle\phi(z)\phi(z')\rangle_{\mathbb{C}}} \tag{78}$$

To get the correction due to an arbitrary number of chemical potentials, we shall have to use the $(W\phi)$ OPE again and chose the integration contours as previously. We would then have to integrate the resulting expression over (r+s) contours. The result (upto leading order) in the lower limit is

$$\int_{\Gamma_{m_1}} dz_{m_1}\ldots\int_{\Gamma_{m_r}} dz_{m_r}\int_{\Gamma_{n_1}} dz_{n_1}\int_{\Gamma_{n_s}} dz_{n_s} G_{m_1\ldots m_r n_1\ldots n_s} \simeq \frac{(\vec{\mu}.\vec{q})^{(r+s)}}{r!s!}(\log(-z')-\log(-z))^{(r+s)} \tag{79}$$

Then, summing over the dummy indices, taking into account all coefficients, we get

$$\begin{aligned}\langle\phi(w,\bar{w})\rangle^{\mu}_{strip} &\simeq \sum_{r,s}\frac{1}{r!s!}\Big(\frac{2\pi}{\beta}\Big)^{mr+ns-r-s}\Big[-\frac{1}{2}\vec{\mu}.\vec{q}(\frac{4\pi t}{\beta})\Big]^{r+s} \\ &\simeq \sum_{r,s}\frac{1}{r!s!}\Big(-(\vec{\hat{\mu}}.\vec{q})\hat{t}\Big)^{r+s} \\ &\simeq \exp[-2(\vec{\hat{\mu}}.\vec{q})\hat{t}]\end{aligned} \tag{80}$$

Thus, the one-point function on the strip in the presence of arbitrary chemical potentials is

$$\langle \phi(w, \bar{w}) \rangle^{\mu}_{strip} = \exp(-2(\vec{\hat{\mu}}.\vec{q})\hat{t})(1 + O(\mu\hat{t}) + O(\mu^2\hat{t}^2) + \dots) \tag{81}$$

6 Decay of perturbations of a thermal state

In this section, we will find that the long time behaviour of an operator $\phi_k(0,t)$ in the quenched state (see (10)) is the same as that of its two-point function (82) in the thermal state (2) (with chemical potentials), which measures the thermal decay of a perturbation. Throughout this section, we will assume that the conformal dimensions of ϕ_k satisfy $h_k = \bar{h}_k$.

We define the thermal two-point function as [21]

$$G_+(t; \beta, \mu) \equiv \frac{1}{Z} \mathrm{Tr}(\phi_k(0,t)\phi_k(0,0) e^{-\beta H - \sum_n \mu_n W_n}) \tag{82}$$

By the techniques developed in the earlier sections, a computation of this quantity amounts to calculating the following correlator on the plane

$$\langle \phi_k(z, \bar{z}) \phi_k(y, \bar{y}) e^{-\sum_n \mu_n W_n} \rangle, \quad z = ie^{-2\pi t/\beta}, \bar{z} = -ie^{2\pi t/\beta}, y = i, \bar{y} = -i \tag{83}$$

where the μ_n-deformations are understood as an infinite series of contours as in the previous section.

For $\mu = 0$, the above two-point function is

$$\langle \phi_k(z, \bar{z}) \phi_k(y, \bar{y}) \rangle = \left[(ie^{-2\pi t/\beta} - i)(-ie^{2\pi t/\beta} + i) \right]^{-2h_k} \xrightarrow{t \to \infty} (-1)^{-2h_k} e^{-4\pi t h_k/\beta} \tag{84}$$

which clearly matches the long time behaviour of the one-point function (97) in the quenched state for $\mu = 0$.

The effect of turning on the chemical potentials can be dealt with as in the previous sections. At $O(\mu_n)$, we will have, as before, a holomorphic contribution and an antiholomorphic contribution. The former is proportional to

$$\langle \phi_k(\bar{z}) \phi_k(\bar{y}) \rangle \times \int_{\Gamma} dz_1 z_1^2 \langle \mathcal{W}_n(z_1) \phi_k(z) \phi_k(y) \rangle \tag{85}$$

As we see, the structure of the integral is the same as in the previous section. As before, logarithmic terms appear in the above integrals which give the leading, linear, t-dependence. Similar remarks also apply to the antiholomorphic contour. Since the calculations are very similar to those in the previous two sections, we do not provide all details. By resumming the series over the infinite number of contours, we find in a straightforward fashion that

$$G_+(t; \beta, \mu) \xrightarrow{t \to \infty} G_+(t; \beta, 0) b(\mu) e^{-\gamma t} \tag{86}$$

where $b(\mu)$ is time-independent, and is of the form $b(\mu) = 1 + O(\mu)$. This proves the statement made in the beginning of this section.

[21] We use the same notations as in [17].

7 Higher spin black holes

- It has been shown by [12] that the equilibrium ensemble (2), in the context of W_n identified with W_∞ charges, correspond to a higher spin bulk dual which is a black hole.

- The result (11), or alternatively (12) implies, therefore, that after the thermalization time, bulk scalars in a local region effectively start perceiving a higher spin black hole geometry.

- Small enough geodesics start seeing black hole geometry [3].

- The result (7) implies

$$\begin{aligned}\hat{\rho}_{dyn,A}(t) &= \left[1 - \frac{1}{2}\frac{C_{k,k}}{C_{0,0}}\left(\langle\phi_k(u,\bar{u})\rangle_{str}\right)^2\right]\hat{\rho}_{eqm,A} + \hat{Q}f(t) \\ f(t) &\propto \langle\phi_k(u,\bar{u})\rangle_{str}\end{aligned} \tag{87}$$

  This means that the dynamical reduced density matrix differs from the equilibrium reduced density matrix by an amount proportional to the expectation value of the field ϕ_k. This has the bulk interpretation that so far as observables in a small local region are concerned, they perceive the geometry as that of a black hole perturbed by a normalizable mode of the bulk scalar dual to ϕ_k. The decay of ϕ_k therefore corresponds to QNM of the scalar field.

- QNM frequencies: Narain et al indeed show that

$$\omega = -i(1 + \lambda + \text{constant } \mu_3 q_3) \tag{88}$$

  We can show that $1 + \lambda = 2h$, which agrees with our result (and with Cardy's result). We will leave for a forthcoming publication a full quantitative comparison between CFT exponents we found and the exact expression (88). The comparison involves matching the normalization of the charge q_3 between Narain et al and our calculation.

8 Discussion

Mention other global quenches. Our equilibration with non-uniform quench (caused by conformal deformations of (6) [14].

W4 and GGE.
(i) QNM
(ii) Caldeira-Leggett
(iii) Dynamical EE [8]

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A The coefficients C_{k_1,k_2}

As explained in [16] (see also Section 2.1), the coefficients C_{k_1,k_2} are determined by the equation

$$C_{k_1,k_2} = \frac{Z_2}{Z_1^2}(n_{k_1}n_{k_2})^{-\frac{1}{2}} \lim_{z_1\to\infty_1, z_2\to\infty_2} (z_1z_2)^{2(h_{k_1}+h_{k_2})}(\bar{z}_1\bar{z}_2)^{2(\bar{h}_{k_1}+\bar{h}_{k_2})}\langle\phi_{k_1}(z_1,\bar{z}_1)\phi_{k_2}(z_2,\bar{z}_2)\rangle_{\mathbb{C}_2} \quad (89)$$

where $\mathbb{C}_2$ reprsents two infinite planes glued along a cut A, Z_2 is the functional integral such a glued geometry and Z_1 is the functional integral over a single plane. This equation can be easily proved by inserting quasiprimary a operator at infinity in each plane in an equation like (16) or (17). The two point function in the glued geometry is to be determined by using the uniformizing map:

$$y = \sqrt{(z+l/2)/(z-l/2)} \quad (90)$$

The normalization constants n_k are determined by the following orthogonality condition of the quasiprimary operators

$$\langle\phi_{k_1}(z_1,\bar{z}_1)\phi_{k_2}(z_2,\bar{z}_2)\rangle_{\mathbb{C}} = \frac{n_{k_1}\delta_{k_1,k_2}}{z_{12}^{h_{k_1}+h_{k_2}}\bar{z}_{12}^{\bar{h}_{k_1}+\bar{h}_{k_2}}} \quad (91)$$

where n_{k_1} is a normalization constant. Note that $C_{k_1,k_2} = C_{k_2,k_1}$. Below we will use the notation

$$\hat{C}_{k_1,k_2} = C_{k_1,k_2}/C_{0,0} \quad (92)$$

<u>Case $(k_1,k_2) = (0,0)$</u>: We will denote the identity operator as $\phi_0 = 1$. It is obvious that

$$C_{0,0} = Z_2/Z_1^2 \quad (93)$$

<u>Case $(k_1,k_2) = (k,0)$</u>: The only case where $C_{k,0} \neq 0$ is when $\phi_k(z,\bar{z})$ is a descendent of the identity operator, e.g. $T(z)$, $\bar{T}(\bar{z})$, $:T(z)\bar{T}(\bar{z}):$, $\Lambda(z)$, $\Lambda(\bar{z})$ etc. [22] E.g.

$$\hat{C}_{T,0} = C_{T,0}/C_{0,0} = \hat{C}_{\bar{T},0} = \frac{l^2}{16}; \hat{C}_{T\bar{T},0} = \frac{l^4}{256}; ... \quad (94)$$

All other $C_{k,0}$ vanish as they are proportional to a one-point function of a primary operator on the Riemann surface (and hence to that on the complex plane).
<u>Case $(k_1,k_2) =$ (primary, primary)</u>: In case ϕ_{k_1}, ϕ_{k_2} are primary operators, (89) gives

$$\hat{C}_{k_1,k_2} = \frac{1}{n_{k_1}}\delta_{k_1,k_2}\left(\frac{le^{i\pi/2}}{4}\right)^{2(h_{k_1}+\bar{h}_{k_1})} \quad (95)$$

<u>Case $(k_1,k_2) =$ (descendent, descendent)</u>: In case ϕ_{k_1} is of the form $L_{-n_1}L_{-n_2}...\bar{L}_{-m_1}\bar{L}_{-m_2}...\phi_{l_1}$ and ϕ_{k_2} is of the form $L_{-r_1}L_{-r_2}...\bar{L}_{-s_1}\bar{L}_{-s_2}...\phi_{l_2}$, we can show that

$$\hat{C}_{k_1,k_2} = \delta_{l_1,l_2}\ \delta_{\sum n,\sum r}\ \delta_{\sum m,\sum s}\ A(n_1,n_2,...,m_1,m_2,...;r_1,r_2,...,s_1,s_2,...)\ l^{2(h_{k_1}+\bar{h}_{k_1})},$$
$$h_{k_1} = h_{l_1} + \sum n, \quad h_{k_2} = h_{l_2} + \sum m \quad (96)$$

where $A(...)$ is a numerical coefficient.

[22] Here $\Lambda(z) = \ :TT:(z) - \frac{3}{10}\partial_z^2 T$ is the level 4 quasiprimary descendent of the identity.

B One-point function with $\mu = 0$

We will first discuss the case with zero chemical potential. We will compute the $\langle \phi_k(w, \bar{w}) \rangle_{cyl}$ or $\langle \phi_k(w, \bar{w}) \rangle_{str}$ by transforming to the complex plane by using the map (20).
Case k = primary:
1. *cylinder:* In this case $\langle \phi_k(w, \bar{w}) \rangle_{cyl} \propto \langle \phi_k(z, \bar{z}) \rangle_{\mathbb{C}} = 0$, unless $k = 0$, in which case $\langle \phi_0(w, \bar{w}) \rangle_{cyl} \equiv \langle \mathbf{1} \rangle_{cyl} = 1$
2. *strip:* The one-point function on the strip, however, is non-trivial (for $h_k = \bar{h}_k$):

$$\langle \phi_k(w, \bar{w}) \rangle_{str} = \left(\frac{\partial z}{\partial w} \right)^{h_k} \left(\frac{\partial \bar{z}}{\partial \bar{w}} \right)^{\bar{h}_k} \langle \phi_k(z, \bar{z}) \rangle_{UHP} = A_k \left[e^{i\pi/2} (\beta/2\pi)(e^{2\pi t/\beta} + e^{-2\pi t/\beta}) \right]^{-2h_k} \tag{97}$$

where we have used the following result for the one-point function on the UHP:

$$\langle \phi_k(z, \bar{z}) \rangle_{UHP} = A_k (z - \bar{z})^{-2h_k}, \quad h_k = \bar{h}_k \tag{98}$$

which follows by using the method of images. A_k is a normalization constant, *defined by* (98) which can be determined by standard CFT methods [18].
Case k = descendent of identity: In this case, $\phi_k(w, \bar{w})$ is of the form $T, \bar{T}$, or $:T\bar{T}:$ or some descendents thereof. Under a conformal transformation (20), these operators pick up a c-number term in addition to a term proportional to the corresponding operator on the plane/UHP. We will give some examples to illustrate the calculation
1. *cylinder:* In this case

$$\begin{aligned} \langle T(w) \rangle_{cyl} &= \langle \left(-\frac{c\pi^2}{6\beta^2} - \frac{4\pi^2}{\beta^2} z^2 T(z) \right) \rangle_{UHP} = -\frac{c\pi^2}{6\beta^2} \\ \langle :T\bar{T}:(w, \bar{w}) \rangle_{cyl} &= \langle \left([-\frac{c\pi^2}{6\beta^2} - \frac{4\pi^2}{\beta^2} z^2 T(z)][-\frac{c\pi^2}{6\beta^2} - \frac{4\pi^2}{\beta^2} \bar{z}^2 \bar{T}(\bar{z})] \right) \rangle_{UHP} = (\frac{c\pi^2}{6\beta^2})^2 \end{aligned} \tag{99}$$

2. *strip:* In this case

$$\begin{aligned} \langle T(w) \rangle_{str} &= \langle \left(-\frac{c\pi^2}{6\beta^2} - \frac{4\pi^2}{\beta^2} z^2 T(z) \right) \rangle_{UHP} = -\frac{c\pi^2}{6\beta^2} = \langle T(w) \rangle_{cyl} \\ \langle :T\bar{T}:(w, \bar{w}) \rangle_{str} &= \langle \left([-\frac{c\pi^2}{6\beta^2} - \frac{4\pi^2}{\beta^2} z^2 T(z)][-\frac{c\pi^2}{6\beta^2} - \frac{4\pi^2}{\beta^2} \bar{z}^2 \bar{T}(\bar{z})] \right) \rangle_{UHP} \\ &= (\frac{c\pi^2}{6\beta^2})^2 + A_{T\bar{T}} (z - \bar{z})^{-4} \end{aligned} \tag{100}$$

where $A_{T\bar{T}}$ is a normalization constant as in (98).
Case k = descendent of other primaries: In this case,
1. *cylinder:* The one-point function vanishes as in the case of primaries.
2. *strip:* The one-point function can be related to one-point function of primaries which is dealt with above.

C One-point function with $\mu_3 \neq 0$

In this section, we shall explicitly evaluate the integral $I_n(z,z')$ in (56) for n=3. We shall put an upper cut-off A on the integral to show that the terms, other than the odd function $R_3(z/z')$, don't give any polynomial time t dependence.
n=3:

$$\begin{aligned} I_3(z,z') &= \int dz_1\, z_1^2 \frac{q_3(z-z')^3}{(z_1-z)^3(z_1-z')^3} \\ &= q_3\Big[\frac{(z^2+4zz'+z'^2)}{(z-z')^2}(\log(z_1-z)-\log(z_1-z')) - \frac{z^2}{2(z_1-z)^2} + \frac{z'^2}{2(z_1-z')^2} \\ &\quad + \frac{z'(2z+z')}{(z-z')(z_1-z')} + \frac{z(2z'+z)}{(z-z')(z_1-z)}\Big]_{z_1=0}^{z_1=A} \end{aligned}$$

The $z_1 = A$ limit gives us,

$$\frac{1}{2(z-z')^2}\Bigg(\frac{(z-z')}{(A-z)^2(A-z')^2}(-6(zz')^2(z+z') + (2A^3 - 3A^2(z+z'))(z^2+4zz'+(z')^2)$$
$$+2A(zz'(5z^2+8zz'+5(z')^2))) + 2(z^2+4zz'+(z')^2)(\log[A-z]-\log[A-z'])\Bigg)$$

For finite t values, putting $A \to \infty$ ensures that the above expression is 0.
However, the lower limit with $z_1 = 0$ gives,

$$-3\frac{(z+z')}{(z-z')} + \frac{(z^2+4zz'+(z')^2)}{(z-z')^2}(\log(-z)-\log(-z')) \tag{101}$$

The factor that comes with $(\log(z_1-z)-\log(z_1-z'))$ is

$$R_3(z/z') = \frac{q_3(z^2+4zz'+z'^2)}{(z-z')^2}$$

After analytically continuing z and z' according to 21 and 22 respectively and substituting into the terms in 101, we find the first term to be

$$-3\frac{(z+z')}{(z-z')} = 3\tanh\left(\frac{2\pi t}{\beta}\right) \tag{102}$$

and the second term to be

$$R_3\left(\frac{z}{z'}\right) = \tanh^2\left(\frac{2\pi t}{\beta}\right) \tag{103}$$

In the limit of $t \to \infty$, $\tanh\left(\frac{2\pi t}{\beta}\right) \to 1$. Thus, the first term goes to 3 and the coefficient of the second term $R_3(z/z') \to 1$. Therefore,

$$I_3(z,z') = \int dz_1\, z_1^2 \frac{q_3(z-z')^3}{(z_1-z)^3(z_1-z')^3} \xrightarrow{t\to\infty} q_3\left(\frac{4\pi t}{\beta} - 3\right) \tag{104}$$

The important point to be made here is that the coefficient of the product of the 'n' log terms $R_n(z/z')$ (in the presence of a W_n current) always goes to 1 in the $t \to \infty$ limit.

D $\delta\beta$ and $\delta\beta^2$ calculations

We calculate the one-point function on a strip of width $(\beta+\delta\beta)/4$ with $\delta\beta$ small as compared to β. The fractional change in the one point function in powers of $\delta\beta$ is

$$\begin{aligned}\frac{\delta\langle\phi_k(w,\bar{w})\rangle_{str}}{\langle\phi_k(w,\bar{w})\rangle_{str}} &= -\frac{2h}{\beta}\Big(1-\frac{2\pi t}{\beta}\tanh\frac{2\pi t}{\beta}\Big)\delta\beta+\frac{h(1+2h)}{\beta^2}\delta\beta^2-\Big(\frac{4\pi h(1+2h)}{\beta^3}\tanh\frac{2\pi t}{\beta}\Big)\delta\beta^2 t \\ &\quad +\Big(-\frac{4\pi^2 h}{\beta^4}+\frac{4\pi^2 h(1+2h)}{\beta^4}\tanh^2\frac{2\pi t}{\beta}\Big)\delta\beta^2 t^2+O(\delta\beta^3 t^3) \qquad (105)\end{aligned}$$

We do an analogous calculation in each order of $\delta\beta$ by considering insertions of $e^{-\delta\beta H/4}$ operators at the two boundaries of the strip of width $\beta/4$.

$$\begin{aligned}&\frac{\langle B|e^{-(\beta/4+\delta\beta/4-\tau)H}\phi_k(w,\bar{w})e^{-(\beta/4+\delta\beta/4+\tau)H}|B\rangle}{\langle B|e^{-(\beta+\delta\beta)H/2}|B\rangle}=\frac{\langle\Psi_0(\beta)|e^{(-\delta\beta/4+\tau)H}\phi_k(w,\bar{w})e^{(-\delta\beta/4-\tau)H}|\Psi_0(\beta)\rangle}{\langle\Psi_0(\beta)|e^{(-\delta\beta/2)H}|\Psi_0(\beta)\rangle} \\ &=\langle\phi_k(w,\bar{w})\rangle_{str(\beta)}+\frac{1}{4}[\langle\{H,\phi_k(w,\bar{w})\}\rangle_{str(\beta)}-2\langle H\rangle_{str(\beta)}\langle\phi_k(w,\bar{w})\rangle_{str(\beta)}]\delta\beta \\ &+\frac{1}{32}\Big[\langle\{H^2,\phi_k(w,\bar{w})\}\rangle+\langle 2H\phi_k(w,\bar{w})H\rangle-4\langle H\rangle\langle\{H,\phi_k(w,\bar{w})\}\rangle-4\langle H^2\rangle\langle\phi_k(w,\bar{w})\rangle\Big]\delta\beta^2+\mathcal{O}(\delta\beta^3)\end{aligned}$$

We are only interested in computing the coefficients of the $\delta\beta^2 t^2$ term, after analytically continuing to real time $\tau=it$, in the above expression.

D.1 $\delta\beta$

The coefficient of the $\delta\beta$ term is

$$\frac{1}{4}[\langle H,\phi_k(w,\bar{w})\rangle_{str,\beta}-2\langle H\rangle_{str,\beta}\langle\phi_k(w,\bar{w})\rangle_{str,\beta}] \qquad (106)$$

This is just the connected correlator that can be denoted as

$$\frac{1}{4}[\langle H,\phi_k(w,\bar{w})\rangle_{str,\beta}\rangle_{str,\beta}]_{(c)} \qquad (107)$$

In the integral form the above expression is,

$$\begin{aligned}\langle H\phi_k(w,\bar{w})+\phi_k(w,\bar{w})H\rangle_{str,\beta} &= -\frac{1}{2\pi}\int_{\Gamma+\Gamma'}dw_1\langle T_{ww}(w_1)\phi_k(w,\bar{w})\rangle_{str,\beta} \\ &\quad +T_{ww}(w_1)\longleftrightarrow\bar{T}_{\bar{w}\bar{w}}(\bar{w}) \qquad (108)\end{aligned}$$

Considering $\phi_k(w,\bar{w})$ to be a conformal primary, the transformation (20) gives

$$\langle T_{ww}(w_1)\phi_k(w,\bar{w})\rangle_{str,\beta}=\Big(\frac{\partial z}{\partial w}\Big)^h\Big(\frac{\partial\bar{z}}{\partial\bar{w}}\Big)^{\bar{h}}\Big[\Big(\frac{\partial z_1}{\partial w_1}\Big)^2\langle T_{zz}(z_1)\phi_k(z,\bar{z})\rangle_{\rm UHP}\Big]$$

It is easy to show that all the other terms are equal to the above term and hence contribute an overall factor of 4. Using the Ward identity for the stress tensor,

$$\begin{aligned}\langle T_{zz}(z_1)\phi_k(z,\bar{z})\rangle_{UHP} &= \langle T_{zz}(z_1)\Phi_k(z)\Phi_k(z^*)\rangle_{\mathbb{C}} \\ &= \Big(\frac{h}{(z_1-z)^2}+\frac{\partial_z}{(z_1-z)}+\frac{h}{(z_1-z^*)^2}+\frac{\partial_{z^*}}{(z_1-z^*)}\Big)\langle\Phi_k(z)\Phi_k(z^*)\rangle_{\mathbb{C}}\end{aligned}$$

Simplifying the above, we get

$$\frac{\langle T_{zz}(z_1)\phi_k(z,\bar{z})\rangle_{UHP}}{\langle \phi_k(z,\bar{z})\rangle_{UHP}} = \frac{h(z-z^*)^2}{(z_1-z)^2(z_1-z^*)^2} \tag{109}$$

Now, evaluating the integral,

$$\int_{\Gamma+\Gamma'} dz_1 \frac{z_1\langle T_{zz}(z_1)\phi_k(z,\bar{z})\rangle_{UHP}}{\langle \phi_k(z,\bar{z})\rangle_{UHP}} = \frac{h}{(\bar{z}-z)}(\log(z_1-z)-\log(z_1-\bar{z})) + h\left(\frac{z}{z-z_1}+\frac{\bar{z}}{(\bar{z}-z_1)}\right) \tag{110}$$

Thus, taking into account all terms and overall factors, the coefficient of $\delta\beta$t from 106 turns out to be

$$\frac{1}{4}[\langle H, \phi_k(w,\bar{w})\rangle_{str,\beta}\rangle_{str,\beta}]_{(c)} = \frac{4\pi h}{\beta^2}\tanh\frac{2\pi t}{\beta} \tag{111}$$

This is in exact agreement with the coefficient obtained from (106) at first order in $\delta\beta t$.

D.2 $\delta\beta^2$

The coefficient of the $\delta\beta^2$ term is

$$\frac{1}{32}\left[\langle\{H^2,\phi_k\}\rangle_{str,\beta} - 4\langle H\rangle_{str,\beta}\langle\{H,\phi_k\}\rangle_{str,\beta} + 2\langle H\phi_k H\rangle_{str,\beta}\right] \tag{112}$$

The above expression is essentially a connected correlator that can be denoted as

$$\frac{1}{32}\left[\langle\{H^2,\phi_k\}\rangle_{str,\beta} + 2\langle H\phi_k H\rangle_{str,\beta}\right]_{(\mathrm{c})} \tag{113}$$

The first term in the above expression is,

$$\begin{aligned}\langle H^2\phi_k(w,\bar{w}) + \phi_k H^2\rangle_{str,\beta} = \quad & 2\Bigg(\frac{1}{4\pi^2}\int_{\Gamma_1} dw_1 \int_{\Gamma_2} dw_2 \langle T_{ww}(w_1)T_{ww}(w_2)\phi_k(w,\bar{w})\rangle_{str,\beta} \\ & + \frac{1}{4\pi^2}\int_{\Gamma_1} dw_1 \int_{\Gamma_2} d\bar{w}_2 \langle T_{ww}(w_1)\bar{T}_{\bar{w}\bar{w}}(\bar{w}_2)\phi_k(w,\bar{w})\rangle_{str,\beta} \\ & + T_{ww}(w) \leftrightarrow \overline{T}_{\bar{w}\bar{w}}(\bar{w})\Bigg)\end{aligned} \tag{114}$$

The factor of 2 takes into account the insertion of T and $\bar{T}$ before the operator ϕ_k on the strip in the time ordered correlation function. Assuming $\phi_k(w,\bar{w})$ is a conformal primary, the following gives the transformation of the connected part of the above 3-pt. function from the strip to the UHP/LHP;

$$\langle T_{ww}(w_1)T_{ww}(w_2)\phi_k(w,\bar{w})\rangle_{str,\beta} = \left(\frac{\partial z}{\partial w}\right)^h \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{\bar{h}} \left\langle \left(\frac{\partial z_1}{\partial w_1}\right)^2 T_{zz}(z_1)\left(\frac{\partial z_2}{\partial w_2}\right)^2 T_{zz}(z_2)\phi_k(z,\bar{z})\right\rangle_{\mathrm{UHP}}$$

$$\langle \bar{T}_{\bar{w}\bar{w}}(\bar{w}_1)\bar{T}_{\bar{w}\bar{w}}(\bar{w}_2)\phi_k(\bar{w},\bar{w})\rangle_{str,\beta} = \left(\frac{\partial z}{\partial w}\right)^h \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{\bar{h}} \left\langle \left(\frac{\partial \bar{z}_1}{\partial \bar{w}_1}\right)^2 \bar{T}_{\bar{z}\bar{z}}(\bar{z}_1)\left(\frac{\partial \bar{z}_2}{\partial \bar{w}_2}\right)^2 \bar{T}_{\bar{z}\bar{z}}(\bar{z}_2)\phi_k(\bar{z},z)\right\rangle_{\text{LHP}}$$

$$\langle T_{ww}(w_1)\bar{T}_{\bar{w}\bar{w}}(\bar{w}_2)\phi_k(w,\bar{w})\rangle_{str,\beta} = \langle T_{ww}(w_1)T_{ww}((w_2)^*)\Phi_k(w)\bar{\Phi}_k(w^*)\rangle_{cyl,\beta} \tag{115}$$

The 2nd term in the connected part is,

$$\begin{aligned}\langle H\phi_k(w,\bar{w})H\rangle_{str,\beta} &= \frac{1}{4\pi^2}\int_{\Gamma_1} dw_1 \int_{\Gamma_2} dw_2 \langle T_{ww}(w_1)\phi_k(w,\bar{w})T_{ww}(w_2)\rangle_{str,\beta} \\ &+\frac{1}{4\pi^2}\int_{\Gamma_1} dw_1 \int_{\Gamma_2} dw_2 \langle T_{ww}(w_1)\phi_k(w,\bar{w})\bar{T}_{\bar{w}\bar{w}}(\bar{w}_2)\rangle_{str,\beta} \\ &+T_{ww}(w) \longleftrightarrow \bar{T}_{\bar{w}\bar{w}}(\bar{w})\end{aligned} \tag{116}$$

Using the Ward identity for the stress tensor,

$$\begin{aligned}&\langle T_{zz}(z_1)T_{zz}(z_2)\phi_k(z,\bar{z})\rangle_{UHP} = \langle T_z(z_1)T_z(z_2)\Phi_k(z)\bar{\Phi}_k(z^*)\rangle_{\mathbb{C}} \\ &= \Bigg[\left(\frac{h}{(z_1-z)^2}+\frac{\partial_z}{(z_1-z)}\right)\left(\frac{h}{(z_2-z)^2}+\frac{\partial_z}{(z_2-z)}\right) \\ &+\left(\frac{h}{(z_1-z^*)^2}+\frac{\partial_{z^*}}{(z_1-z^*)}\right)\left(\frac{h}{(z_2-z')^2}+\frac{\partial_{z'}}{(z_2-z')}\right) \\ &+\left(\frac{h}{(z_1-z)^2}+\frac{\partial_z}{(z_1-z)}\right)\left(\frac{h}{(z_2-z^*)^2}+\frac{\partial_{z^*}}{(z_2-z^*)}\right) \\ &+\left(\frac{h}{(z_1-z^*)^2}+\frac{\partial_{z^*}}{(z_1-z^*)}\right)\left(\frac{h}{(z_2-z)^2}+\frac{\partial_z}{(z_2-z)}\right)\Bigg]\langle\Phi_k(z)\bar{\Phi}_k(z^*)\rangle_{\mathbb{C}}\end{aligned} \tag{117}$$

We obtain similar expressions for objects on the lower half plane.
All terms coming from 114 and 116 are essentially given by the term above. All other terms in 114 and 116 supply a numerical factor of 8 each. So we get an overall factor of 16 multiplying the above term.
The entire calculation 113 then reduces to calculating the following indefinite integral,

$$h^2(z-\bar{z})^4\int_{\Gamma_1} dz_1 \int_{\Gamma_2} dz_2 \frac{z_1 z_2\langle T_{zz}(z_1)T_{zz}(z_2)\phi_k(z,\bar{z})\rangle_{\text{UHP}}}{\langle\phi_k(z,\bar{z})\rangle_{\text{UHP}}} \tag{118}$$

Since we are interested in calculating the linear and quadratic t terms, we shall focus on just the log terms in the above expression.
The double log terms in the above integral are

$$\frac{h(2z\bar{z}+h(z+\bar{z})^2)}{(z-\bar{z})^2}(\log(z_1-z)-\log(z_1-\bar{z}))(\log(z_2-z)-\log(z_2-\bar{z})) \tag{119}$$

To calculate the coefficient, we put the analytically continued values of z and $\bar{z}$ into the above expressions and consider the overall coefficients. From 119, we find the coefficient of $\delta\beta^2 t^2$ to be

$$\begin{aligned}\frac{1}{32}\left[\langle\{H^2,\phi_k\}\rangle_{str,\beta}+2\langle H\phi_k H\rangle_{str,\beta}\right]_{(c)} &= \left(-\frac{4\pi^2 h}{\beta^4}\frac{1}{\cosh^2\left(\frac{2\pi t}{\beta}\right)}+\frac{8\pi^2h^2}{\beta^4}\tanh^2\left(\frac{2\pi t}{\beta}\right)\right) \\ &= \left(-\frac{4\pi^2 h}{\beta^4}+\frac{4\pi^2 h(1+2h)}{\beta^4}\tanh^2\left(\frac{2\pi t}{\beta}\right)\right)\end{aligned}$$

This coefficient is in exact agreement with the coefficient obtained from 105.

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