Last year at RRI (which is just down the road from here) I lectured on entangle-entropy, and in connection with that, I tried to make more explicit my intuition about why (and when) one would expect an area law to hold.

However aside from that, I've not been working on E-E recently and have no particularly new results to report. Instead I thought I'd try to convey some of this intuition, especially since it conflicts in some points with the intuition that C.M. folks seem to have developed for the subject.

This is all heuristic, but some of it comes from a calculation that, if completed, would rigorously establish an area law. Completing this calc. offers a nice math-physic problem(s) that someone here might enjoy, and I'll describe that second.

Originally I planned to throw out also some stray thoughts relating to E-E and BH's etc, but there won't be time.

So here's the talk as it would have been - - -

1° Bangalore talk — Arinda's ~~co~~ meet

(0) Just before leaving home for Bangalore, I realized that my talk was to be part of a conference on entanglement S. Last yr I gave some lectures on that at RRI, down the street, but I haven't been working on it recently. So I thought I would gather up some stray thoughts & loose ends from previous work, some of which represent possible research problems which maybe some of you will take up.

If time permits my talk will have 5 parts:

1. Why area law (you don't need a gap!)
2. An unfinished calculation (rigourously derive area law with coeff.)
3. A logical gap in a proof of the semiclassical GSL
4. QNM as horizon deg of freedom — what choice of B.C.?
5. Fractal structure of BH horizon? (implications for Sentangle)

ask later — just mention these

(1) my CM friends seemingly have a very different intuition about Sentang than I do. They seem to associate an area law with a gapped (massive) deg of freedom, and to think of it as otherwise rather exceptional. Whereas I think of area law as typical. Let me try to explain why.

(EL

1A ~~V~~ Why area law? Why cutoff?

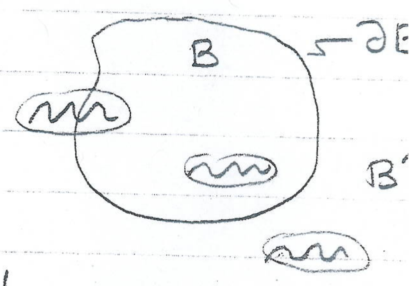
(Va) Need for cutoff (heuristic)

$\mathcal{L} = -\frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2$ $\left((\nabla\phi)^2 = \eta^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right)$
(temporarily $m \neq 0$) $(c = \hbar = 1)$

S = entanglement across sphere of radius R (at fixed t)
dimensional analysis $\Rightarrow$ $S = f(mR)$
As $R \to 0$ expect $S \to 0$ unless $S = \infty$.
But then $S = f(mR) \to 0$ as $m \to 0$ with R fixed,
which is nonsense. Hence $S = \infty$.
$\Rightarrow$ Need UV cutoff
Does nature provide one?
Must be covariant (respecting Lorentz symmetry)
Only known covariant cutoff is causet.

(Vb) Why expect an area law (heuristic)

Σ = surface of $t = $ cst
Within Σ, expand ϕ in "wavelets" or similar "modes".



First Assume $m = 0$. Plausible that each mode that straddles ∂B contributes ≈ 1 bit of entanglement.

The modes are quasi-localized, each has an approximate location and wavelength (ie $\vec{k}$)

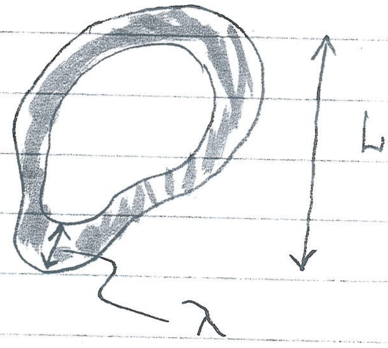
Recall $k = \|\vec{k}\| = 2\pi/\lambda$

In range d^3V and d^3k, $\# \approx \dfrac{d^3V\, d^3k}{(2\pi)^3}$

$(2\pi = 2\pi\hbar = h)$

Let $V = |B| =$ volume of B

$A = |\partial A| =$ area of ∂B



For <u>all</u> modes in B, $dN \approx \dfrac{V\, d^3k}{(2\pi)^3}$

Count the straddlers, N_S.

They are a fraction $\lambda A / V$.

$\left(\begin{array}{l} \lambda \ll L = \text{diameter of } B \\ \lambda \ll \text{rad. curv. } \partial B \end{array}\right.$

ie

$$dN_S \approx \frac{\lambda A\, d^3k}{(2\pi)^3}$$

Integrate from $\boxed{k_{min} \approx \frac{2\pi}{L}}$ to $\boxed{k_{max} \approx \frac{2\pi}{l}}$ ← cutoff!

$\Rightarrow$

$$N_S \approx \int_{min}^{max} dN_S \approx \int \frac{2\pi}{k} \frac{A\, 4\pi k^2 dk}{(2\pi)^3}$$

$$= \frac{A}{\pi} \int k\, dk = \frac{A}{2\pi}\left(k_{max}^2 - k_{min}^2\right)$$

$$\approx \boxed{2\pi A \left(1/l^2 - 1/L^2\right)} \approx \frac{2\pi A}{l^2} \quad (\because l \ll L)$$

This was 3+1. We see $3+1 \to \int k\, dk$

$2+1 \to \int dk$

Hence in 2+1: $\quad 1+1 \to \int dk/k$

$$S \sim A \int^{1/l} dk \sim \boxed{\frac{A}{l}} \longleftarrow \text{circumference } (2+1D)$$

But the area-law is modified in $D=1+1$.
First notice "A" = const. However

$$S \sim 1 \int_{1/L}^{1/\ell} \frac{dk}{k} \sim \log L/\ell \qquad (1+1\ D) \qquad (13.1)$$

Thus S diverges logarithmically as $L \to \infty$.

Later we will want the generalization to $m^2 > 0$.
Expect then that correlations extend only over a distance of $\lambda^{max} \sim 1/m$ (ie $\hbar/mc$), whence $k_{min} \sim m$ (<u>not</u> L^{-1}) $\Rightarrow$ $S(m) \sim \log \frac{1/m}{\ell} = \log 1/m\ell$

$$\boxed{S(m) \sim \log \frac{1}{m\ell}} \qquad (\text{in } 1+1\ D) \qquad (13.2)$$

1B <u>REMARK</u> Treating the vacuum as a relativistic gas in a static gravitational field (viewpoint of "Rinder observer") leads to the same conclusion, $S \sim 1/6 \log L/\ell$. (Does it then yield $1/6 \log 1/m\ell$ when $m \neq 0$?) $\boxed{n}$

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VI Apply our formula to scalar field in $\mathbb{M}^4$

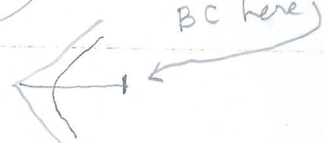
We had $\Lambda^a{}_c = -W^{a\beta} W_{\beta c}$. What is W ?

$$L = \int \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}|\vec{\nabla}\phi|^2 \, d^3x \qquad (m=0)$$ $\boxed{n}$

$$H = 1/2 \int \underbrace{\dot{\phi}(x)\dot{\phi}(x)}_{G_{AB}\dot{\phi}^A\dot{\phi}^B} + \underbrace{\vec{\nabla}\phi(x)\cdot\vec{\nabla}\phi(x)}_{V_{AB}\phi^A\phi^B}$$

$A, B \leftrightarrow x$

Put Dirichlet BC here



$$G_{AB} \leftrightarrow G(x,y) = \delta^{(3)}(x-y)$$

Rindler case

{ [(see details with lecture notes)] }

$$T = \frac{1}{2\pi z} \qquad (\underline{\text{local}} \text{ temp})$$

$m=0 \Rightarrow$ entropy density $\sim \frac{1}{T^3} \sim \frac{1}{z^3}$

$$\Rightarrow \quad S \sim \int_\ell^L \frac{A\,dz}{z^3} \sim \frac{A}{\ell^2}$$

In $\mathbb{M}^{1+1}$ we get density $\sim \frac{1}{T}$

$$S \sim \int_\ell^L \frac{dz}{z} \sim \ln(L/\ell)$$

(even coeff of 1/6 comes out right!) (*)

(limited to half-space, Lor-invar. Theory)

(*) 2/6 = 1/3 if 2 bdries

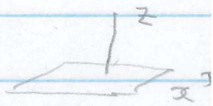
(16)

A third way to see (also for ½-space)

Consider massive scalar again
translational symm along x^1, x^2 ⇒ entanglement due to
mode $e^{i p_\perp \cdot x_\perp}$ decouple $p_\perp$ $(p \cdot x = p_1 x^1 + p_2 x^2)$
⇒ sum of $(1+1)d$ problems

Now all these are massive with $m^2 = m_0^2 + p_\perp \cdot p_\perp$

$$\phi(t, x^\perp, z) = e^{i p_\perp x^\perp} f(t, z)$$



$$(\Box - m^2)\phi = 0$$

$$-\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial z^2} - \underbrace{(m^2 + p_\perp^2)}_{m^2 \to m^2 + p^2} f = 0$$

⇒ $S(p) \sim \log(1/\ell p)$ from 1d result

<u>and</u> $S \to 0$ for $p \gg 1/\ell$

$$\Rightarrow S \sim \int^\infty A\, d^2p\, S(p_\perp) \sim A \int^{1/\ell} p\, dp \log\frac{1}{\ell p} = \frac{A}{4\ell^2}$$

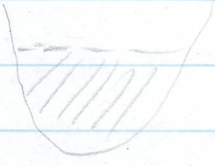
So this a third derivation, but it also helps to explain a "mystery": why get area law "even in this gapless case"

But mystery is in eye of beholder!
In my eyes the mystery is opposite:
why fermions at <u>finite</u> <u>density</u> $(\mu_{chem} > 0)$
do <u>not</u> follow area law. Rather $S \sim A \ln A$.
Whence the log?
I'm not sure following explanation is best possible...

Dirac sea

for excitations around fermi surface

dispersion relation looks massless (linear)



Problem again splits into $S = \sum_p S(p)$ $\quad (p = p_\perp)$

p = transverse momentum shifts E_f but disp rel remains linear

hence $\log L/l$ remains for all p.

($l \sim 1/p_{Fermi}$)

See forthcoming paper with Arushya Chandran and Chris Laumann for details.

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② Exact formula for $S_{\text{entanglement}}$

limit to half-space in $\mathbb{R}^3$ $\quad (M^4)$

relativistic scalar field $(m^2 = 0)$

We just saw splits into 1d problems. Want $S(m = \|p_\perp\|)$

Now $\exists$ general formula (me, Abhishek Dhar)

(last yr's lectures at RRI) (also $\exists$ 4D form: S from W
(Goa talk)

We'll use non-covariant cutoff (pfui!)

Without cutoff we have

$$S = A \int \frac{2\pi p\, dp}{(2\pi)^2} S(p) = \frac{A}{2\pi} \int m\, dm\, S(m)$$

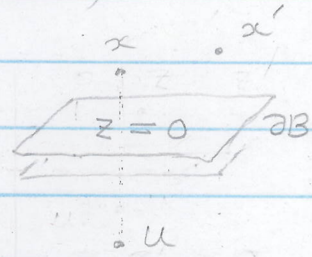
where $S(m) = \sum_\lambda \nu \ln \nu - (\nu - 1) \ln(\nu - 1)$, where $\nu = \dfrac{\sqrt{\lambda + 1} + 1}{2}$

② Exact formula for $S_{\text{entanglement}}$

Limit to half-space in $\mathbb{R}^3$ ($\mathbb{M}^4$)
relativistic scalar field, $m=0$

∃ general formula (me, Abhishek Dhar)
(last yr lectures at RRI)
(also ∃ 4D expression: S from W — Goa talk)

We will use non-covariant cutoff (pfui!)
Before cutoff we have



$$\Lambda(x,x') = -\iiint_{-\infty}^{0} W^{-1}(xu)\, W(ux')$$

$S = \Sigma /$ eigenvalues of Λ.

Of course $S=\infty$
Introduce cutoff via $z<-\ell$: $\int_{-\infty}^{0} \to \int_{-\infty}^{-\ell}$

$$S = \operatorname{tr}\left\{ \frac{\sqrt{\Lambda+1}+1}{2} \log \frac{\sqrt{\Lambda+1}+1}{2} - \frac{\sqrt{\Lambda+1}-1}{2} \log \frac{\sqrt{\Lambda+1}-1}{2} \right\}$$

Decomposes into 1d problems as before: $e^{i p_\perp \cdot x^\perp}$

Result is then
$$S = \frac{A}{2\pi} \int_0^\infty \xi \, d\xi \, \sigma(\xi)$$ (2π ok?)

where ($\xi = \|p_\perp\| \ell$) and

$\sigma(\xi)$ computed from

$$\Lambda_\xi(z,z') = \frac{1}{\pi^2}\int_{-\infty}^{-\xi} K_0(z-u)\,\frac{K_1(u-z')}{u-z'}\,du \qquad \frac{1}{\pi^2}\,?$$

(which is just massive in M^2 with $\xi = m\ell$)

One knows spec(Λ_ξ) discrete, accum. at $\lambda = 0$.

Conjectures/questions

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Research problems

① Prove that $\lambda_n \to 0$ sufficiently fast that $\sigma(\xi) < \infty \quad \forall \xi > 0$.

② Prove $\sigma(\xi) \to 0$ exponentially as $\xi \to \infty$

③ Prove $\sigma(\xi) \sim c \ln 1/\xi$ for $\xi \to 0$ and find c.

④ Evaluate $\int_0^\infty \xi \, d\xi \, \sigma(\xi)$ to get coeff of A/ℓ^2 in area law. (convergence alone $\Rightarrow$ area law

(for this particular cutoff)

⑤ Generalize the Rindler-gas estimate to $m^2 > 0$

⑥ " our derivation of $\Delta(x,y)$ " "

Probs 1-4 are nice math-phys problems !
and 6