PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Entanglement & C-theorems

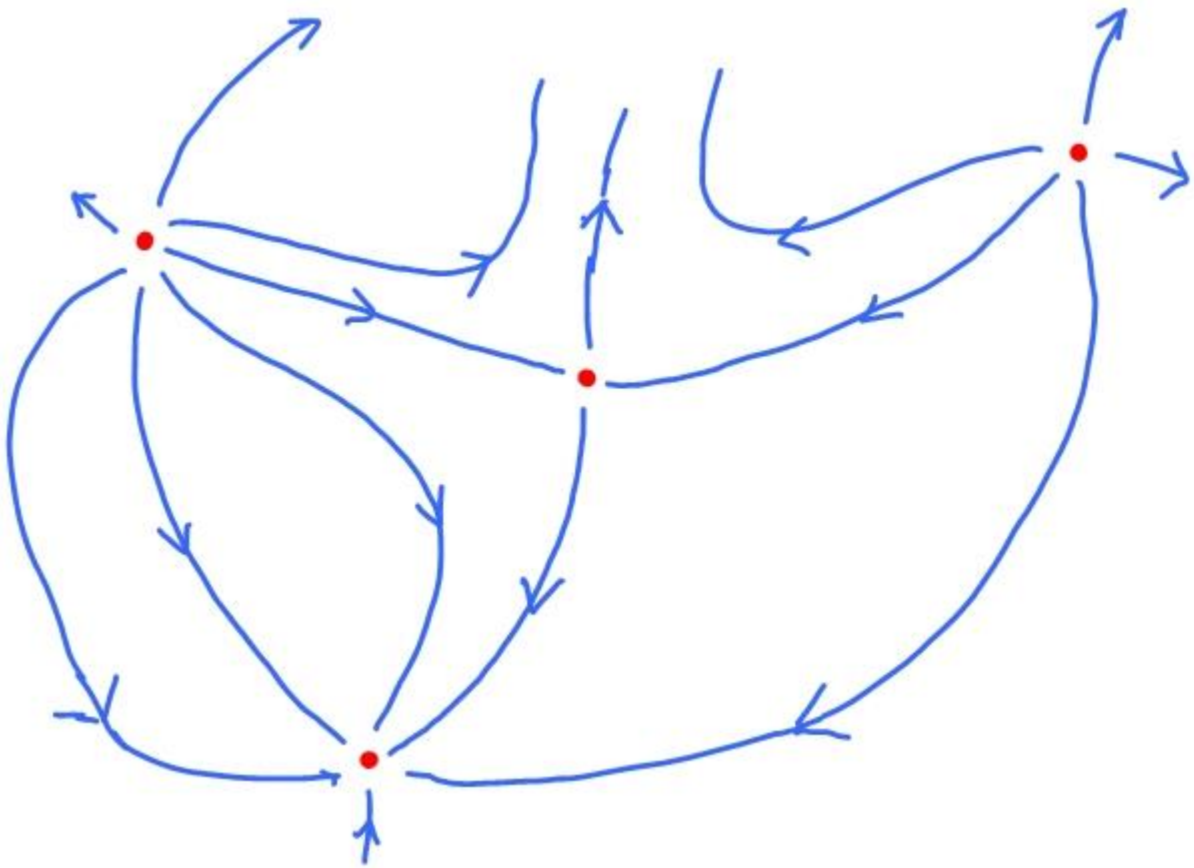
Chandrasekhar Lecture 3

ICTS, December 12, 2014

Overview:

1. **Introductory remarks on c-theorems**
2. Holographic c-theorems
3. Entanglement Entropy and new “c-theorems”
4. Progress beyond Holography
5. Beyond Entanglement Entropy
6. Concluding remarks

RG flows:



Renormalization Group:

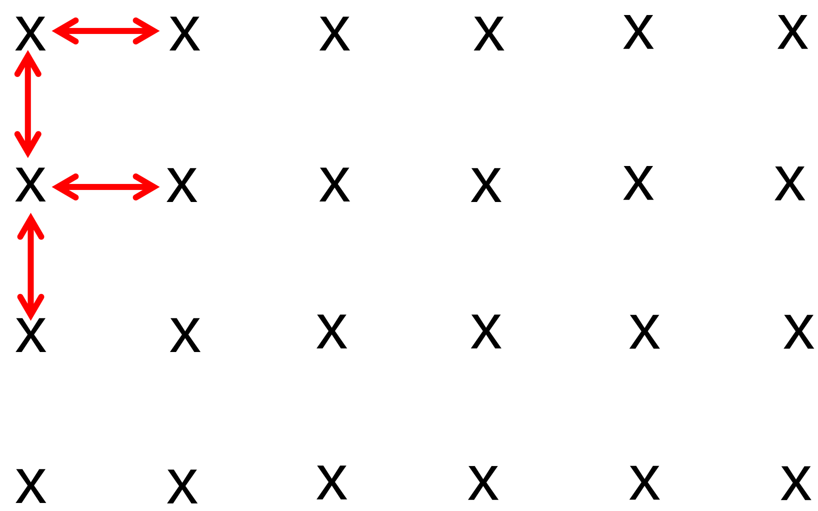
mathematical apparatus that allows systematic investigation of the changes of a physical system as viewed at different *distance/energy scales*"

Classic example of spin system:

Hamiltonian:

$$H = -K_0 \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$

nearest neighbours



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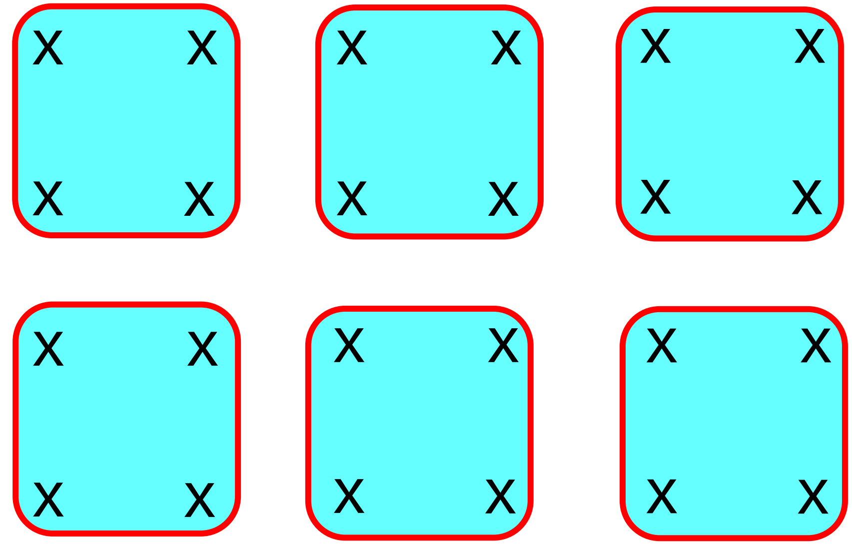
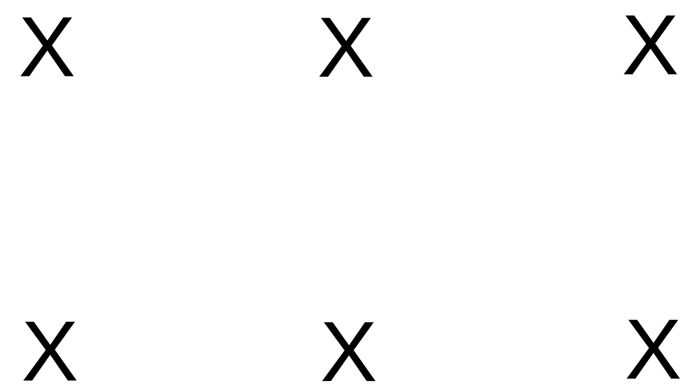


- replace effective spins on blocks in original lattice



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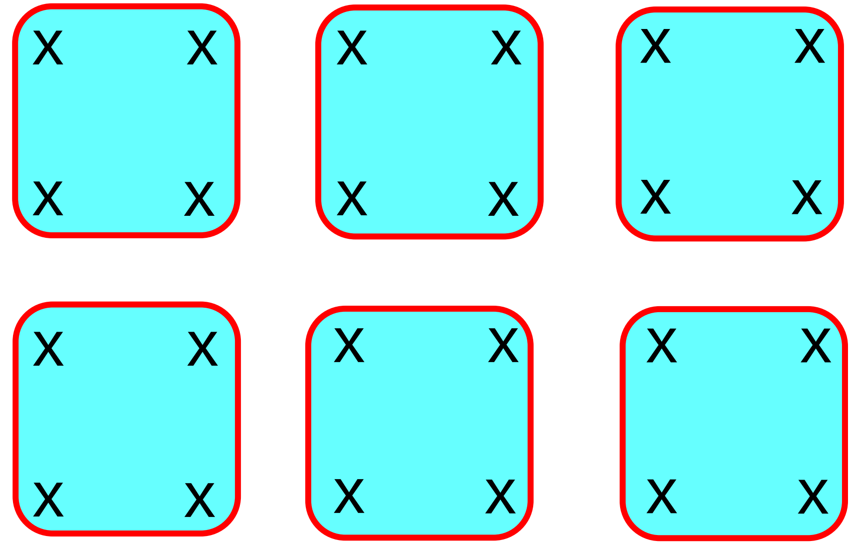
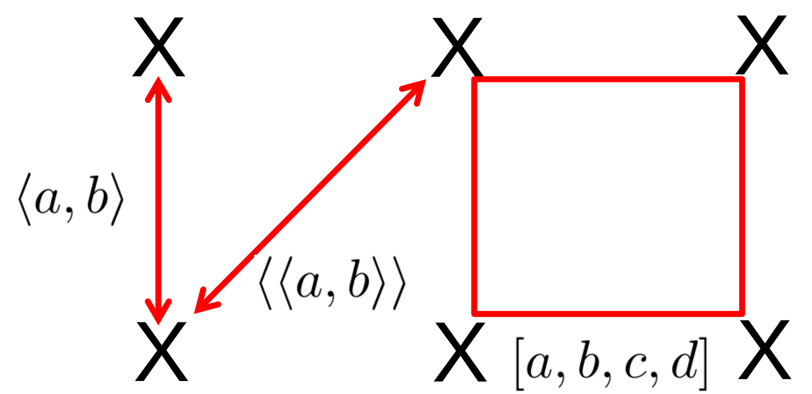


- replace effective spins on blocks in original lattice

new Hamiltonian:

$$H' = -K'_0 \sum_{\langle a,b \rangle} \vec{S}_a \cdot \vec{S}_b - K'_1 \sum_{\langle\langle a,b \rangle\rangle} \vec{S}_a \cdot \vec{S}_b - K'_2 \sum_{[a,b,c,d]} \vec{S}_a \cdot \vec{S}_b \, \vec{S}_c \cdot \vec{S}_d + \cdots$$



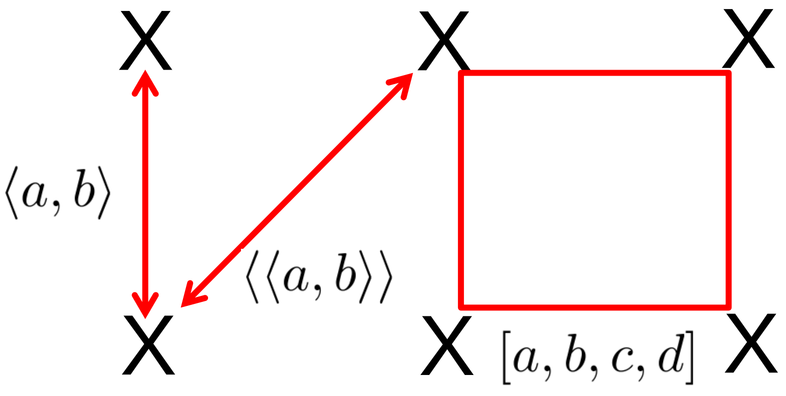
no reason the form of the Hamiltonian should be unchanged

Classic example of spin system:

- replace effective spins on blocks in original lattice

new Hamiltonian:

$$H' = -K_0' \sum_{\langle a,b \rangle} \vec{S}_a \cdot \vec{S}_b$$
$$-K_1' \sum_{\langle\langle a,b \rangle\rangle} \vec{S}_a \cdot \vec{S}_b$$
$$-K_2' \sum_{[a,b,c,d]} \vec{S}_a \cdot \vec{S}_b \, \vec{S}_c \cdot \vec{S}_d + \cdots$$



no reason the form of the Hamiltonian should be unchanged

- in general, H characterized by a collection of couplings

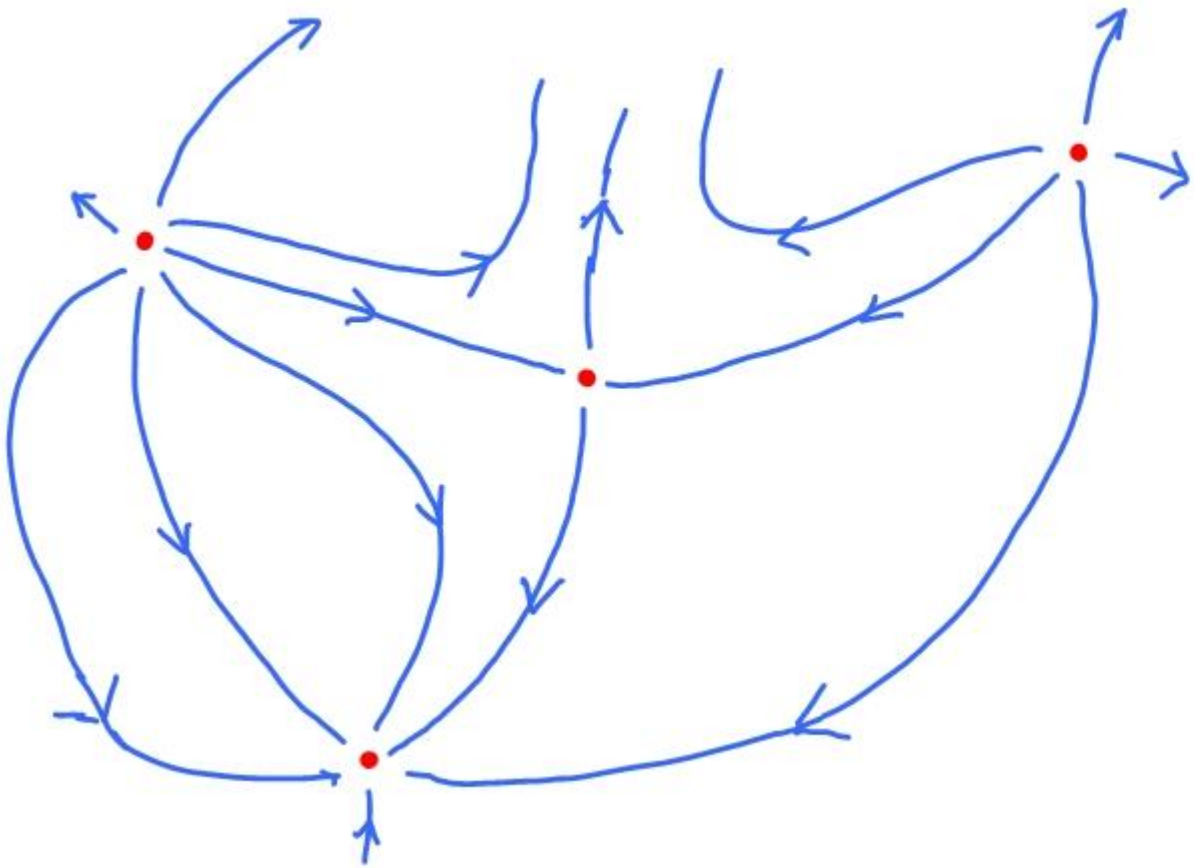
$$\mu = [K_0, K_1, K_2, \cdots]$$

- under spin block replacements, couplings change

$$\mu \to \mu' = R_b \, \mu$$

$\longrightarrow$ systematic computations give RG flow

RG flows:



Renormalization Group:

mathematical apparatus that allows systematic investigation of the changes of a physical system as viewed at different *distance/energy scales*”

Zamolodchikov c-theorem (1986):

- renormalization-group (RG) flows can seen as one-parameter motion

$$\frac{d}{dt} \equiv -\beta^i(g)\,\frac{\partial}{\partial g^i}$$

in the space of (renormalized) coupling constants $\{g^i,\ i = 1,2,3,\cdots\}$ with beta-functions as "velocities"

- for unitary, Lorentz-inv. QFT's in two dimensions, there exists a positive-definite real function of the coupling constants $C(g)$:

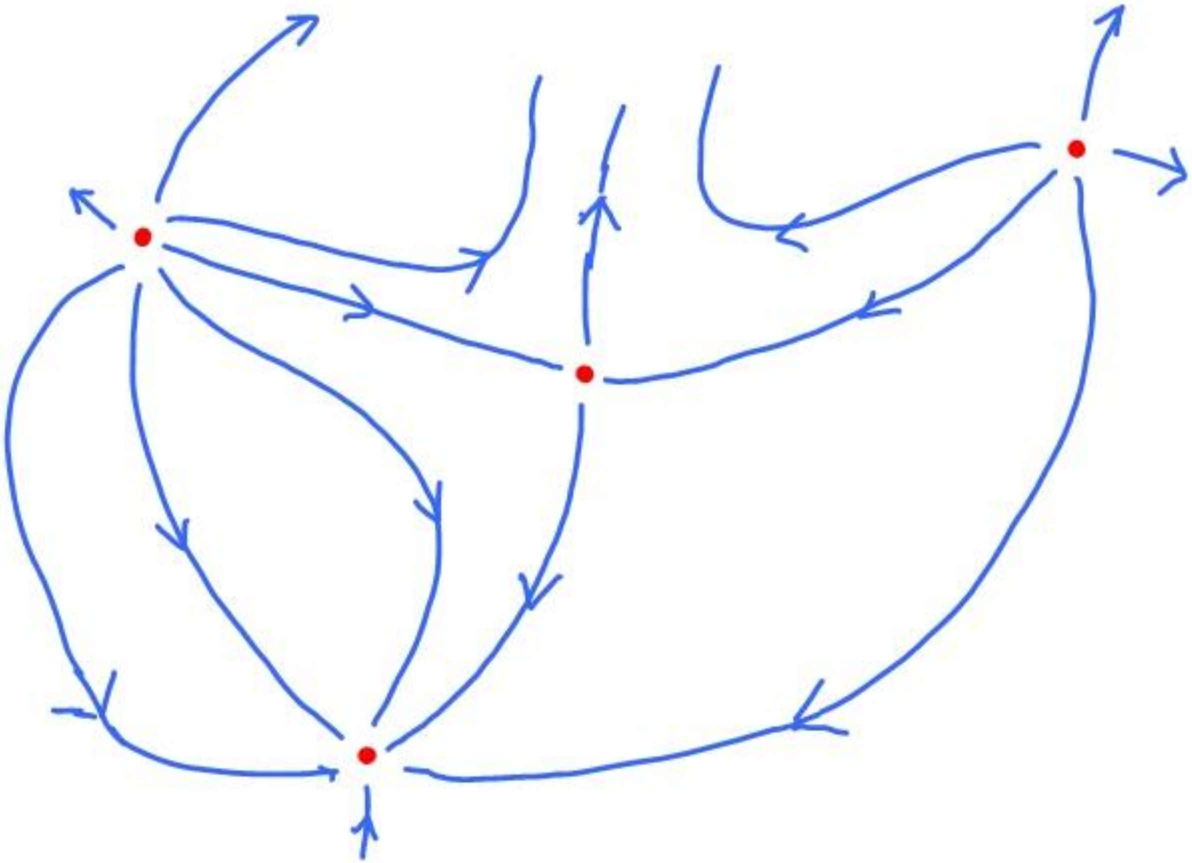
1. monotonically decreasing along flows: $\frac{d}{dt}C(g) \leq 0$
2. "stationary" at fixed points $g^i = (g^*)^i$:

$$\beta^i(g^*) = 0 \longleftrightarrow \frac{\partial}{\partial g^i}C(g) = 0$$

3. at fixed points, it equals central charge of corresponding CFT

$$C(g^*) = c$$

Zamolodchikov's C-function adds a dimension to RG flows:



BECOMES

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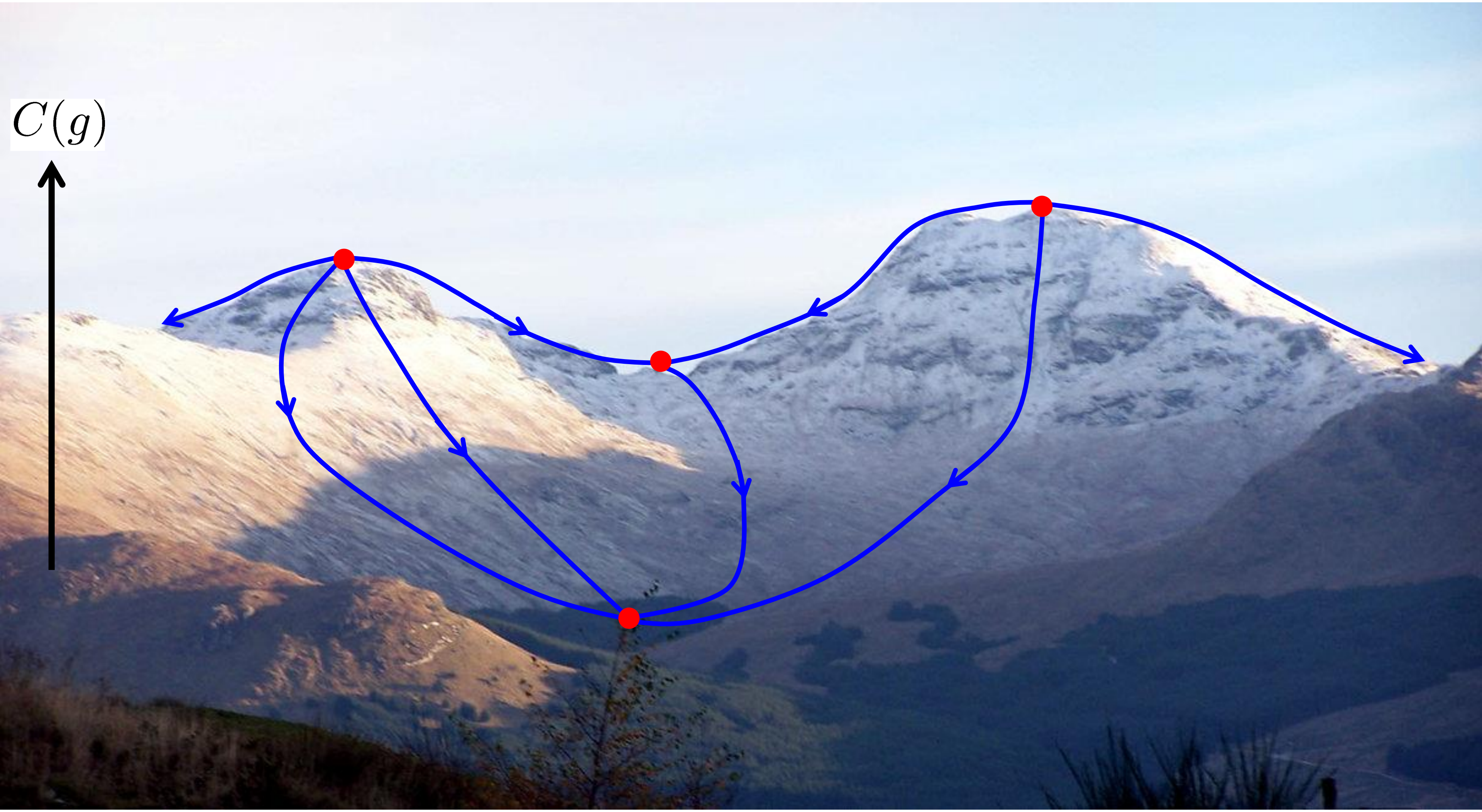

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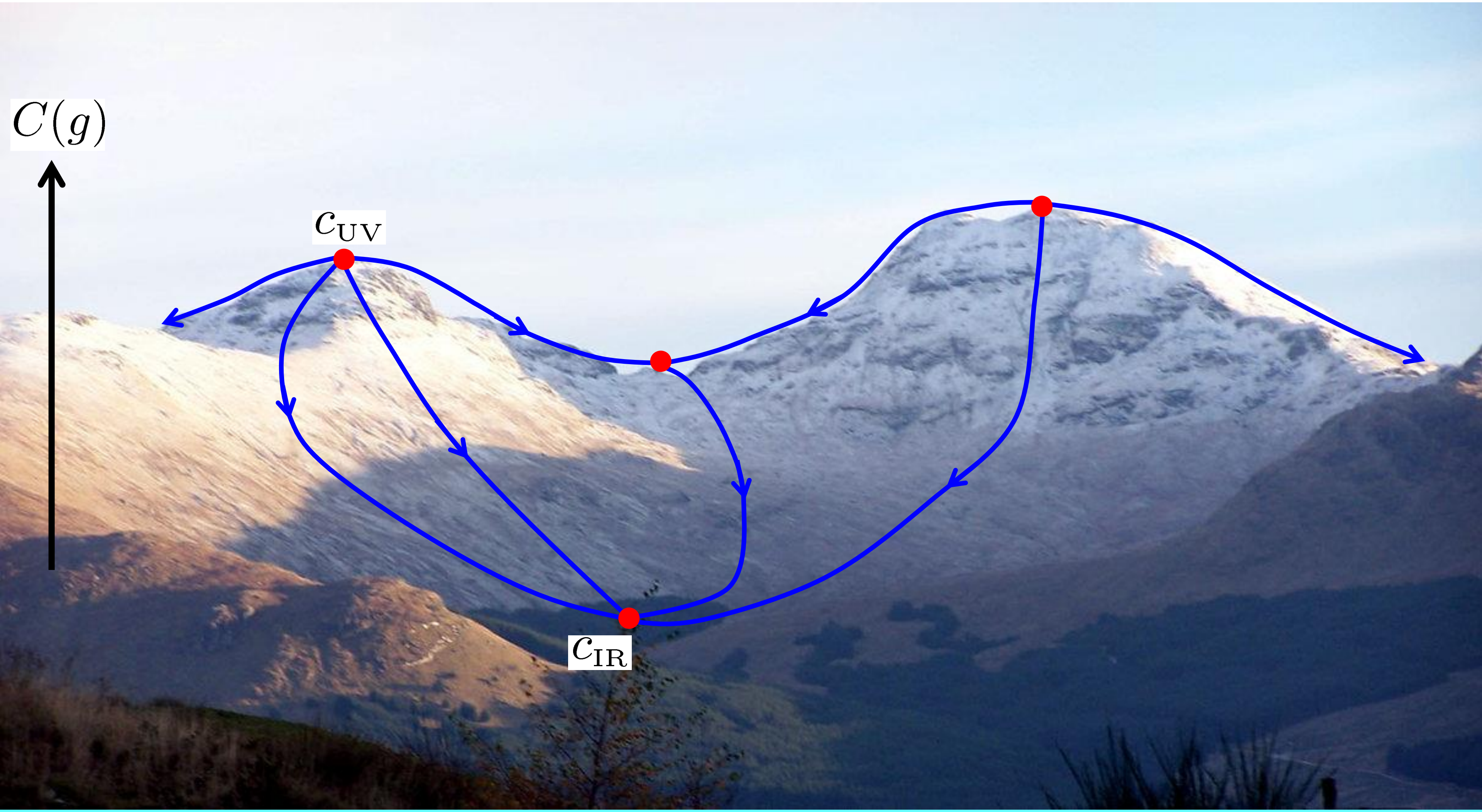

Simple consequence for any RG flow in d=2: $c_{\mathrm{UV}} > c_{\mathrm{IR}}$

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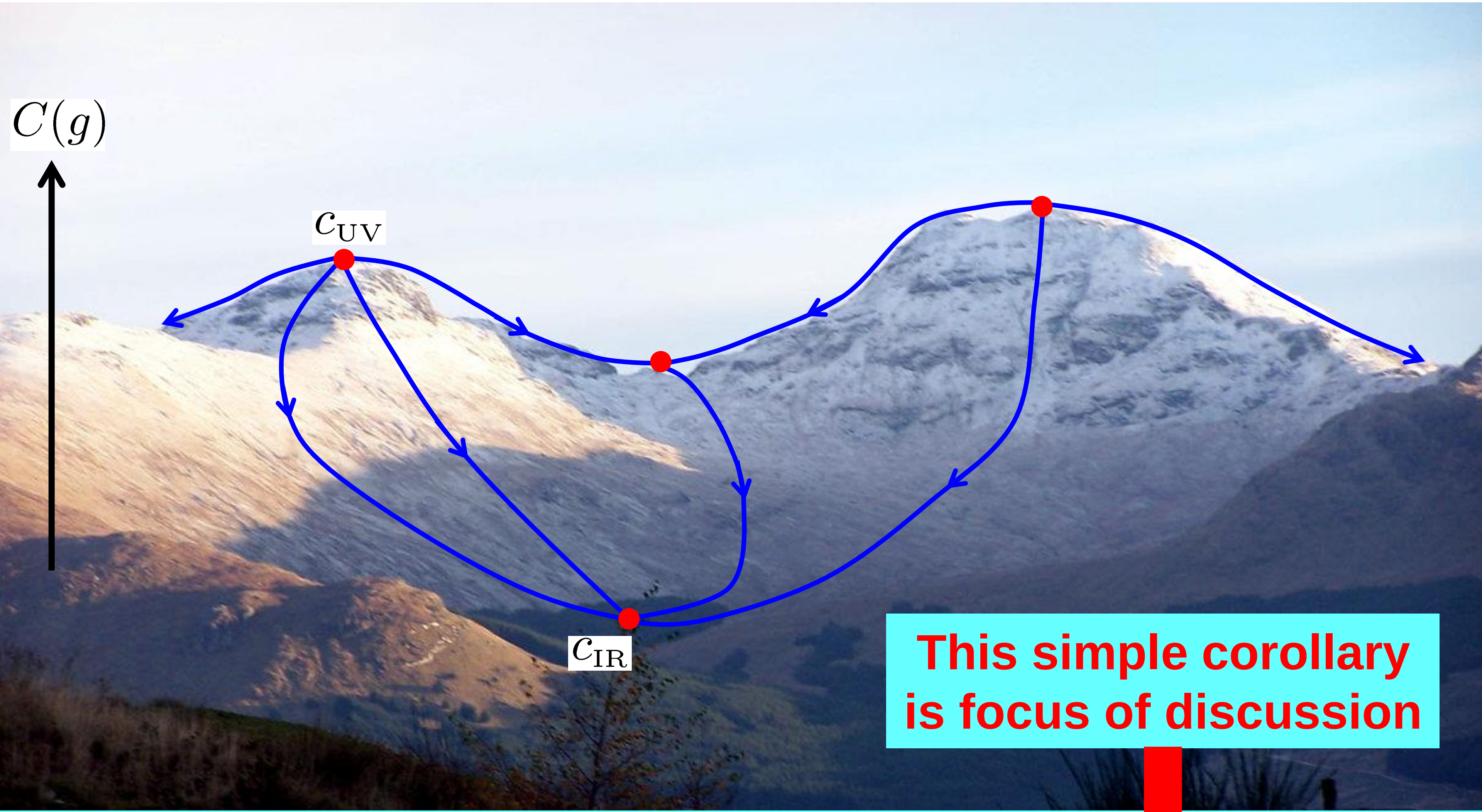


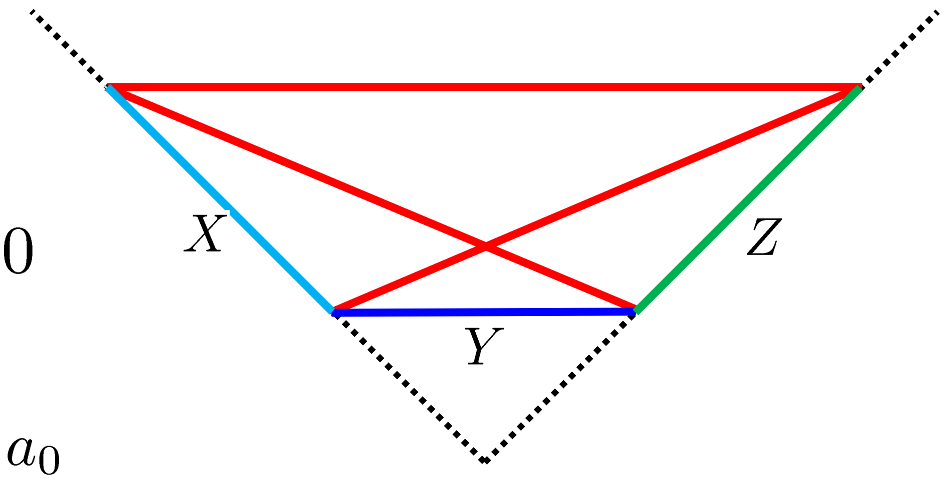
Simple consequence for any RG flow in d=2: $c_{\mathrm{UV}} > c_{\mathrm{IR}}$

(Casini & Huerta '04)

RG flows Meet Entanglement:

- c-theorem for d=2 RG flows can be established using unitarity, Lorentz invariance and **strong subaddivity inequality**:

$$S(X\cup Y\cup Z)-S(X\cup Y)-S(Y\cup Z)+S(Y)\leq 0$$



- define: $C(\ell)=3\,\ell\;\partial_\ell S(\ell)$

$$\longrightarrow \quad \partial_\ell C(\ell)\leq 0$$

- for d=2 CFT: $S=\dfrac{c}{3}\,\log(\ell/\delta)+a_0$

$$\longrightarrow \quad C_{\rm CFT}(\ell)=c$$

(Calabrese & Cardy)

(Holzhey, Larsen & Wilczek)

- hence it follows that: $\boxed{c_{\rm UV}\;>\;c_{\rm IR}}$

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RG flows Meet Entanglement: (more detail)

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- for d=2 CFT: $S_{\rm CFT}=\frac{c}{3}\log(\ell/\delta)+a_0$ (Calabrese & Cardy) (Holzhey, Larsen & Wilczek)

⟶ isolate central charge with: $3\,\ell\,\partial_\ell S_{\rm CFT}(\ell)=c$

- in general, define: $C(\ell)=3\,\ell\,\partial_\ell S(\ell)$

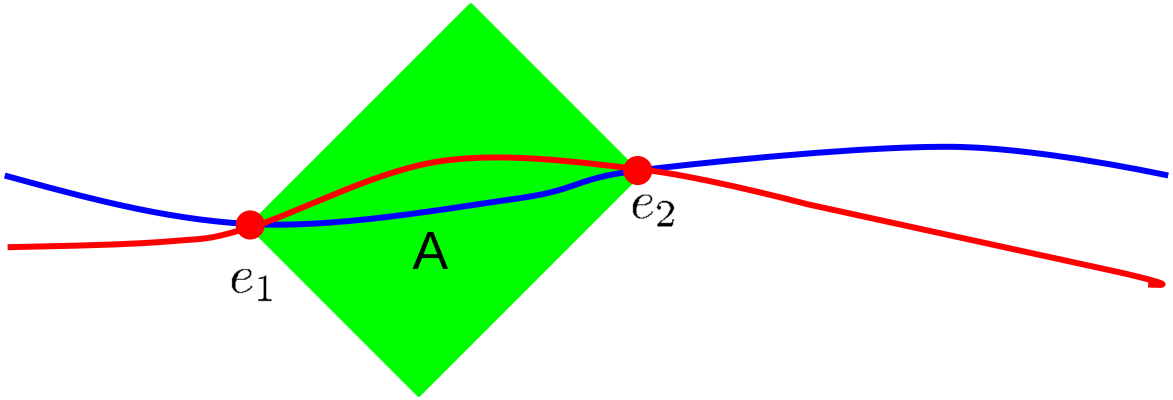
⟶ $C_{\rm CFT}(\ell)=c$

⟶ ℓ appears as proxy for energy scale

(Casini & Huerta '04)

RG flows Meet Entanglement: (more detail)

- interval A with endpoints e_1 and e_2 on some Cauchy surface



- by causality, ρ_A describes physics in causal diamond

- by unitarity, $S(e_1,e_2)$ independent of details of Cauchy surface

- by translation invariance (in vacuum), $S(e_1,e_2)$ only depends on proper distance between e_1 and e_2

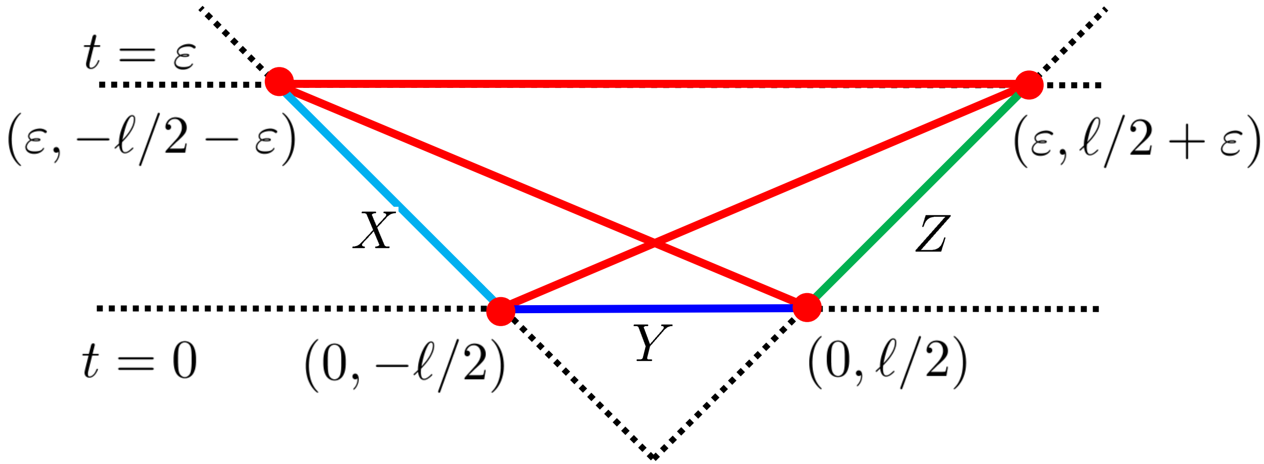
$$\ell_{12} = \left[(x_2 - x_1)^2 - (t_2 - t_1)^2\right]^{1/2}$$

(Casini & Huerta '04)

RG flows Meet Entanglement: (more detail)

- apply strong subaddivity inequality in following geometry:

$$S(X\cup Y\cup Z)-S(X\cup Y)-S(Y\cup Z)+S(Y)\le 0$$



$$S(Y)=S(\ell)\,,\ \ S(X\cup Y\cup Z)=S(\ell+2\varepsilon)$$

$$S(X\cup Y)=S(Y\cup Z)=S(\sqrt{\ell(\ell+2\varepsilon)})$$

SSA $\longrightarrow$ $S(\ell+2\varepsilon)+S(\ell)-2S(\sqrt{\ell(\ell+2\varepsilon)})\le 0$

$\varepsilon\to 0\ :\quad S''+S'/\ell\le 0 \longrightarrow \partial_\ell(\ell S')\le 0 \longrightarrow \boxed{\partial_\ell C(\ell)\le 0}$

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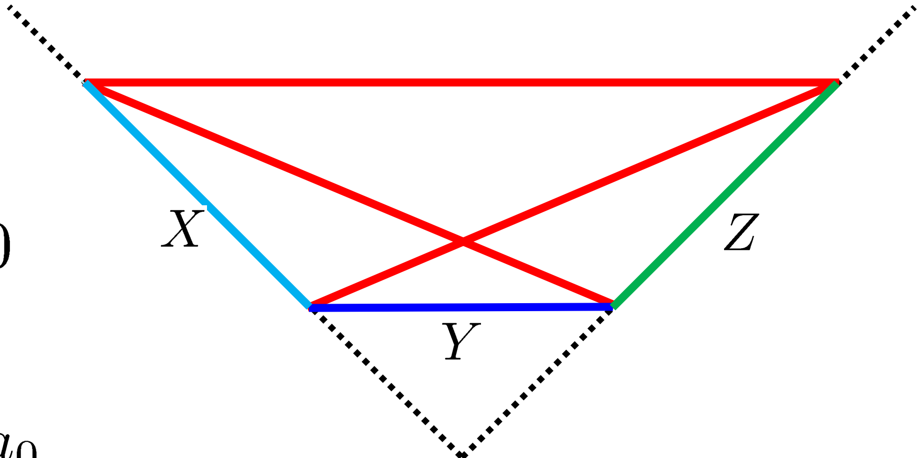

- define: $C(\ell) = 3\,\ell\ \partial_\ell S(\ell)$

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- hence it follows that: $\boxed{c_{\rm UV} > c_{\rm IR}}$

C-theorems in higher dimensions??

d=2: $$\langle\, T_\mu{}^\mu\, \rangle = -\frac{c}{12}\, R$$

d=4: $$\langle\, T_\mu{}^\mu\, \rangle = \frac{c}{16\pi^2}\, I_4 - \frac{a}{16\pi^2} E_4 - \frac{a'}{16\pi^2}\, \nabla^2 R$$

$I_4 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$ and $E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$

- in 4 dimensions, have three central charges: $c,\ a,\ a'$
- do any of these obey a similar “c-theorem” under RG flows?

ie, $[\,??\,]_{\rm UV} > [\,??\,]_{\rm IR}$

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<u>a' -theorem</u>: a' is scheme dependent (not globally defined)

<u>c -theorem</u>: there are numerous counter-examples

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a -theorem: proposed by Cardy (1988)

- numerous nontrivial examples, eg, perturbative fixed points (Jack & Osborn), SUSY gauge theories (Anselmi et al; Intriligator & Wecht)
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- numerous nontrivial examples, eg, perturbative fixed points (Jack & Osborn), SUSY gauge theories (Anselmi et al; Intriligator & Wecht)
- holographic field theories with Einstein gravity dual
- generalized holographic models & conjectures (RM & Sinha), d=3 F-theorem (Jafferis, Klebanov, Pufu & Safdi), proof of a-theorem (Komagordski &Schwimmer), proof of F-theorem (Casini & Huerta)

Overview:

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2. **Holographic c-theorems**
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AdS/CFT Correspondence:

Bulk:

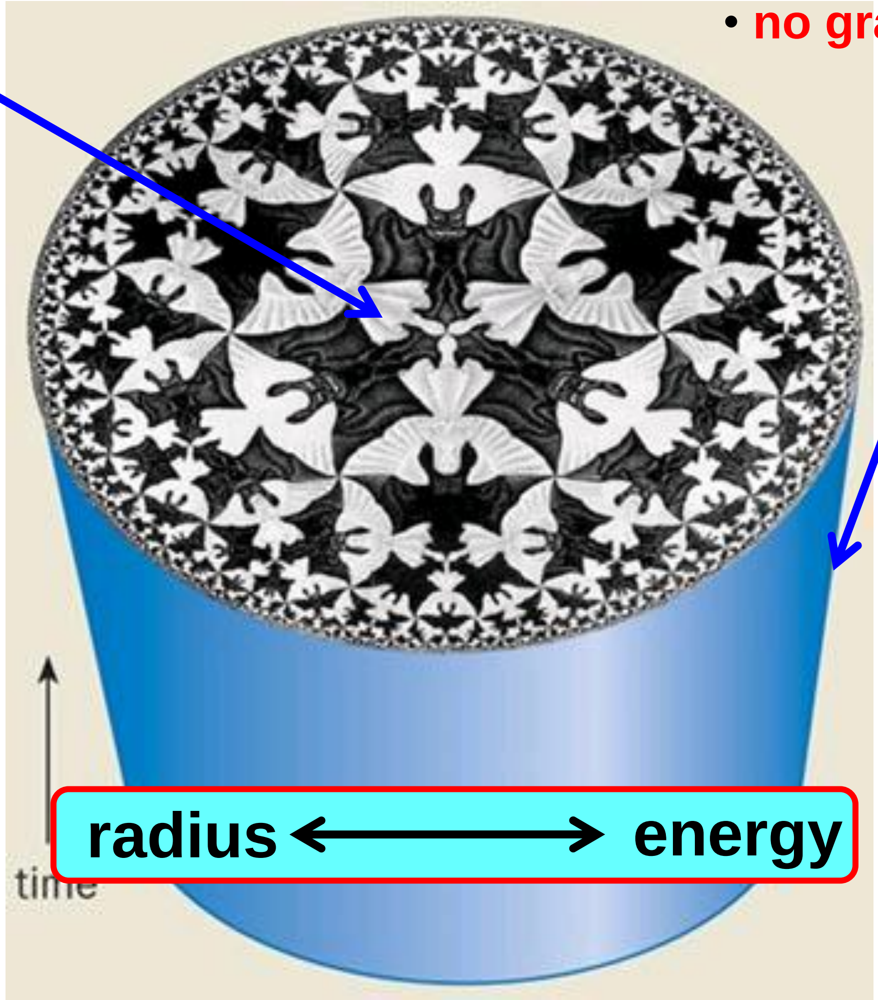
- quantum gravity
- negative cosmological constant
- **d+1** spacetime dimensions

Holography

Boundary:

- quantum field theory
- no scale (at quantum level)
- **d** spacetime dimensions
- **no gravity!**

anti-de Sitter space



conformal field theory

(Maldacena '97)

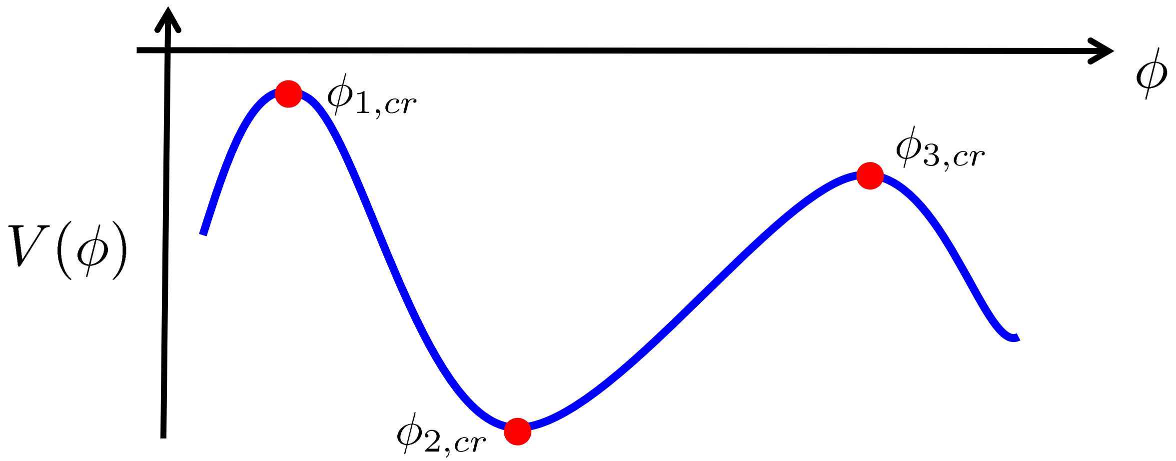
(Freedman, Gubser, Pilch & Warner, hep-th/9904017)
(Girardello, Petrini, Porrati and Zaffaroni, hep-th/9810126)

Holographic RG flows:

$$I = \frac{1}{2\ell_P^3}\int d^5x\,\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]$$

- imagine potential has stationary points giving negative Λ

$$\longrightarrow \quad V(\phi_{i,cr}) = -\frac{12}{L^2}\alpha_i^2$$



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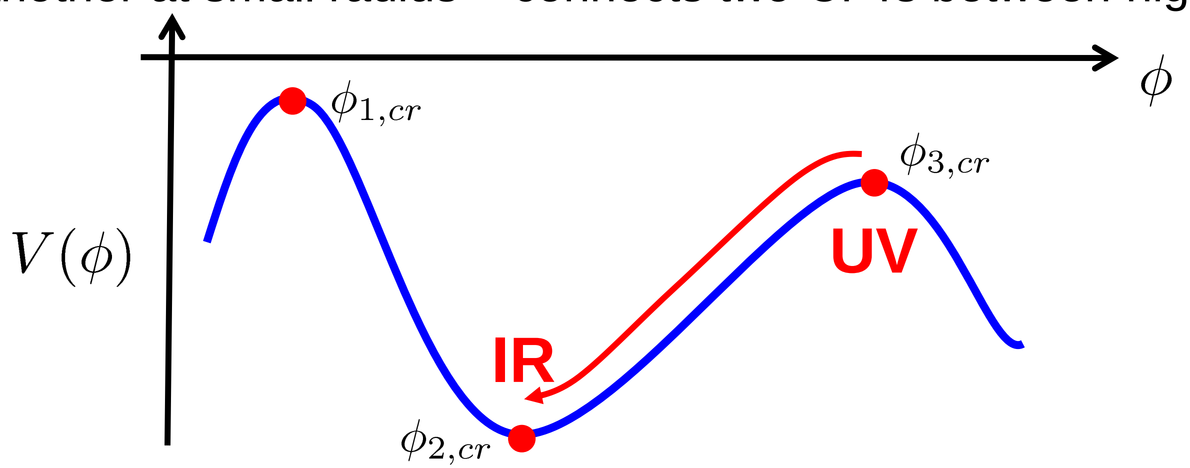
• Holographic RG flows are solutions starting at one stationary point at large radius and ending at another at small radius – connects two CFTs between high and low energies



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- consider metric: $ds^2 = e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2$
- at stationary points, AdS_5 vacuum: $A(r) = r/\tilde{L}$ with $\tilde{L} = L/\alpha_i$
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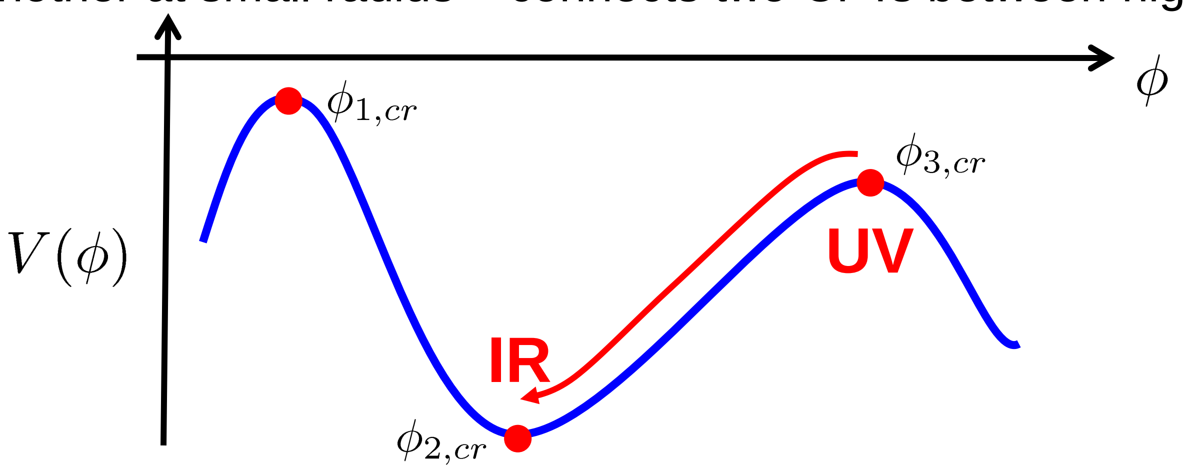

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Holographic RG flows:

- for general flow solutions, define: $a(r) \equiv \dfrac{\pi^2}{\ell_P^3 A'(r)^3}$

$$a'(r) = -\frac{3\pi^2}{\ell_P^3 A'(r)^4} A''(r) = -\frac{\pi^2}{\ell_P^3 A'(r)^4} \left(T^t{}_t - T^r{}_r \right) \geq 0$$

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Einstein equations

null energy condition
$(\, T_{\mu\nu}\, \ell^\mu\, \ell^\nu \geq 0\,)$

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Einstein equations **null energy condition**
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- at stationary points, $a(r) \to a^* = \pi^2\, \tilde{L}^3/\ell_P^3$ and hence

$$\boxed{a^*_{UV} \geq a^*_{IR}}$$

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- using holographic trace anomaly: $a^* = a$

(e.g., Henningson & Skenderis)

$\longrightarrow$ supports Cardy's conjecture

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- for Einstein gravity, central charges equal$(a = c)$: $c_{UV} \geq c_{IR}$

(Freedman, Gubser, Pilch & Warner, hep-th/9904017)

Holographic RG flows:

$$I = \frac{1}{2\ell_P^{d-1}} \int d^{d+1}x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right]$$

- same story is readily extended to (d+1) dimensions

- defining: $a_d(r) \equiv \frac{\pi^{d/2}}{\Gamma(d/2)\,(\ell_P A'(r))^{d-1}}$

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(for even d! what about odd d?) (e.g., Henningson & Skenderis)

Improved Holographic RG Flows:

- add higher curvature interactions to bulk gravity action

  → provides holographic field theories with, eg, $a \neq c$
  so that we can clearly distinguish evidence of a-theorem

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- constrain gravitational couplings with consistency tests
  (positive fluxes; causality; unitarity) and use best judgement

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- ultimately one needs to fully develop string theory for interesting holographic backgrounds!

- *"if certain general characteristics are true for all CFT's, then holographic CFT's will exhibit the same features"*

(Myers & Robinsion, 1003.5357)

Quasi-Topological gravity:

$$I = \frac{1}{2\ell_P^3}\int d^5x\,\sqrt{-g}\left[\frac{12}{L^2} + R + L^2\frac{\lambda}{2}\,\chi_4 + L^4\frac{7\mu}{4}\,\mathcal{Z}_5\right]$$

with $\chi_4 = R^{abcd}R_{abcd} - 4R_{ab}R^{ab} + R^2$

$$\mathcal{Z}_5 = R_a{}^c{}_b{}^d R_d{}^e{}_c{}^f R_e{}^a{}_f{}^b + \frac{1}{56}\big(21R_{abcd}R^{abcd}R - 72R_{abcd}R^{abc}{}_e R^{de} + 120R_{abcd}R^{ac}R^{bd} + 144R_a{}^bR_b{}^cR_c{}^a - 132R_a{}^bR_b{}^aR + 15R^3\big)$$

- three dimensionless couplings, $L/\ell_P\,,\ \lambda\,,\ \mu\,,$ allow us to explore dual CFT's with most general three-point function $\langle T_{ab}\,T_{cd}\,T_{ef}\rangle$

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- three dimensionless couplings, L/ℓ_P , λ , μ , allow us to explore dual CFT's with most general three-point function $\langle T_{ab}\,T_{cd}\,T_{ef}\rangle$

"maintain control of calculations"

- analytic black hole solutions
- linearized eom in AdS$_5$ are second order (in fact, Einstein eq's!)
- can be extended to higher dimensions (D≥5)
- gravitational couplings constrained (Myers, Paulos & Sinha, 1004.2055)

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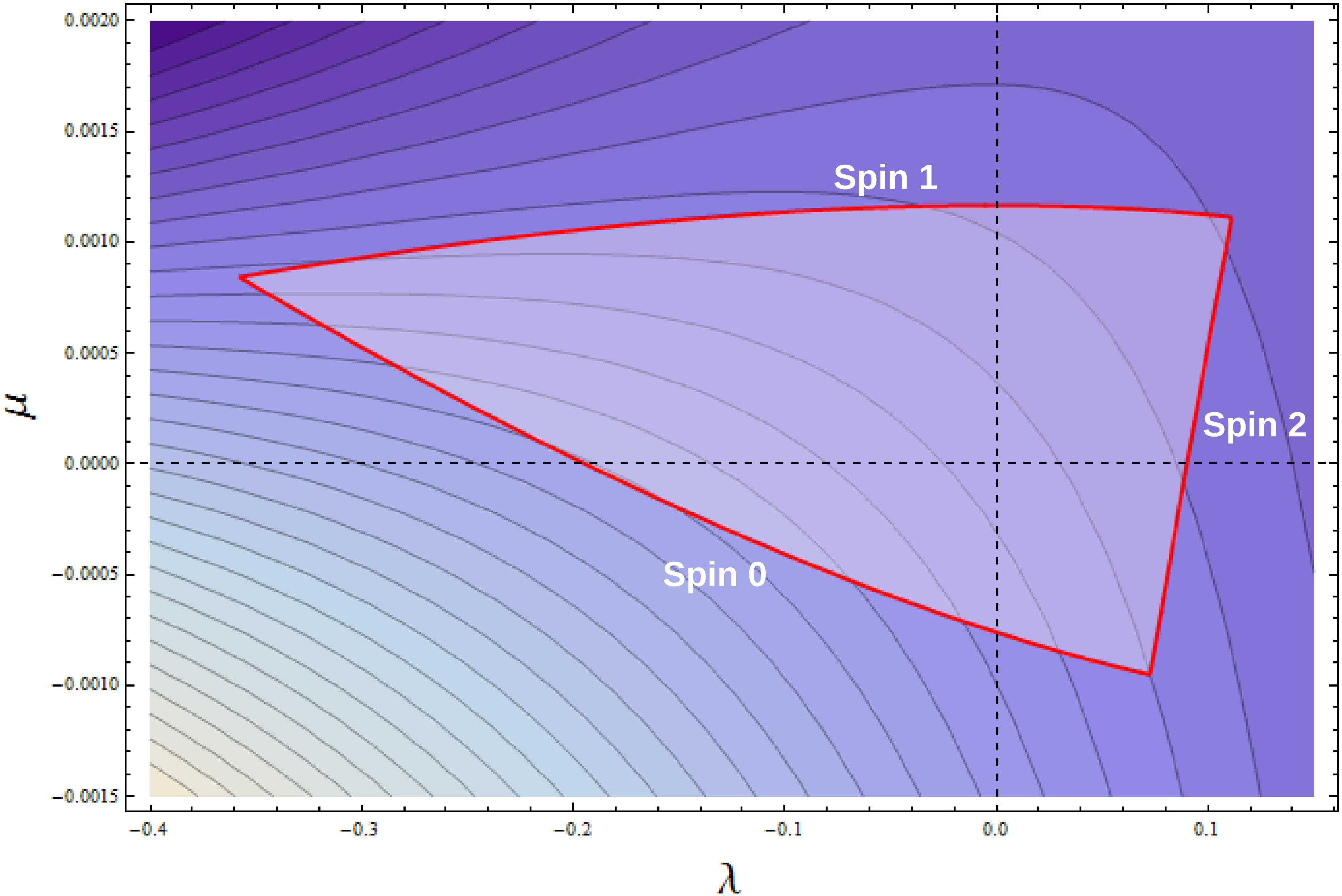
Quasi-Topological gravity:



More recent constraints:

Tunneling between vacua:

- implicitly we focus on backgrounds that reduce to solution of Einstein gravity as $\lambda\,,\ \mu \rightarrow 0$
- typically other "pathological" branches also exist, eg, gravitons are ghosts (Boulware & Deser)
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Time delay & causality violations:

- modified 3-pt graviton vertex may produce causality violations
- may be resolved by new higher spin fields with $m^2 \sim 1/L^2$

(Camanho, Edelstein, Maldacena & Zhiboedov)

$\longrightarrow$ perhaps higher spin fields not excited in holo-RG flows? (backgrounds are simple & symmetric, eg, conformally flat)

(Myers & Robinsion, 1003.5357)

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RG flows in Quasi-Topological gravity:

- consider metric: $ds^2 = e^{2A(r)}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + dr^2$

$\longrightarrow$ AdS$_5$ vacua: $A(r) = r/\tilde{L} \quad = r\, f_\infty^{1/2}/L$

- recall central charges:

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- "simplest" r-dependent functions satisfying this condition

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$$\begin{aligned} a'(r) &= -\frac{3\pi^2}{\ell_P^3 A'(r)^4} A''(r) \left(1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4\right) \\ &= -\frac{\pi^2}{\ell_P^3 A'(r)^4} \left(T^t{}_t - T^r{}_r\right) \geq 0 \qquad ✓ \end{aligned}$$

assume null energy condition

gravitational equations of motion

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- can try to be more creative in defining c(r) but we were unable to find a expression where flow is guaranteed to be monotonic
- our toy model seems to provide support for Cardy's “a-theorem” in four dimensions

Higher Dimensions: $D = d + 1 \quad (\mu = 0 \text{ for } d = 5)$

- straightforward to reverse engineer “a-theorem” flows
- eq’s of motion:

$$T^t{}_t - T^r{}_r = (d-1)\, A''(r)\, \left(1 - 2\lambda L^2 A'(r)^2 - 3\mu L^4 A'(r)^4\right)$$

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- flow between stationary points (where $a_d^* \equiv a_d(r)|_{AdS}$)

$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

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- quasi-topological gravity obeys c-theorem in very nontrivial way
- results can be generalized to include:
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  - general Lovelock interactions
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What do we learn about boundary theory?

Overview:

1. Introductory remarks on c-theorems
2. Holographic c-theorems
3. **Entanglement Entropy and new “c-theorems”**
4. Progress beyond Holography
5. Beyond Entanglement Entropy
6. Concluding remarks

- holography indicates some interesting c-theorems but

what is a_d^* in terms of the boundary theory??

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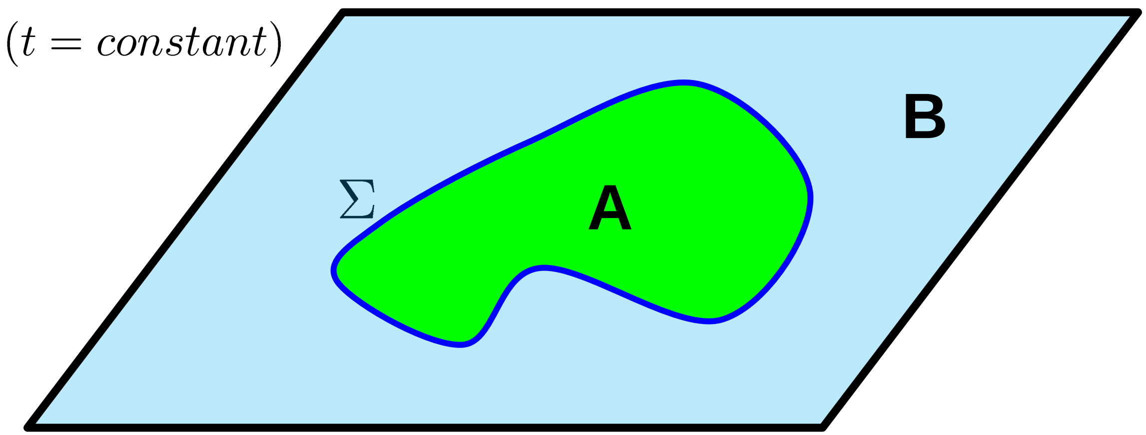
What is a_d^* for odd d??

Entanglement Entropy

- general tool; divide quantum system into two parts and use entropy as measure of correlations between subsystems
- in QFT, typically introduce a (smooth) boundary or entangling surface Σ which divides the space into two separate regions
- integrate out degrees of freedom in "outside" region
- remaining dof are described by a density matrix ρ_A

⟶ calculate von Neumann entropy: $S_{EE} = -Tr\,[\rho_A\,\log\rho_A]$

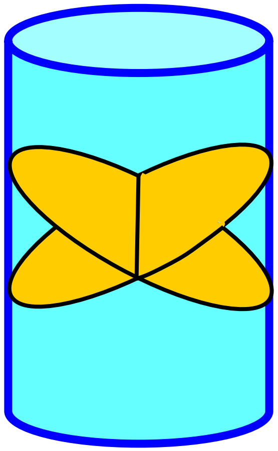


(Casini, Huerta & RM; RM & Sinha)

Holographic Entanglement Entropy:

- S_{EE} for CFT in d-dim. flat space and choose S^{d-2} with radius R
- conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $\mathcal{R} \sim 1/R^2$ and $T = 1/2\pi R$
- holographic dictionary: thermal bath in CFT = black hole in AdS

$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$



- desired "black hole" is a hyperbolic foliation of AdS
- bulk coordinate transformation implements desired conformal transformation on boundary
- apply Wald's formula (for any gravity theory) for horizon entropy

universal contributions:

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\, a_d^* \log(2R/\delta) + \cdots \quad \text{for even d}$$

$$\cdots + (-)^{\frac{d-1}{2}} 2\pi\, a_d^* + \cdots \quad \text{for odd d}$$

Entanglement C-theorem conjecture:

(RM & Sinha)

- identify central charge with universal contribution in entanglement entropy of ground state of CFT across sphere S^{d-2} of radius R:

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, a_d^*\, \log(2R/\delta) & \textbf{for even } \boldsymbol{d} \\ (-)^{\frac{d-1}{2}}\, 2\pi\, a_d^* & \textbf{for odd } \boldsymbol{d} \end{cases}$$

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$$(a_d^*)_{UV} \geq (a_d^*)_{IR}$$

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⟶ behaviour discovered for holographic model but conjectured that result applies generally (outside of holography)

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(Jafferis, Klebanov, Pufu & Safdi)

F-theorem:

- examine partition function for broad classes of 3-dimensional quantum field theories on three-sphere (SUSY gauge theories, perturbed CFT’s & O(N) models)
- in all examples, F= – log Z(S^3)>0 and decreases along RG flows

$\longrightarrow$ **conjecture:** $F_{UV} > F_{IR}$

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- stress-energy fixed by trace anomaly – vanishes for odd d!
- upon passing to Euclidean time with period $2\pi R$:

$$S_{EE} = \log Z|_{S^d} \quad \text{for any odd d}$$

F-theorem:

- focusing on renormalized or universal contributions, eg,

$$F_0 = -\log Z\big|_{finite} = -S_{univ} = 2\pi\, a_3^* \,.$$

- generalizes to general odd d:

$$(-)^{\frac{d-1}{2}}\,\log Z\big|_{finite} = (-)^{\frac{d-1}{2}} S_{univ} = 2\pi\, a_d^* \,.$$

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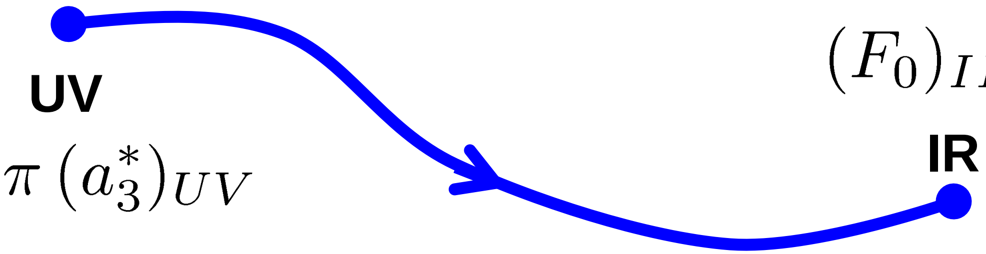
- equivalence shown only for fixed points but good enough:

$$(F_0)_{IR} = 2\pi\,(a_3^*)_{IR}$$



$$(F_0)_{UV} = 2\pi\,(a_3^*)_{UV}$$

- evidence for F-theorem (SUSY, perturbed CFT's & O(N) models) supports entanglement conjecture and our holographic analysis provides additional support for F-theorem

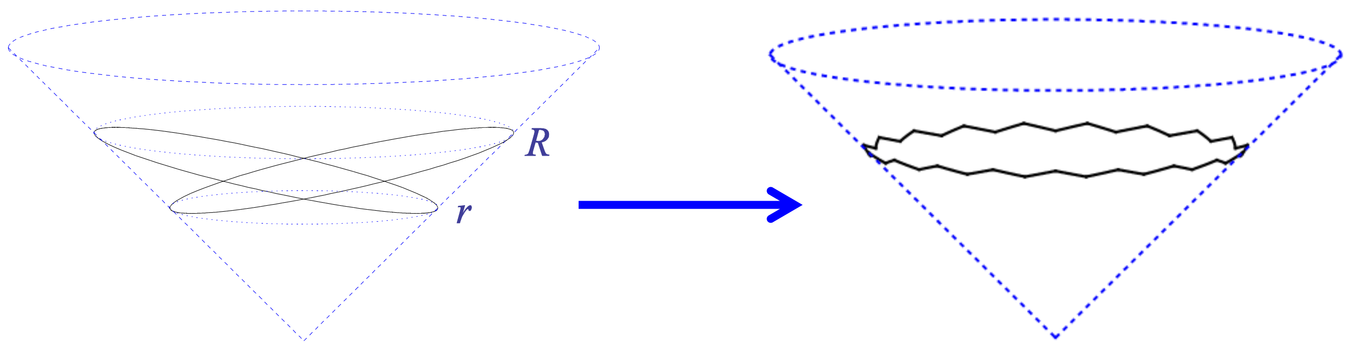
(Casini & Huerta '12)

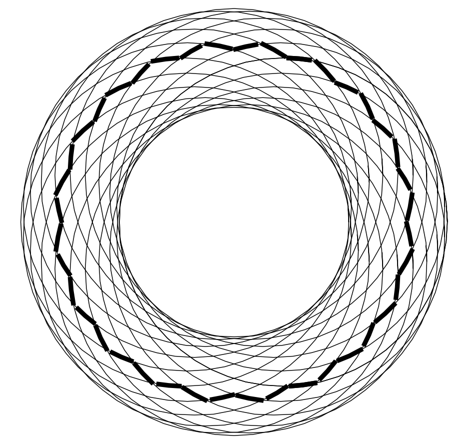
Entanglement proof of F-theorem:

- F-theorem for d=3 RG flows established using unitarity, Lorentz invariance and **strong subadditivity**

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + \ldots + S(\cap_i X_i)$$

- geometry more complex than d=2: consider many circles intersecting on **null** cone





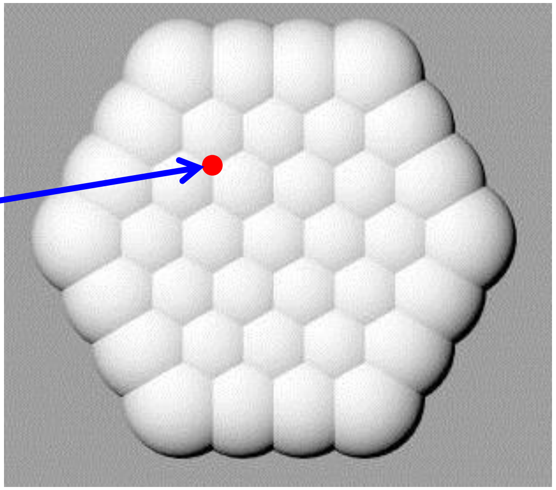
- no corner contribution from intersection in null plane
- define: $C(R) = R S'(R) - S(R)$
- for d=3 CFT: $S(R) = c_0\, R - 2\pi a_3 \longrightarrow C_{\mathrm{CFT}}(R) = 2\pi a_3$
- with SSA and "continuum" limit $\longrightarrow \partial_R C(R) \leq 0$
- hence $C(R)$ decreases monotonically and $[a_3]_{\mathrm{UV}} > [a_3]_{\mathrm{IR}}$

Beyond d=3:

- is there entropic proof of c-theorem in higher dimensions?

$\longrightarrow$ need a new idea?

higher dim. intersections lead
to subleading divergences
which trivialize SSA inequality



Beyond d=3:

(Komargodski & Schwimmer; see also: Luty, Polchinski & Rattazzi)

d=4 a-theorem and Dilaton Effective Action

- analyze RG flow as "broken conformal symmetry" (Schwimmer & Theisen)
- couple theory to "dilaton" (conformal compensator) and organize effective action in terms of $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$

  diffeo X Weyl invariant: $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu} \quad \tau \to \tau + \sigma$

- follow effective dilaton action to IR fixed point, eg,

$$S_{anomaly} = -\delta a \int d^4x \sqrt{-g} \left(\tau E_4 + 4(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) \partial_\mu \tau \partial_\nu \tau - 4(\partial\tau)^2 \Box \tau + 2(\partial\tau)^4 \right)$$

$\delta a = a_{UV} - a_{IR}$: ensures UV & IR anomalies match

- with $g \to \eta$, only contribution to 4pt amplitude with null dilatons:

$$S_{anomaly} = 2\,\delta a \int d^4x\, (\partial\tau)^4$$

- dispersion relation plus optical theorem demand: $\delta a > 0$
- no entanglement in sight?

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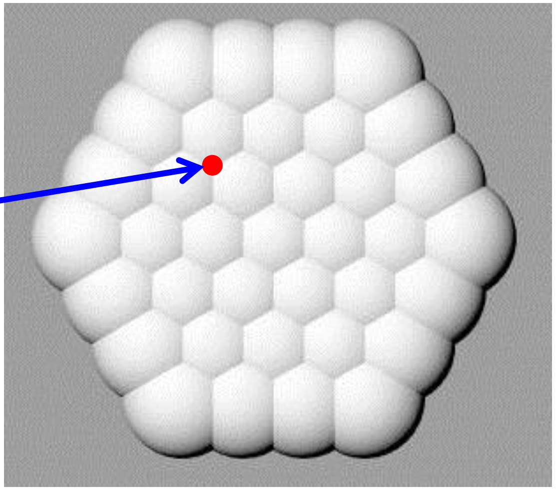

higher dim. intersections lead to subleading divergences which trivialize SSA inequality

• d=4 a-theorem proved with more "standard" QFT techniques

(Komargodski & Schwimmer)

• hybrid approach proposed? (Solodukhin): needs work

• can c-theorems be proved for higher dimensions? eg, d=5 or 6

→ again, entropic approach needs a new idea

→ dilaton-effective-action approach requires refinement for d=6

(Elvang, Freedman, Hung, Kiermaier, RM & Theisen; Elvang & Olson)

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Why is universal term in S_{EE} universal?

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1}\, 4\, a_d^*\, \log(2R/\delta) & \textbf{for even } \boldsymbol{d} \\ (-)^{\frac{d-1}{2}}\, 2\pi\, a_d^* & \textbf{for odd } \boldsymbol{d} \end{cases}$$

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recall: d=2 CFT

$$S_{uni} = \frac{\mathbf{c}}{3} \log\left(\frac{C}{\pi\,\delta} \sin\frac{\pi\ell}{C}\right)$$

(Calabrese & Cardy)

(Holzhey, Larsen & Wilczek)

d=4 CFT

(Solodukhin)

$$S_{uni} = \log(R/\delta)\, \frac{1}{2\pi} \int_\Sigma d^2x \sqrt{h}\, \left[\mathbf{c}\left(C^{ijkl}\, \tilde{g}^{\perp}_{ik}\, \tilde{g}^{\perp}_{jl} - K^{i\,b}_{a} K^{i\,a}_{b} + \frac{1}{2} K^{i\,a}_{a} K^{i\,b}_{b}\right) - \mathbf{a}\, \mathcal{R}\right]$$

d=2m CFT (with symmetry)

(RCM & Sinha)

$$S_{uni} = \log\,(R/\delta)\;2\pi \int_\Sigma d^{d-2}x\, \sqrt{h}\, \frac{\partial\langle T_\lambda{}^\lambda\rangle}{\partial R^{\mu\nu}{}_{\rho\sigma}}\, \hat{\varepsilon}^{\,\mu\nu}\, \hat{\varepsilon}_{\rho\sigma}$$

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(Hertzberg & Wilczek)

see Banerjee talk

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(Hertzberg & Wilczek)

- in regulators, tension between Lorentz inv. and unitarity

$\longrightarrow$ latter emerge in $\delta \to 0$ limit, but regulator exposed in S_{EE}

“Renormalized” Entanglement Entropy: (Liu & Mezei)

- divergences determined by local geometry of entangling surface with covariant regulator, eg,

$$S_{CFT} = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots + (-)^{\frac{d-1}{2}} \, 2\pi \, a_d + O(\delta/R)$$

- can isolate finite term with appropriate manipulations, eg,

d=3: $\mathcal{S}_3(R) = R S'(R) - S(R)$

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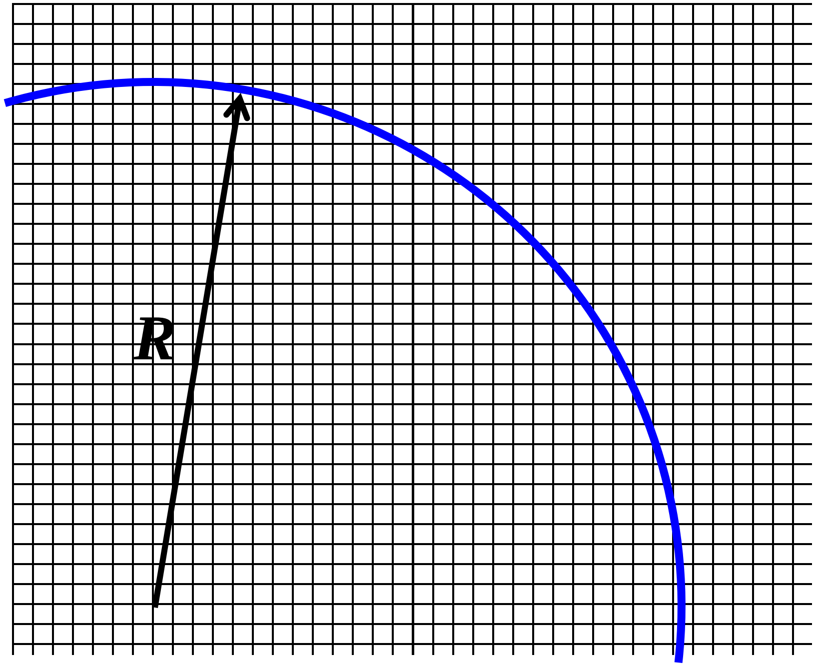
- approach demands special class of regulators: "covariant"

⟶ is result artifact of choosing "nice" regulator??

- if a_d^* is physical, we should be able to use any regularization which defines the continuum QFT

- consider defining a_3 in presence of lattice regulator

$$d = 3: \; S(R) = c_0 \frac{2\pi R}{\delta} - 2\pi a_3$$



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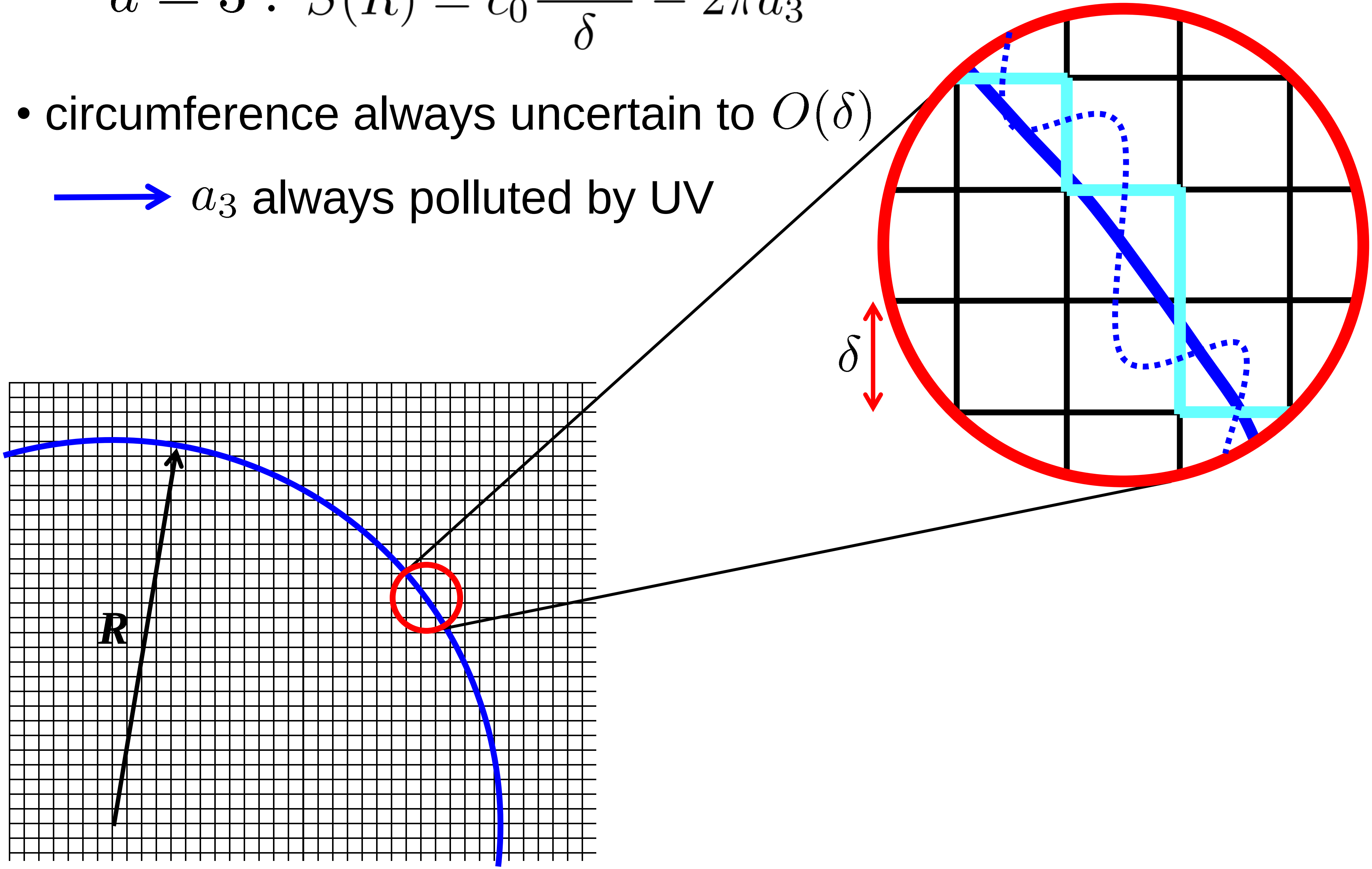
→ a_3 always polluted by UV



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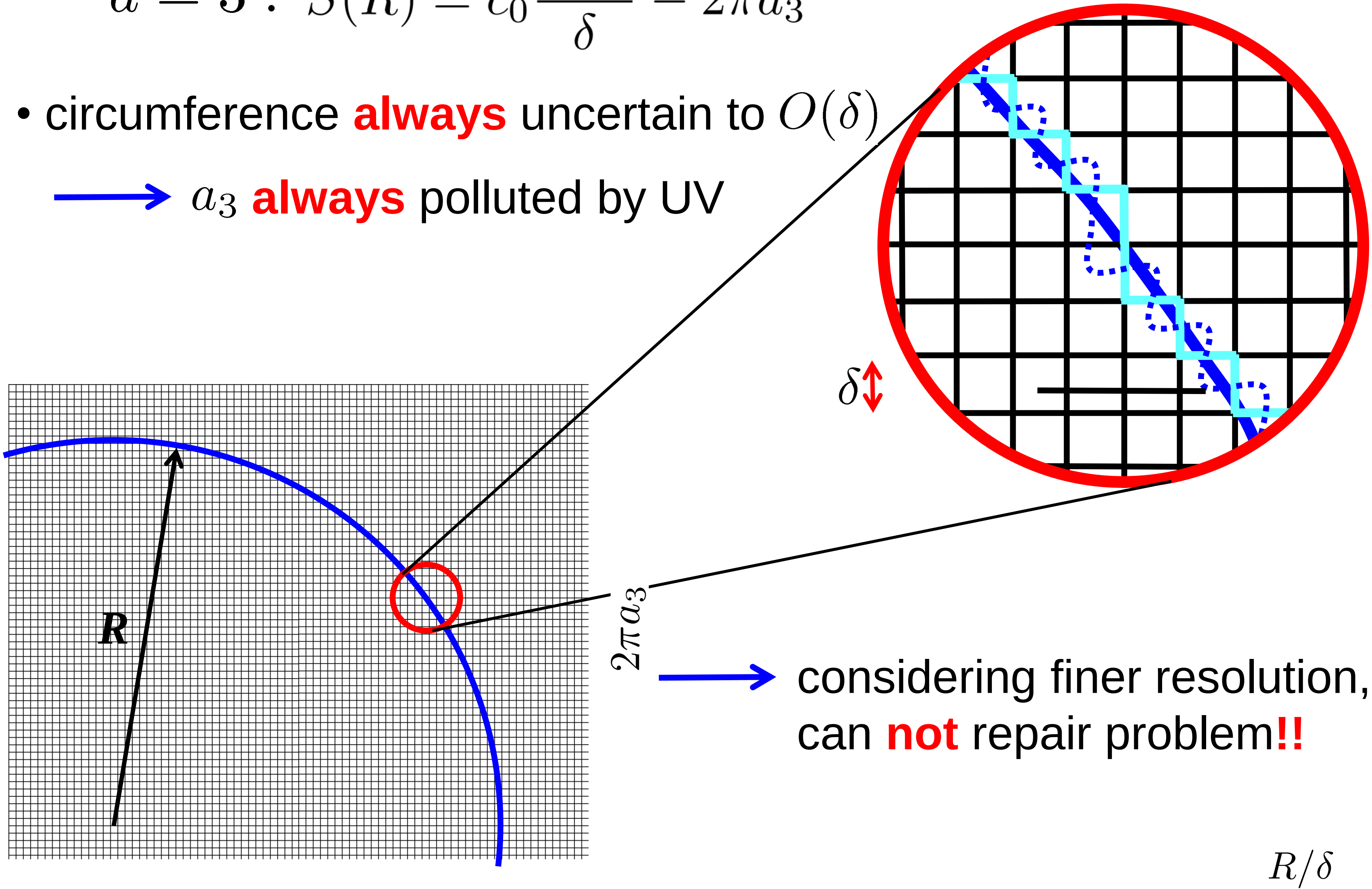

$\longrightarrow$ considering finer resolution, can **not** repair problem**!!**

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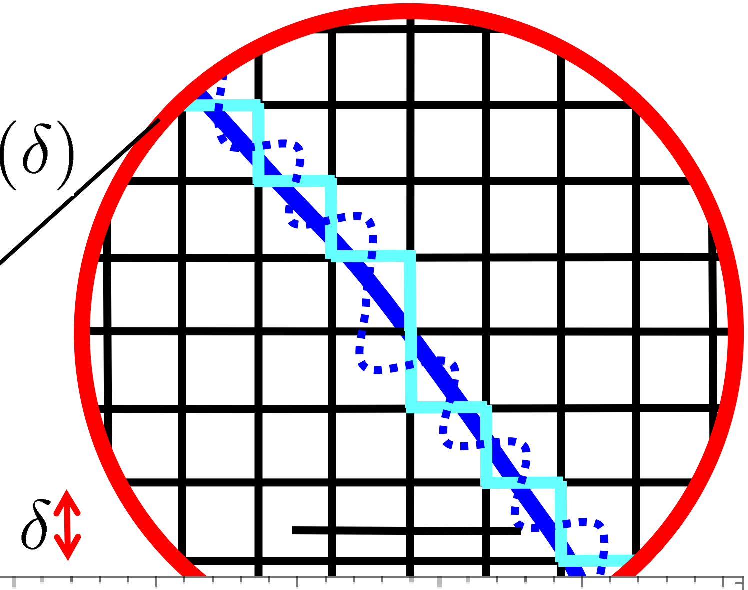

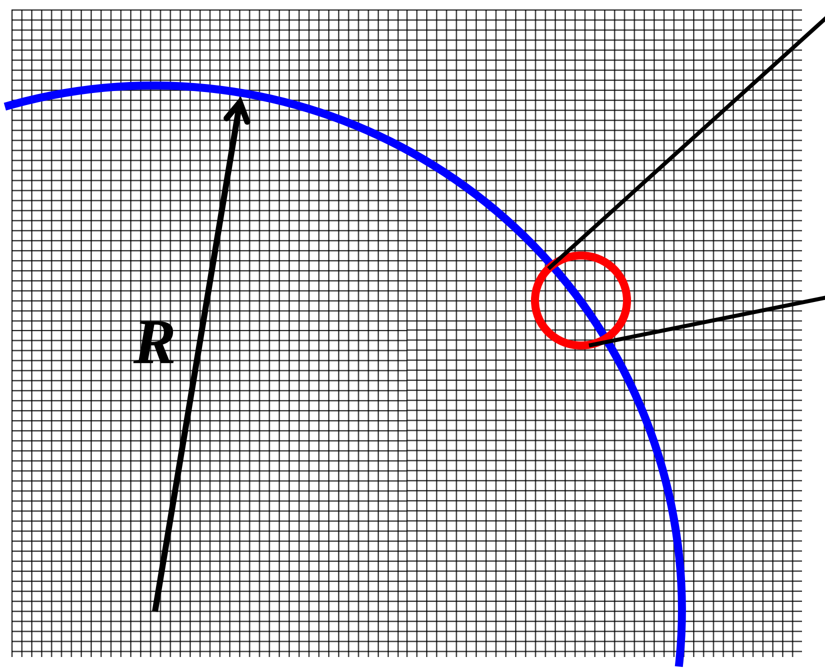

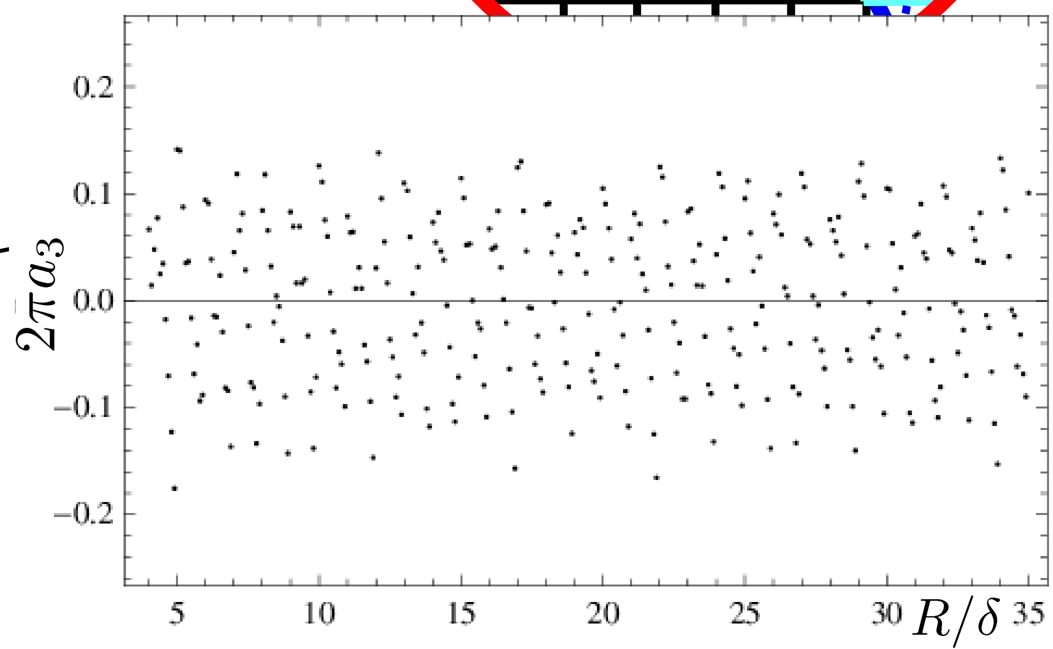

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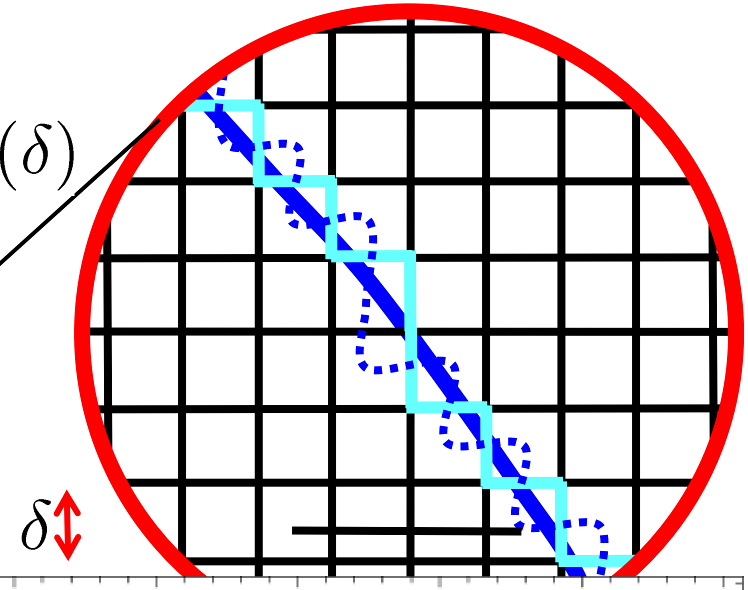

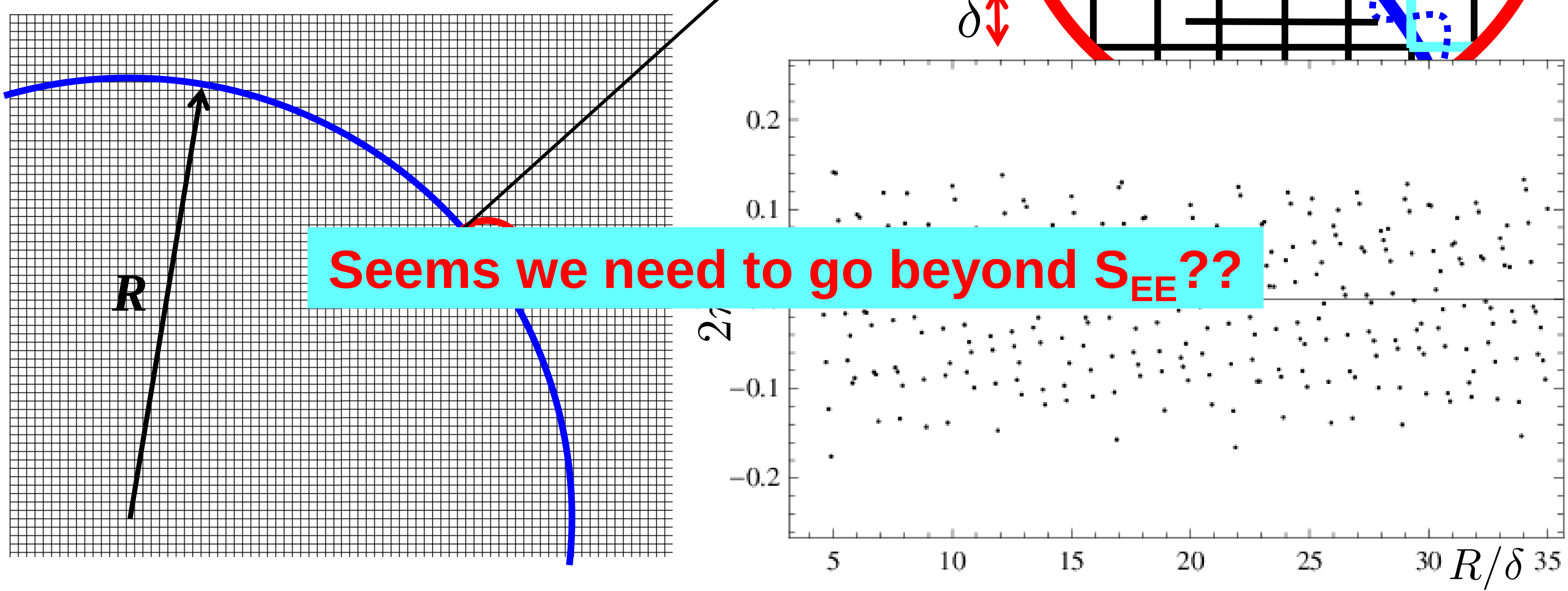

Seems we need to go beyond S_{EE}??

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1. C-function must be dimensionless, well-defined quantity, which is independent of the regularization scheme

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- S_{EE} seems to fail to satisfy criteria 1 & 2
- **alternative choice??**

Mutual Information:

- another measure of entanglement between two systems
- for non-intersecting regions A and B:

$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- can be defined without reference to S_{EE} (Araki; Narnhofer)
- bounds correlators between A and B (Wolf, Verstraete, Hastings & Cirac)

$$I(A,B) \geq \frac{|\langle \mathcal{O}_A \, \mathcal{O}_B \rangle_c|^2}{2\|\mathcal{O}_A\|^2 \, \|\mathcal{O}_B\|^2}$$

- **finite!** UV divergences in S(A) and S(B) canceled by S(A U B)

Mutual Information:

- another measure of entanglement between two systems
- for non-intersecting regions A and B:

$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- can be defined without reference to S_{EE} (Araki; Narnhofer)
- bounds correlators between A and B (Wolf, Verstraete, Hastings & Cirac)

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- **finite!** UV divergences in S(A) and S(B) canceled by S(A U B)
- if c-function defined with mutual information

$\longrightarrow$ criterion 1 will automatically be satisfied

$\longrightarrow$ criterion 2 will be satisfied with further care

C-function from Mutual Information:

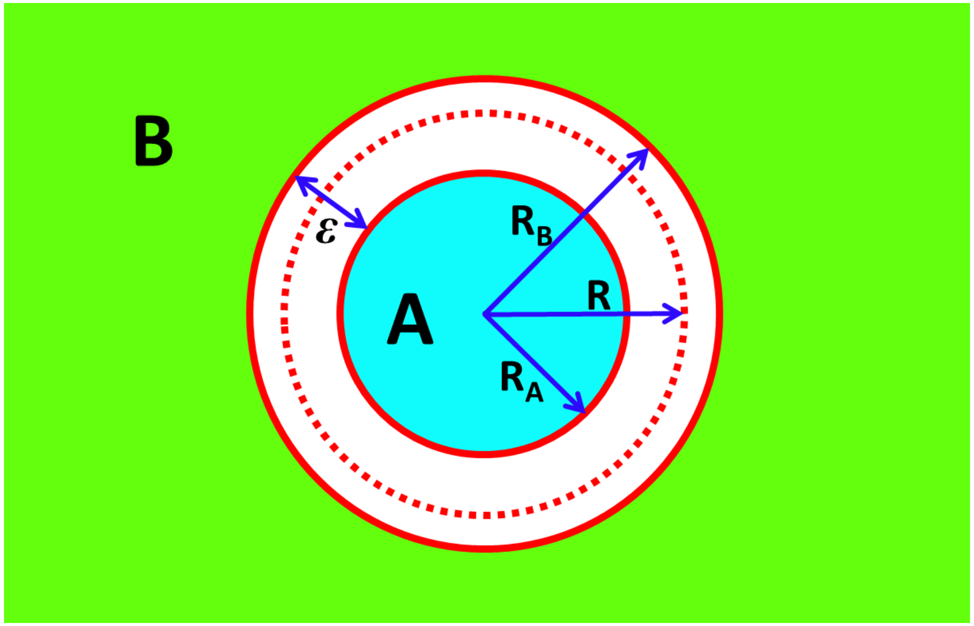
$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- consider following geometry:

$$R_A = R - \left(\frac{1}{2} - \alpha\right)\varepsilon$$

$$R_B = R + \left(\frac{1}{2} + \alpha\right)\varepsilon$$

or $$R = \frac{R_A + R_B}{2} - \alpha\,\varepsilon$$



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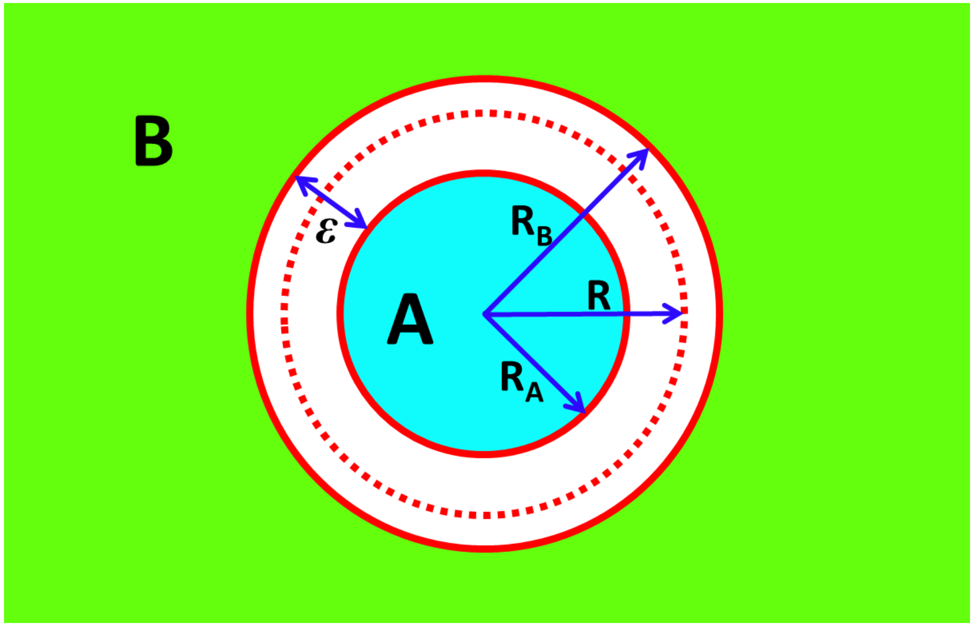

- consider regime: $R \gg \varepsilon \gg \delta$ (R and ε are macro scales)

- mutual information takes form:

$$I(A,B) = 4\pi R\left(\frac{\tilde{a}}{\varepsilon} + \tilde{b}\right) - 4\pi\,\tilde{c}_0 + O(\varepsilon/R)$$

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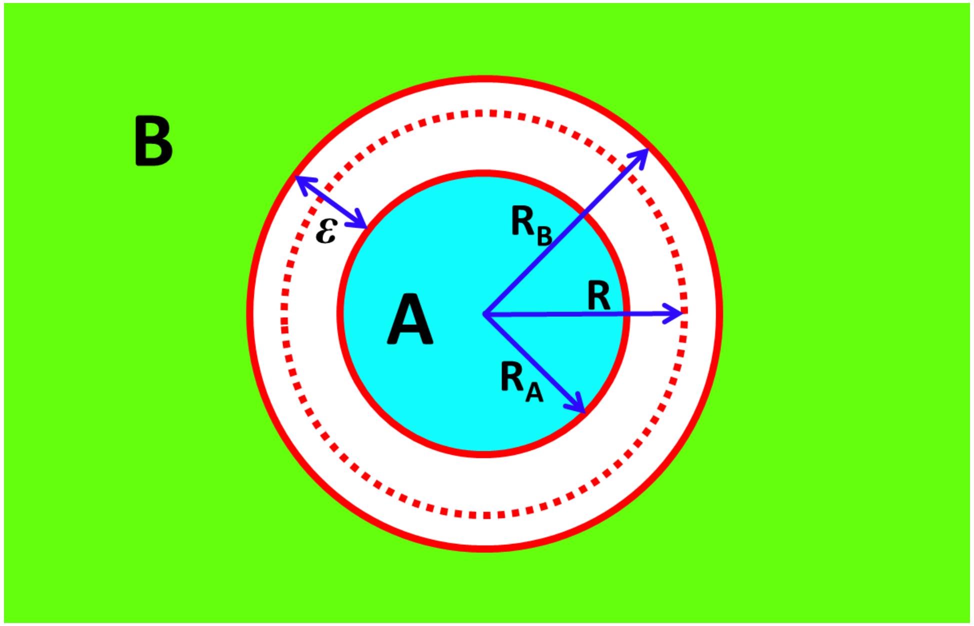

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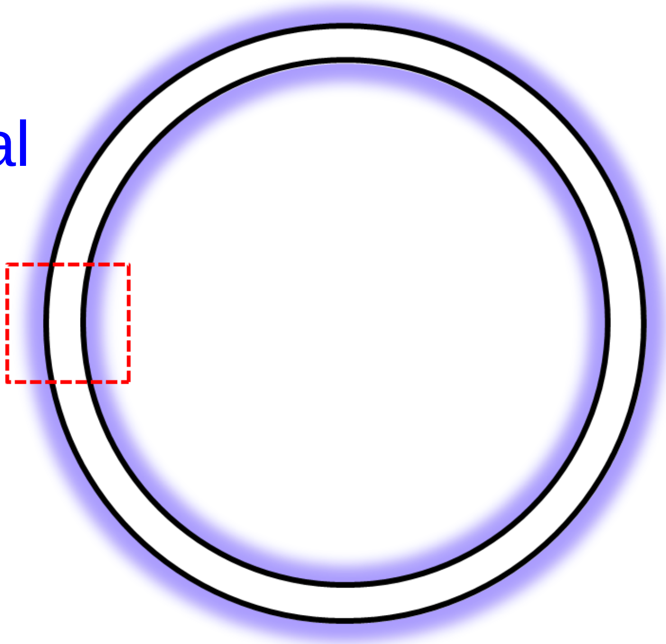
- ambiguity: $\alpha \to \alpha' = \alpha + \delta\alpha\,, \quad \tilde{c}_0 \to \tilde{c}_0' = \tilde{c}_0 + \tilde{a}\,\delta\alpha$

UV independence of $\tilde{c}_0$:

- can we choose α such that $\tilde{c}_0$ is independent of higher scales?
- consider sitting at critical point with *m*, lowest mass scale in RG flow: $R \gg \varepsilon \sim 1/m$
- correlations near boundary **non**conformal
- high energy contribution to I(A,B): "local"

$$I(A,B)_{HE} = 2\pi R \left(\sigma_0 + \frac{\sigma_1}{R} + \frac{\sigma_2}{R^2} + \cdots \right)$$



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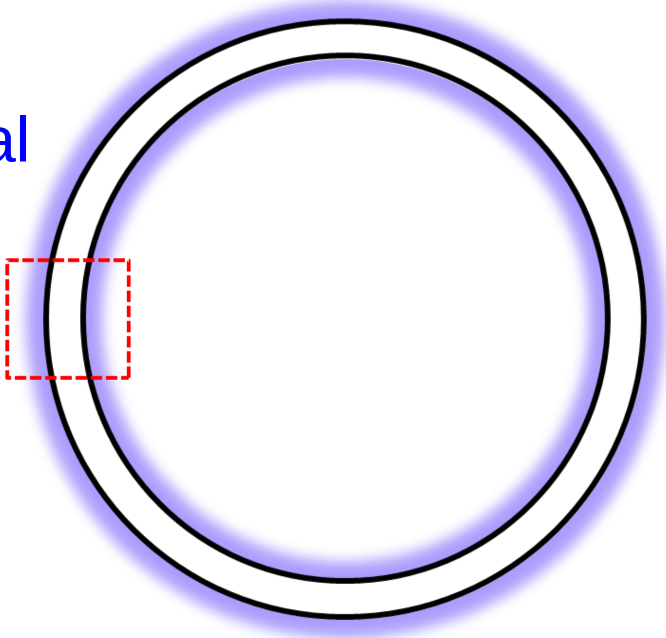

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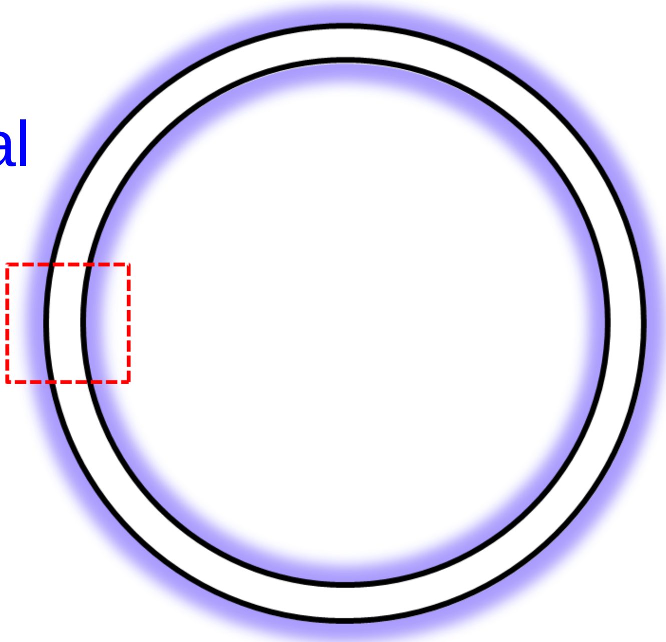

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$$I(A,B)_{HE} = \int ds\ \left(\sigma_0 - \sigma_1\,\mathbf{n}\cdot\partial_s\mathbf{t} - \sigma_2\,\mathbf{t}\cdot\partial_s^2\mathbf{t} + \cdots\right)$$

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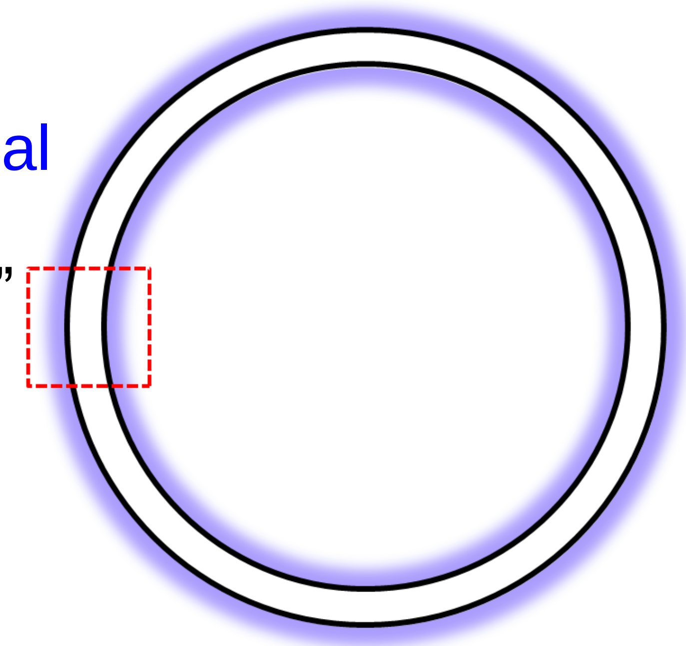

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- σ_1 must vanish if reflection symmetry $\longrightarrow$ $\alpha = 0$

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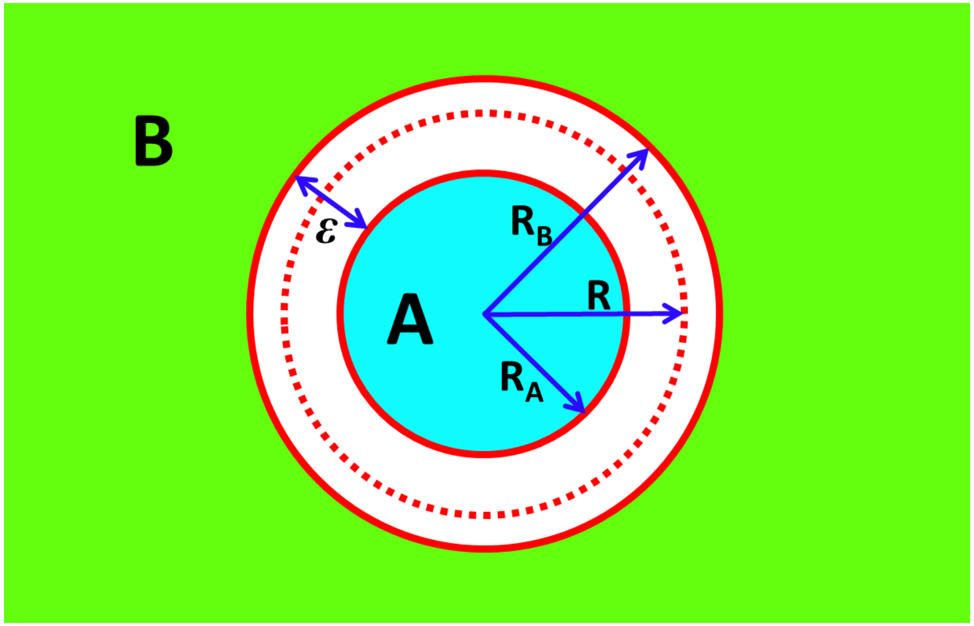

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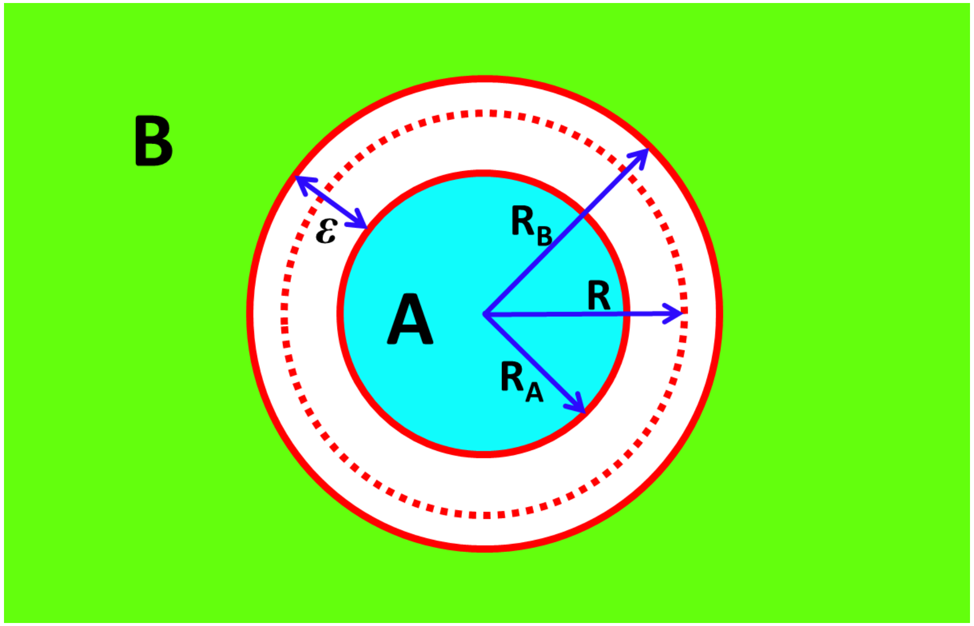

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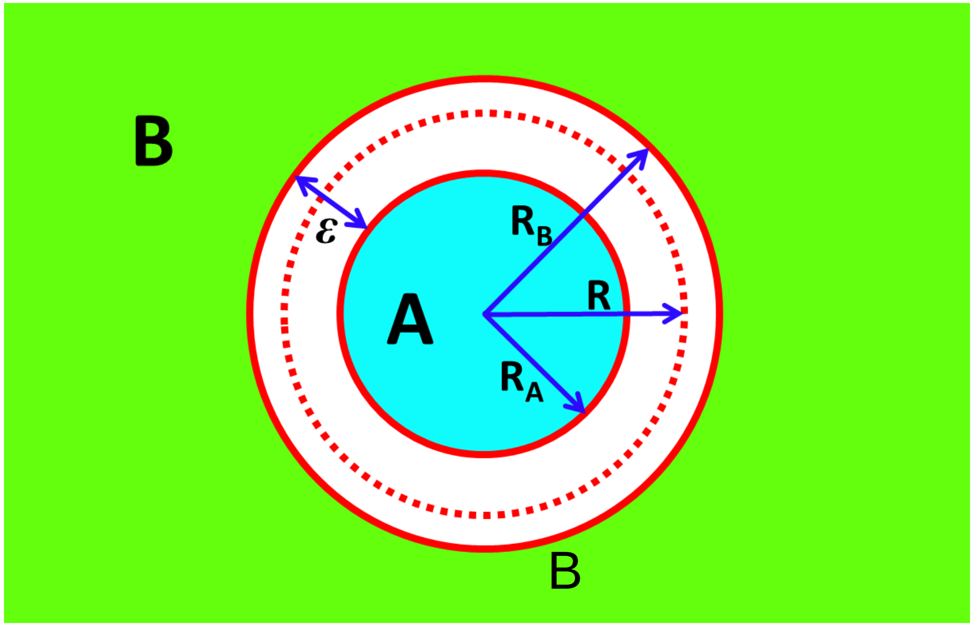
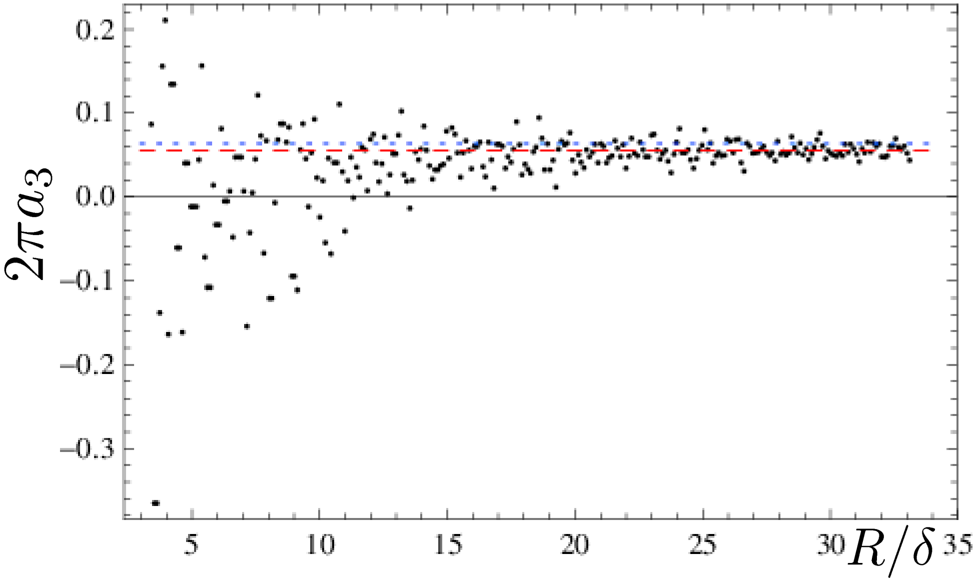


- in regime: $R \gg \varepsilon \gg \delta$

- calculate for a free scalar on a square lattice:



(with $\varepsilon/R = 2/11$, result good to 15%)

Criteria to properly establish c-theorem:

1. C-function must be dimensionless, well-defined quantity, which is independent of the regularization scheme

   → computable with any regulator

2. C-function must be intrinsic to fixed point of interest

   → Independent of details of RG flows

3. C-function must decrease monotonically along any RG flows connecting a UV fixed point to an IR fixed point

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- monotonic flow follows as in entropic proof of F-theorem

  $\longrightarrow$ have properly established F-theorem in d=3

Conclusions and Questions:

- entanglement lends new insights into c-theorems
- using mutual information, properly established d=3 F-theorem
- how much of Zamolodchikov's structure for d=2 RG flows extends higher dimensions?

→ d=3 entropic C-function not always stationary at fixed points

(Klebanov, Nishioka, Pufu & Safdi)

→ same already observed for d=2; special case or generic? need a better C-function?

Zamolodchikov c-theorem (1986):

- renormalization-group (RG) flows can seen as one-parameter motion

$$\frac{d}{dt} \equiv -\beta^i(g)\,\frac{\partial}{\partial g^i}$$

in the space of (renormalized) coupling constants $\{g^i,\ i = 1, 2, 3, \cdots\}$ with beta-functions as “velocities”

- for unitary, Lorentz-inv. QFT’s in two dimensions, there exists a positive-definite real function of the coupling constants $C(g)$:

1. monotonically decreasing along flows: $\frac{d}{dt}C(g) \leq 0$
2. “stationary” at fixed points $g^i = (g^*)^i$:

$$\beta^i(g^*) = 0 \longleftrightarrow \frac{\partial}{\partial g^i}C(g) = 0$$

3. at fixed points, it equals central charge of corresponding CFT

$$C(g^*) = c$$

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- does scale invariance imply conformal invariance beyond d=2?

$\longrightarrow$ "more or less" in d=4

(Luty, Polchinski & Rattazzi; Dymarsky, Komargodski, Schwimmer & Theisen)

- further lessons for RG flows and entanglement from holography?

$\longrightarrow$ translation of "null energy condition" to boundary theory?

- what can entanglement/quantum information really say about RG flows, holography or nonperturbative QFT?