Holographic Entanglement Entropy

Chandrasekhar Lecture 2
ICTS, December 11, 2014

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

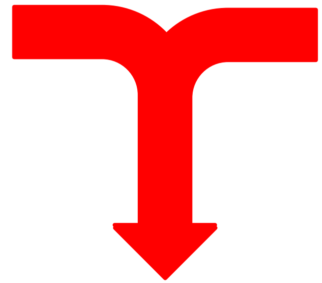
Entanglement Entropy

- condensed matter
- quantum information
- black hole microphysics

AdS/CFT correspondence

(gauge/gravity duality)

- string theory
- quantum gravity



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- proposal by Ryu & Takayanagi (2006)

(New Horizons Prize, 2014)

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Message:

- holographic entanglement entropy is part of a dialogue between string theory and a variety of fields (eg, condensed matter)
- offers potential to learn new lessons both about entanglement in quantum field theory and about quantum gravity

Overview:

1) Entanglement entropy in QFT

2) Primer on AdS/CFT correspondence

3) Introduction to Holographic Entanglement Entropy

4) Derivation of Holographic Entanglement Entropy
(for spherical entangling surfaces)

5) Conclusions

Quantum Entanglement

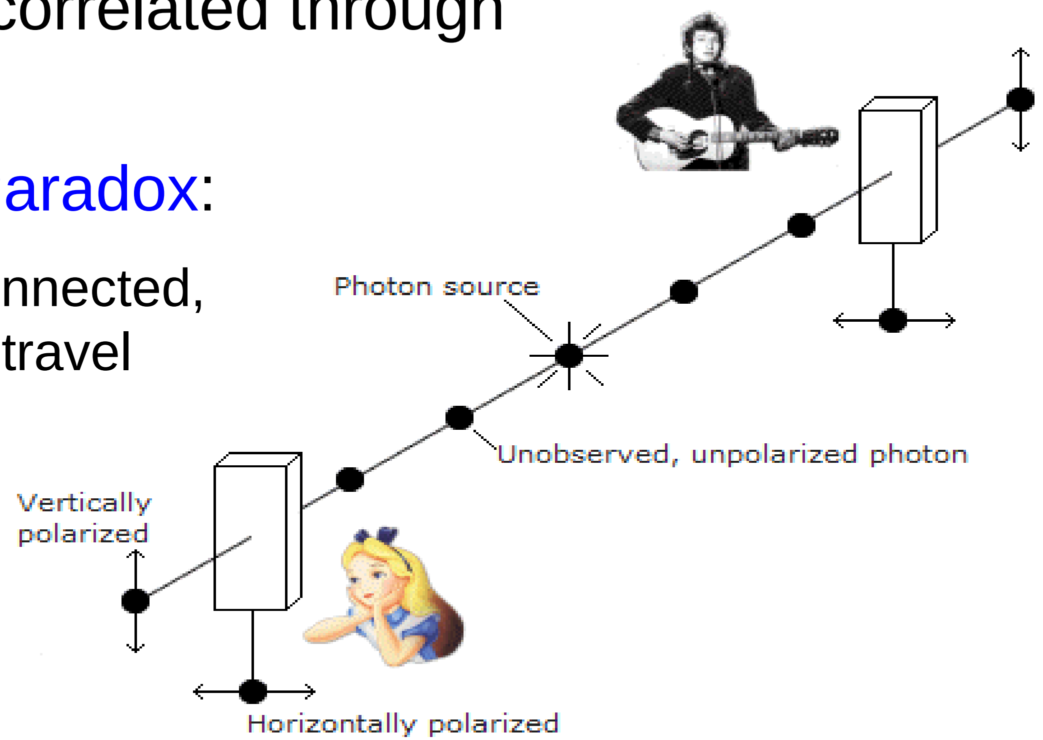
• different subsystems are correlated through global state of full system

Einstein-Podolsky-Rosen Paradox:

• properties of pair of photons connected, no matter how far apart they travel

"*spukhafte Fernwirkung*" = spooky action at a distance

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \Big)$$



Quantum Information: entanglement becomes a resource for (ultra)fast computations and (ultra)secure communications

Condensed Matter: key to "exotic" phases and phenomena, e.g., quantum Hall fluids, unconventional superconductors, quantum spin fluids, ...

Entanglement Entropy

- general diagnostic: divide quantum system into two parts and use entropy as measure of correlations between subsystems
- procedure:
  - divide system into two subsystems, eg, A and B
  - integrate out degrees of freedom in subsystem B
  - remaining dof in A are described by a density matrix ρ_A
  - calculate von Neumann entropy: $S_{EE} = -Tr\,[\rho_A\, \log \rho_A]$
- simple examples found with spin systems:

$$\psi = \alpha\,|\uparrow\downarrow\rangle + \beta\,|\downarrow\uparrow\rangle \qquad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1$$

- “trace” over 2nd spin: $\rho_A = |\alpha|^2 |\uparrow\rangle\langle\uparrow| + |\beta|^2 |\downarrow\rangle\langle\downarrow|$

$$\longrightarrow \quad S_{EE} = -|\alpha|^2 \log|\alpha^2| - |\beta|^2 \log|\beta^2|$$

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$$\begin{aligned}\rho_A \;&=\; |\alpha|^2|\uparrow\rangle\langle\uparrow| + |\beta|^2|\downarrow\rangle\langle\downarrow| \\ &\quad + \alpha\beta^\dagger|\uparrow\rangle\langle\downarrow| + \beta\alpha^\dagger|\downarrow\rangle\langle\uparrow|\end{aligned}$$

$$\longrightarrow \quad S_{EE} = -\sum \lambda_n\, \log \lambda_n = 0 \qquad (\,\lambda_1 = 0\,,\;\lambda_2 = 1\,)$$

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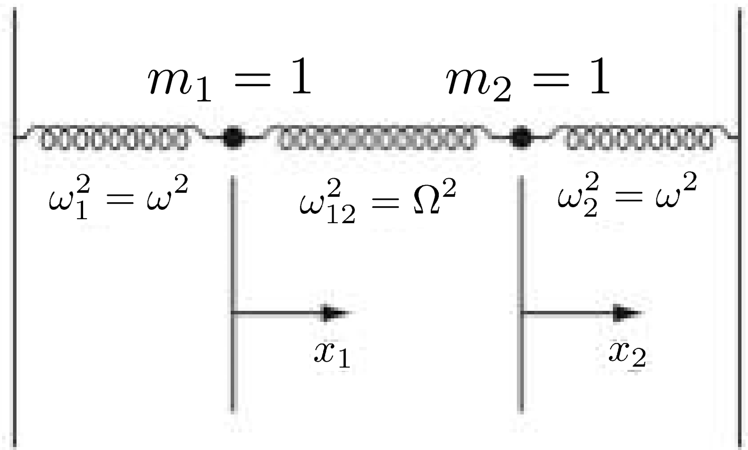
(Srednicki: hep-th/9303048)

Entanglement Entropy

- another example: two coupled harmonic oscillators

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + \omega^2\left(x_1^2 + x_2^2\right) + \Omega^2\left(x_1 - x_2\right)^2\right]$$



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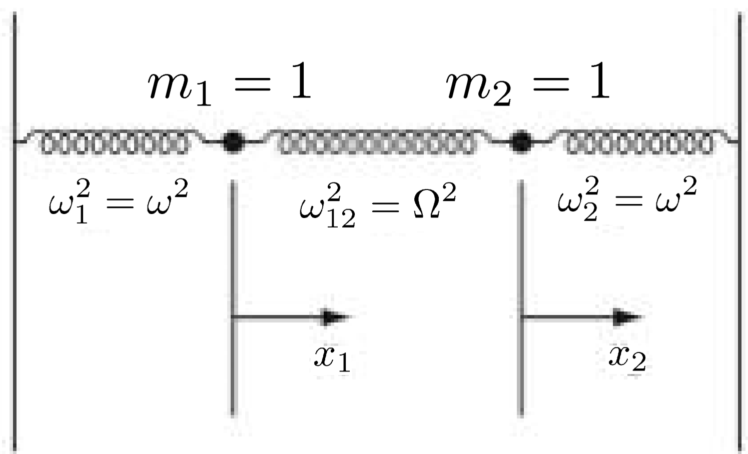
&= \frac{1}{2}\left[p_+^2 + \omega_+^2 x_+^2 + p_-^2 + \omega_-^2 x_-^2\right] \quad \text{with } x_\pm = \frac{1}{\sqrt{2}}\left(x_1 \pm x_2\right) , \\
& \qquad \omega_+^2 = \omega^2 , \quad \omega_- = \omega^2 + 2\Omega^2
\end{aligned}
$$

find normal modes; problem reduces to two independent SHO's



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- ground-state wave-function:

$$\Psi_0(x_+, x_-) = \Psi_0(x_+)\,\Psi_0(x_-) = \frac{1}{\sqrt{\pi}(\omega_+\omega_-)^{1/4}}\,\exp\left[-\frac{1}{4}\left(\omega_+ x_+^2 + \omega_- x_-^2\right)\right]$$

$$\Psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}(\omega_+\omega_-)^{1/4}}\,\exp\left[-\frac{1}{8}\left(\omega_+\left(x_1 + x_2\right)^2 + \omega_-\left(x_1 - x_2\right)^2\right)\right]$$

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- trace over position of oscillator 1:

$$\begin{aligned} \rho(x_2, x_2') &= \int_{-\infty}^{+\infty} dx_1\ \Psi_0(x_1, x_2)\,\Psi_0^\dagger(x_1, x_2') \\ &= \frac{(\gamma - \beta)^{1/2}}{\sqrt{\pi}}\exp\left[-\frac{\gamma}{2}\left(x_2^2 + x_2'^2\right) + \beta\, x_2 x_2'\right] \end{aligned}$$

with $\gamma = \frac{1}{4}(\omega_+ + \omega_-) + \frac{\omega_+\omega_-}{\omega_+ + \omega_-}$ and $\beta = \frac{1}{4}(\omega_+ + \omega_-) - \frac{\omega_+\omega_-}{\omega_+ + \omega_-}$

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$\Omega/\omega \ll 1$ $\quad \sim \omega$ $\quad \sim \Omega^4/8\omega^3$

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entanglement

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$\longrightarrow$ $S_{EE} = -\sum \lambda_n \log \lambda_n$ where $\int_{-\infty}^{+\infty} dx'\, \rho(x, x')\, f_n(x') = \lambda_n\, f_n(x)$

- find: $\lambda_n = (1-\xi)\,\xi^n$ with $f_n(x) = H_n(\alpha^{1/2} x)\, \exp\left[-\alpha\, x^2/2\right]$,

$\xi = \beta/(\gamma+\alpha)$ and $\alpha = (\omega_+\omega_-)^{1/2}$

$$\longrightarrow \quad S_{EE} = -\log(1-\xi) - \frac{\xi}{1-\xi}\log \xi$$

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- notice: as $\Omega/\omega \to 0$, $\xi \sim (\Omega/2\omega)^4 \to 0$ and $S_{EE} \sim \dfrac{1}{4}\left(\dfrac{\Omega}{\omega}\right)^4 \log\left(2\omega/\Omega\right)$

$\to 0$

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- note entanglement arises because I insisted on working with original basis x_1, x_2 ⟶ guided by physical narrative

- no entanglement in normal mode basis x_+, x_-

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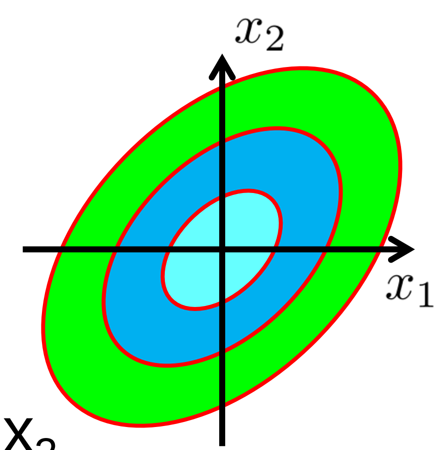
- no entanglement in normal mode basis x_+, x_-

- can interprete eq's as two-dimensional SHO

$$V(x_1, x_2) = \frac{1}{2}\omega^2\left(x_1^2 + x_2^2\right) + \frac{1}{2}\Omega^2\left(x_1 - x_2\right)^2$$

⟶ x_1 and x_2 take less precedence

⟶ $S_{EE} \neq 0$ integrating out any axis except $x_1 = \pm x_2$



Entanglement Entropy 2:

Quantum Field Theory??

- in the context of holographic entanglement entropy, S_{EE} is applied in the context of quantum field theory

- in QFT, typically introduce a (smooth) boundary or entangling surface Σ which divides the space into two separate regions
- integrate out degrees of freedom in “outside” region
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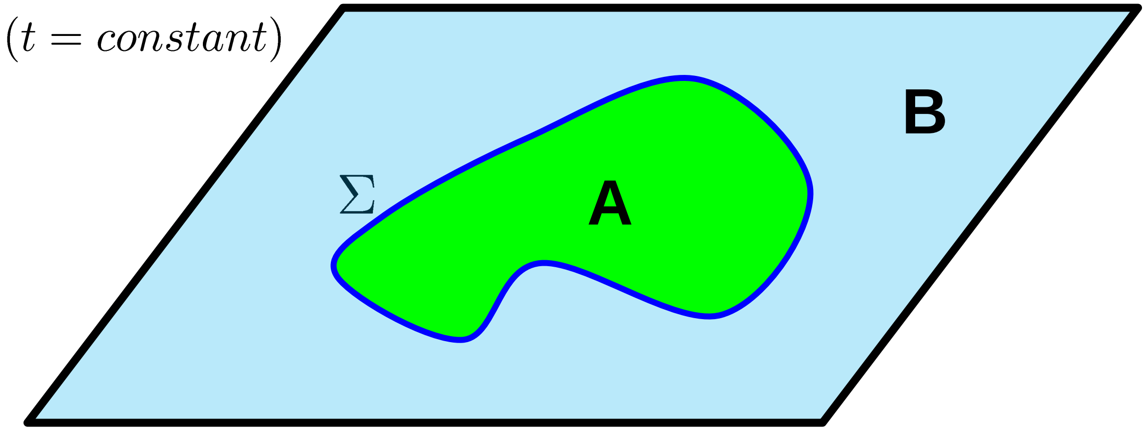
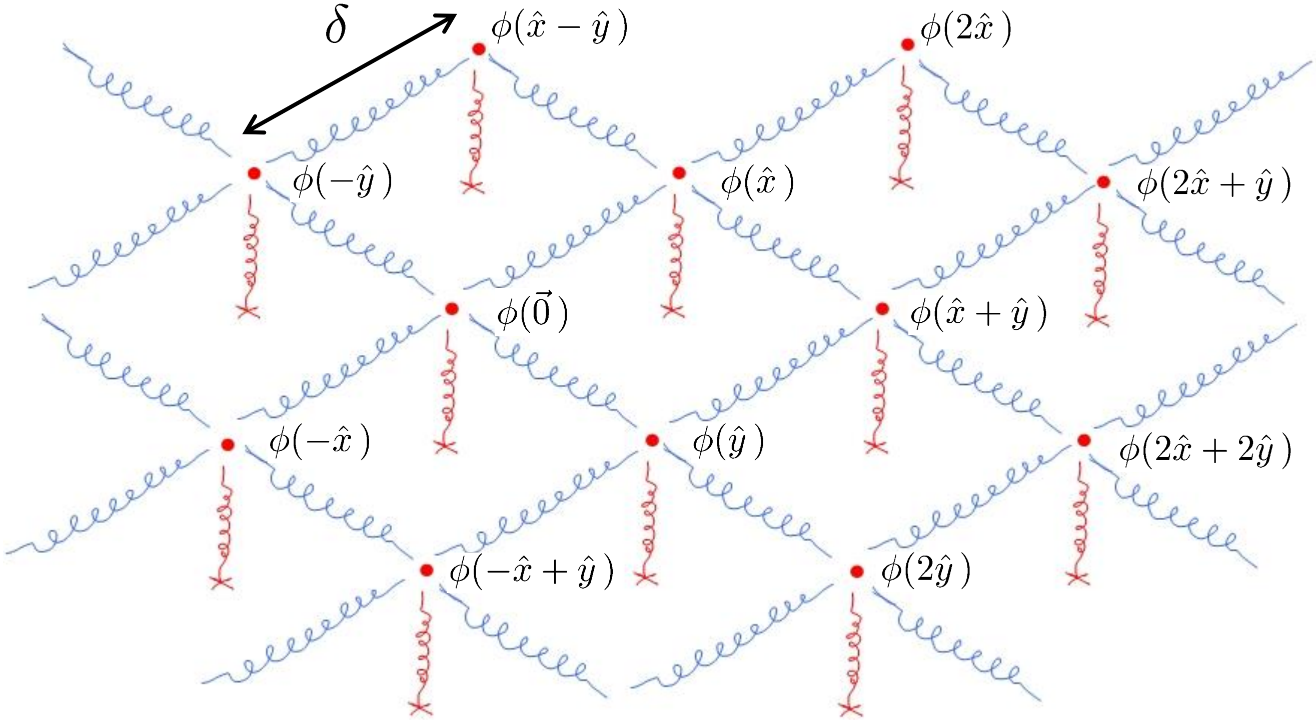


• free scalar field (in d dimensions) = many coupled oscillators

$$H = \frac{1}{2}\sum_{\vec{n}}\left[\frac{p(\vec{n}\,)^2}{\delta^{d-1}} + \delta^{d-1}\left(\frac{1}{\delta^2}\sum_i(\phi(\vec{n}\,) - \phi(\vec{n} - \hat{x}_i\,))^2 + \mu^2\phi(\vec{n}\,)^2\right)\right]$$



with $\omega^2 = \mu^2\,, \quad \Omega^2 = 1/\delta^2$ and reinstated $m = \delta^{d-1}$.

→ physical narrative: have a lattice of oscillators with

δ = lattice spacing = short distance cut-off

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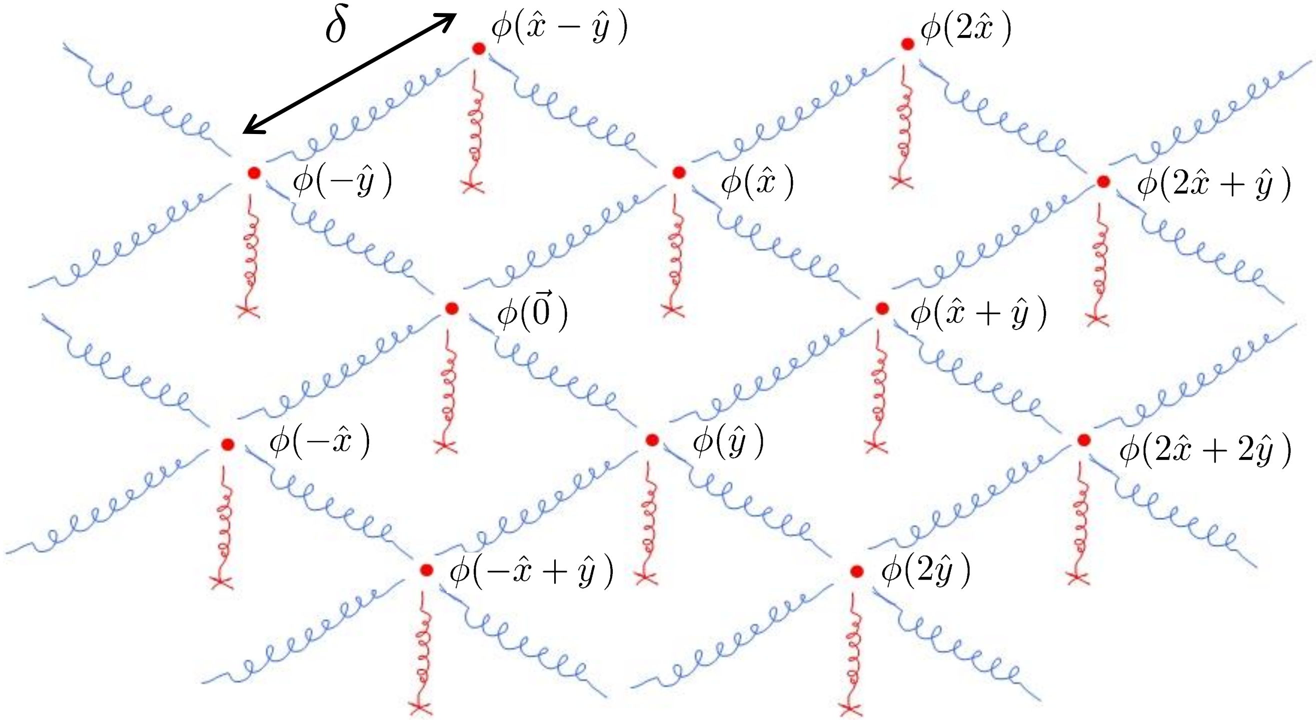

- consider continuum limit: $\delta \to 0$

$$\phi(\vec{n}) \to \phi(\vec{x})\,, \quad p(\vec{n})/\delta^{d-1} \to \pi(\vec{x})\,, \quad \delta^{d-1}\sum_{\vec{\ }} \to \int d^{d-1}x$$

$$H = \frac{1}{2}\int d^{d-1}x\left[|\pi(x)|^2 + |\vec{\nabla}\phi(x)|^2 + \mu^2|\phi(x)|^2\right]$$

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&= \frac{1}{2}\int d^{d-1}k \left[|\pi(k)|^2 + k^2|\phi(k)|^2 + \mu^2|\phi(k)|^2\right]
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$$\phi(\vec{k}) = \int d^{d-1}x \; e^{i\vec{k}\cdot\vec{x}} \; \phi(x)$$

- normal modes given by Fourier transform
- much of “textbook QFT” is perturbing lots of coupled SHO’s

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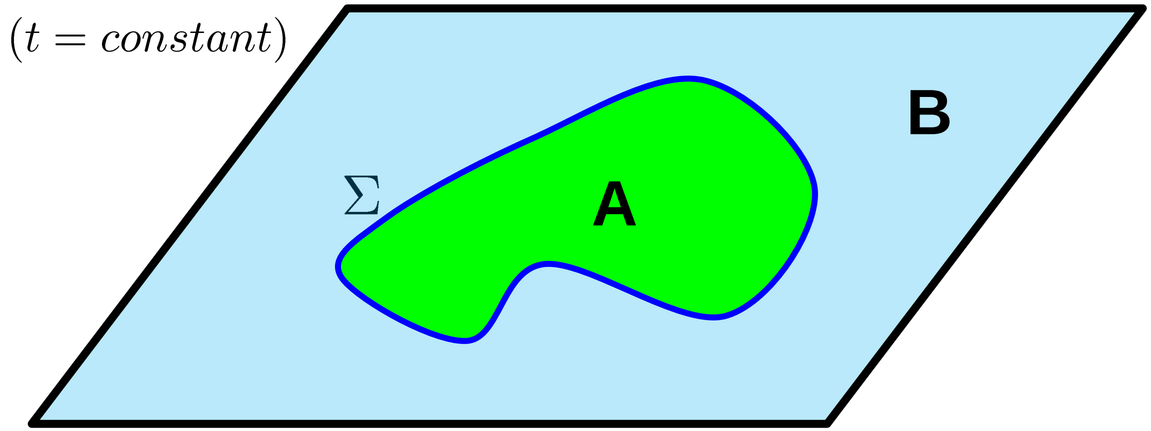

- proposed entanglement calculation done in position basis

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Ground state wave-functional:

$$\Psi_0(\phi(k)) \sim \exp\left[-\frac{1}{2}\int \frac{d^{d-1}k}{(2\pi)^{d-1}}\sqrt{k^2+\mu^2}\phi^\dagger(k)\phi(k)\right]$$

- simple “product” in terms of normal modes

- free scalar field (in d dimensions) = many coupled oscillators

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Ground state wave-functional:

$$\Psi_0(\phi(x)) \sim \exp\left[-\frac{1}{2}\int d^{d-1}x_1\int d^{d-1}x_2\,\phi(x_1)\,W(x_1,x_2)\,\phi(x_2)\right]$$

with $W(x_1,x_2) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}}\sqrt{k^2+\mu^2}\,e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}$

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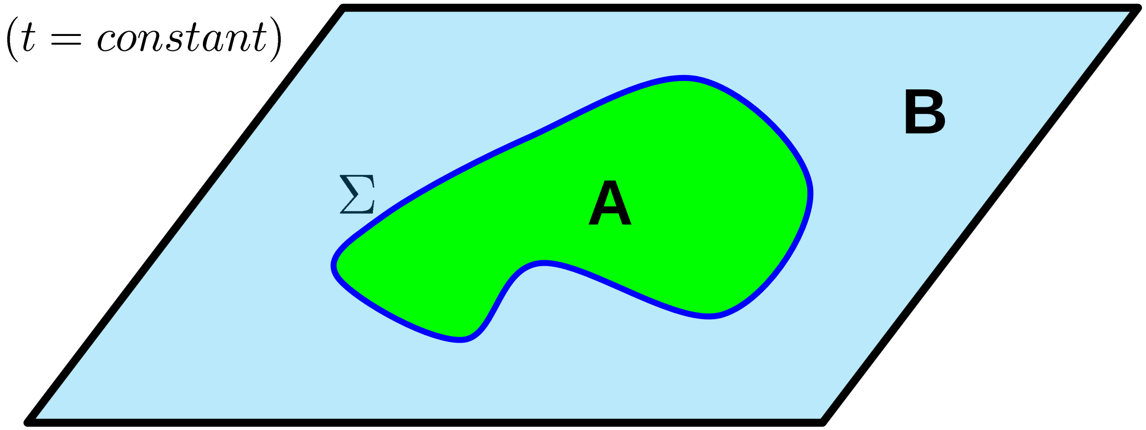

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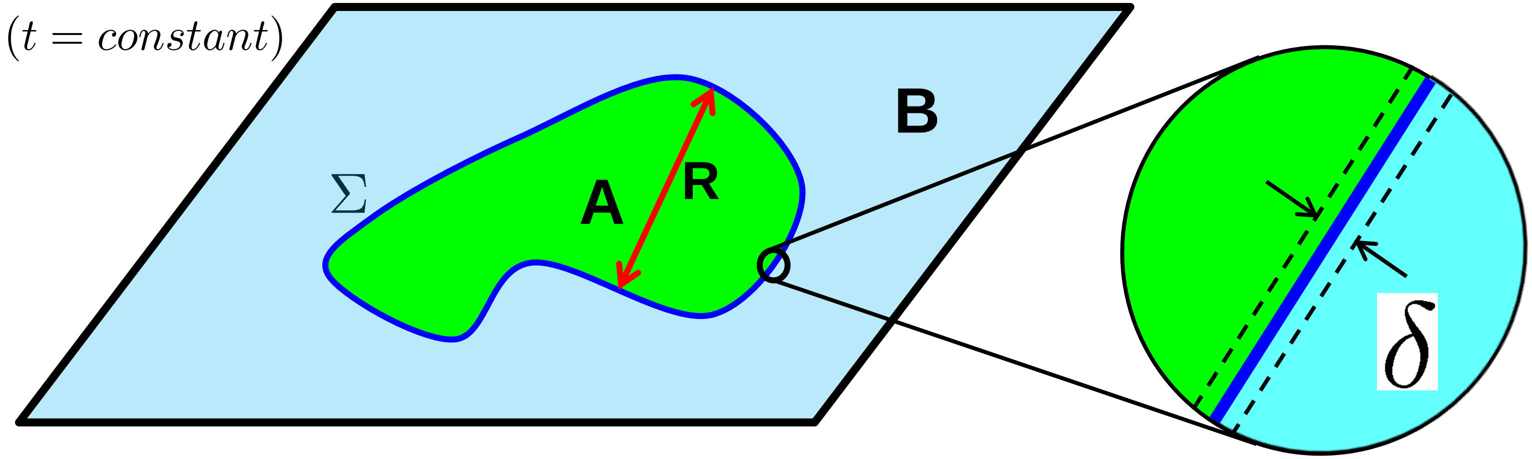


- result is UV divergent! dominated by short-distance correlations
- must regulate calculation: δ = short-distance cut-off

$$S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots$$

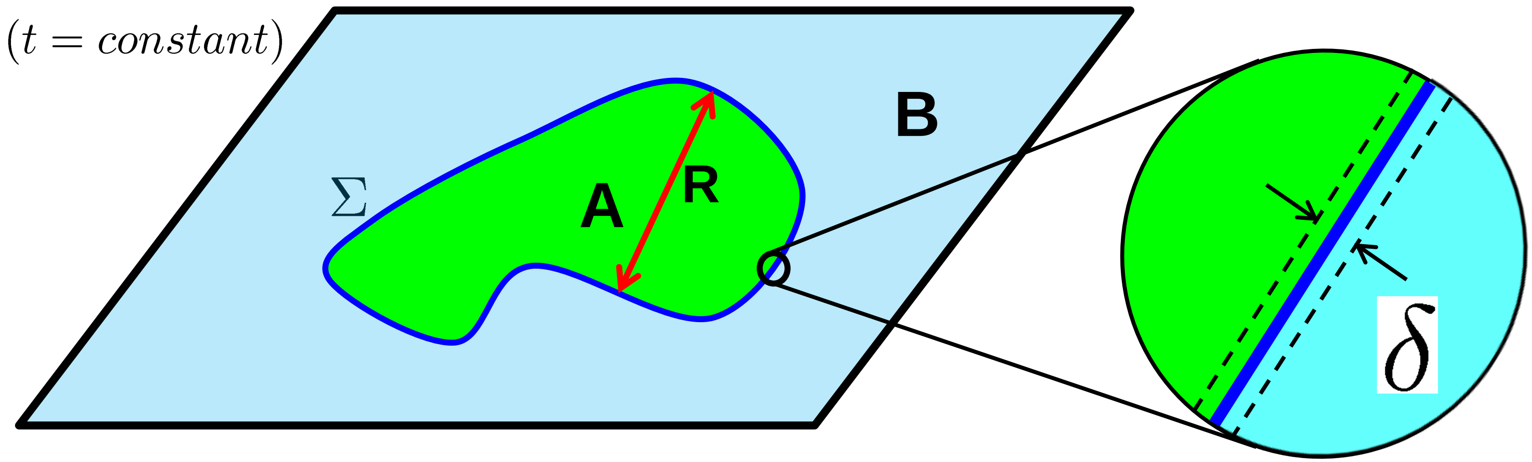
d = spacetime dimension

- careful analysis reveals geometric structure, eg,

$$S = \tilde{c}_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \tilde{c}_2 \frac{\int_\Sigma d^{d-2}\sigma \;\text{“}\mathcal{R}\text{”}}{\delta^{d-4}} + \cdots$$

Entanglement Entropy 2:

→ calculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$



- result is UV divergent! dominated by short-distance correlations
- must regulate calculation: δ = short-distance cut-off

$$S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots$$

d = spacetime dimension

- leading coefficients sensitive to details of regulator, eg, $\delta \to 2\delta$
- find universal information characterizing underlying QFT in subleading terms, eg,

$$S = \cdots + c_d \log\left(R/\delta\right) + \cdots$$

More general comments on **Entanglement Entropy:**

- nonlocal quantity which is (at best) very difficult to measure

  → no accepted experimental procedure

- still useful diagnostic in condensed matter theory and in quantum information theory

More general comments on **Entanglement Entropy:**

Where did hep-th find "Entanglement Entropy"? ⟶ **black holes**

Bekenstein & Hawking: "quantum" black holes are thermal systems leaking blackbody radiation

⟶ event horizon carries entropy!!

$$S_{BH} = \mathcal{A}/4G_N$$

- Sorkin '84: leading term in EE obeys "area law": $S = c_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$

⟶ suggestive of Bekenstein-Hawking formula if $\delta \simeq \ell_P$

(Sorkin `84; Bombelli, Koul, Lee & Sorkin; Srednicki; Frolov & Novikov; . . .)

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- issues?: leading singularity not universal; regulator dependent
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- recently considered in AdS/CFT correspondence

  (Ryu & Takayanagi `06)

Overview:

1) Entanglement entropy in QFT

2) Primer on AdS/CFT correspondence

3) Introduction to Holographic Entanglement Entropy

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5) Conclusions

AdS/CFT Correspondence:

Bulk:

- quantum gravity
- negative cosmological constant
- **d+1** spacetime dimensions

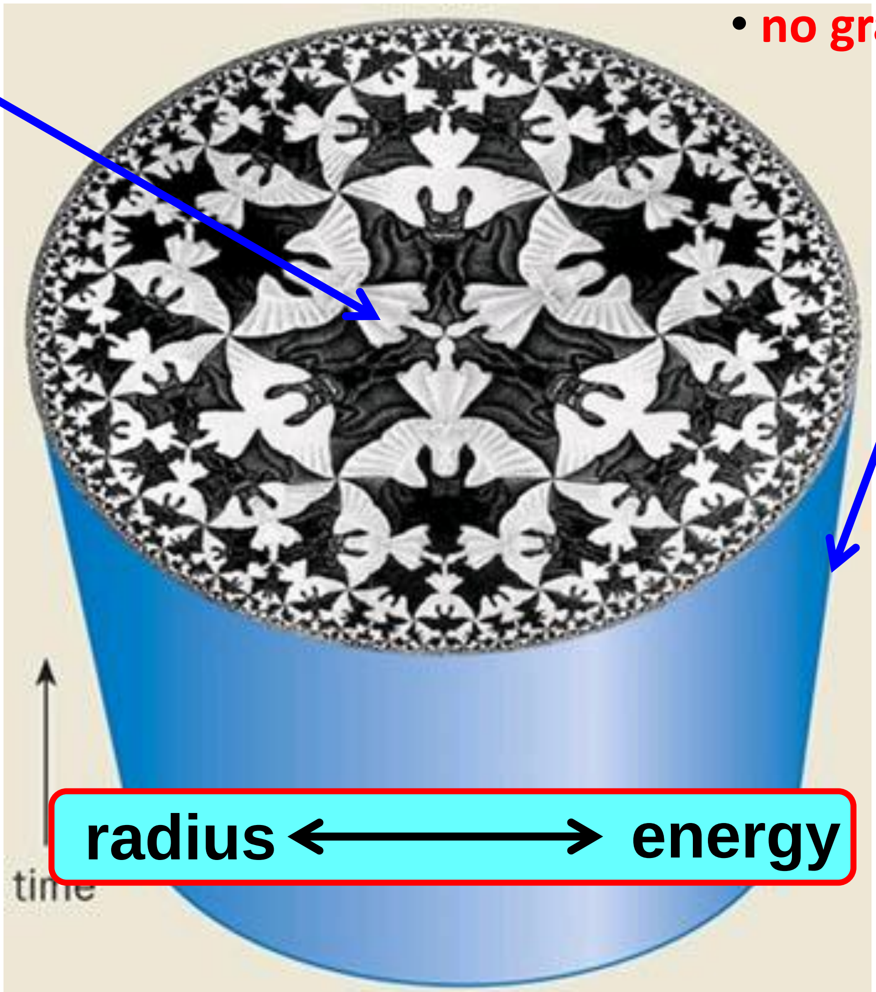
Holography ⟷

Boundary:

- quantum field theory
- no scale (at quantum level)
- **d** spacetime dimensions
- **no gravity!**

anti-de Sitter space

conformal field theory



(Maldacena ‘97)

AdS/CFT correspondence:

anti-de Sitter space ⟺ conformal field theory

quantum gravity
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quantum field theory
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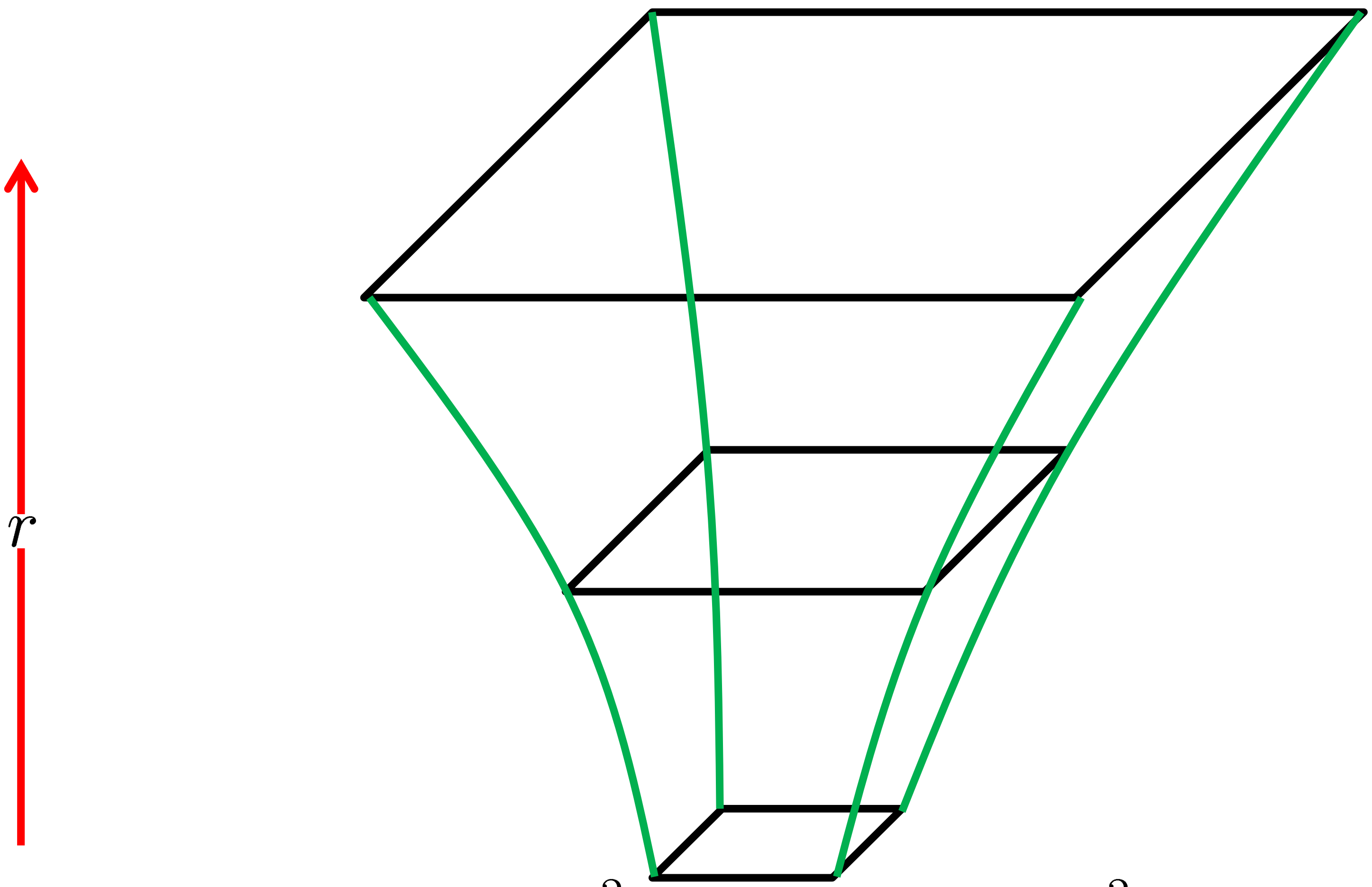
classical gravity
with small curvatures

large central charge ($N_c \to \infty$)

strong coupling ($\lambda \to \infty$)

anti-de Sitter space:

maximally symmetric geometry with negative curvature



$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2}dr^2$$

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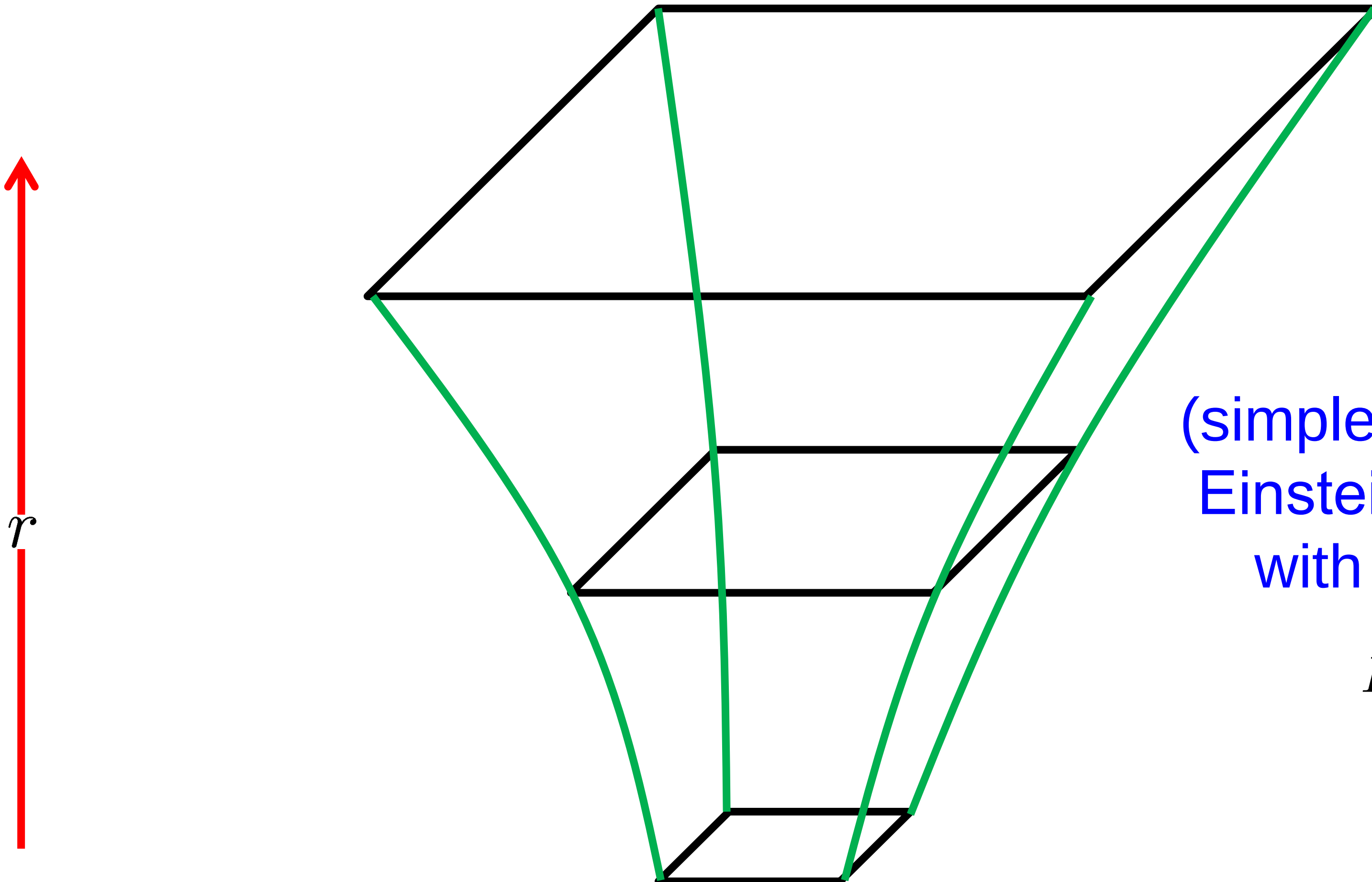

(simplest) solution of Einstein's equations with negative Λ:

$$R_{ab} = -\frac{d}{L^2}$$

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2}dr^2$$

anti-de Sitter space:

$$R \sim -\frac{1}{L^2}$$

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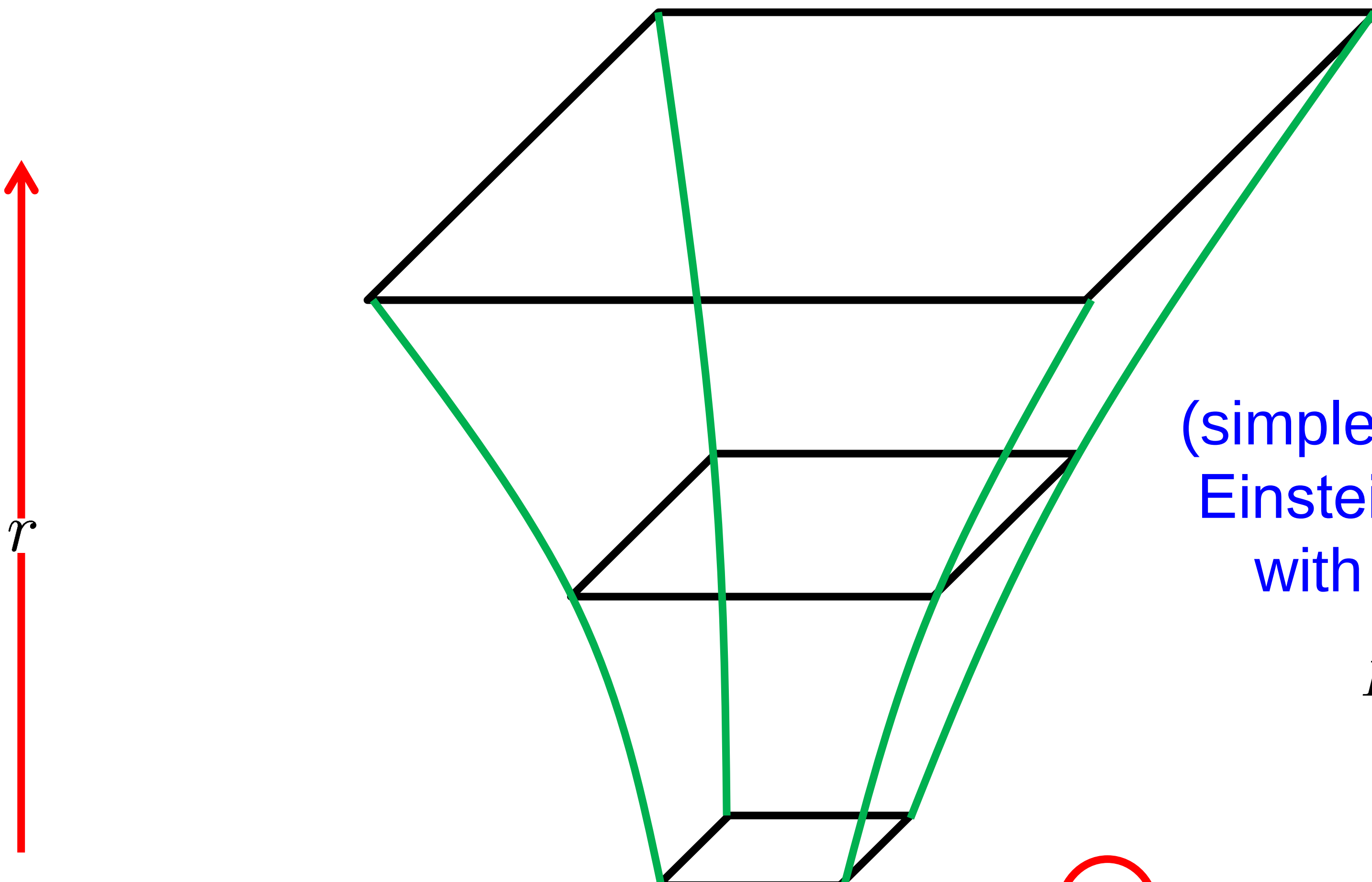

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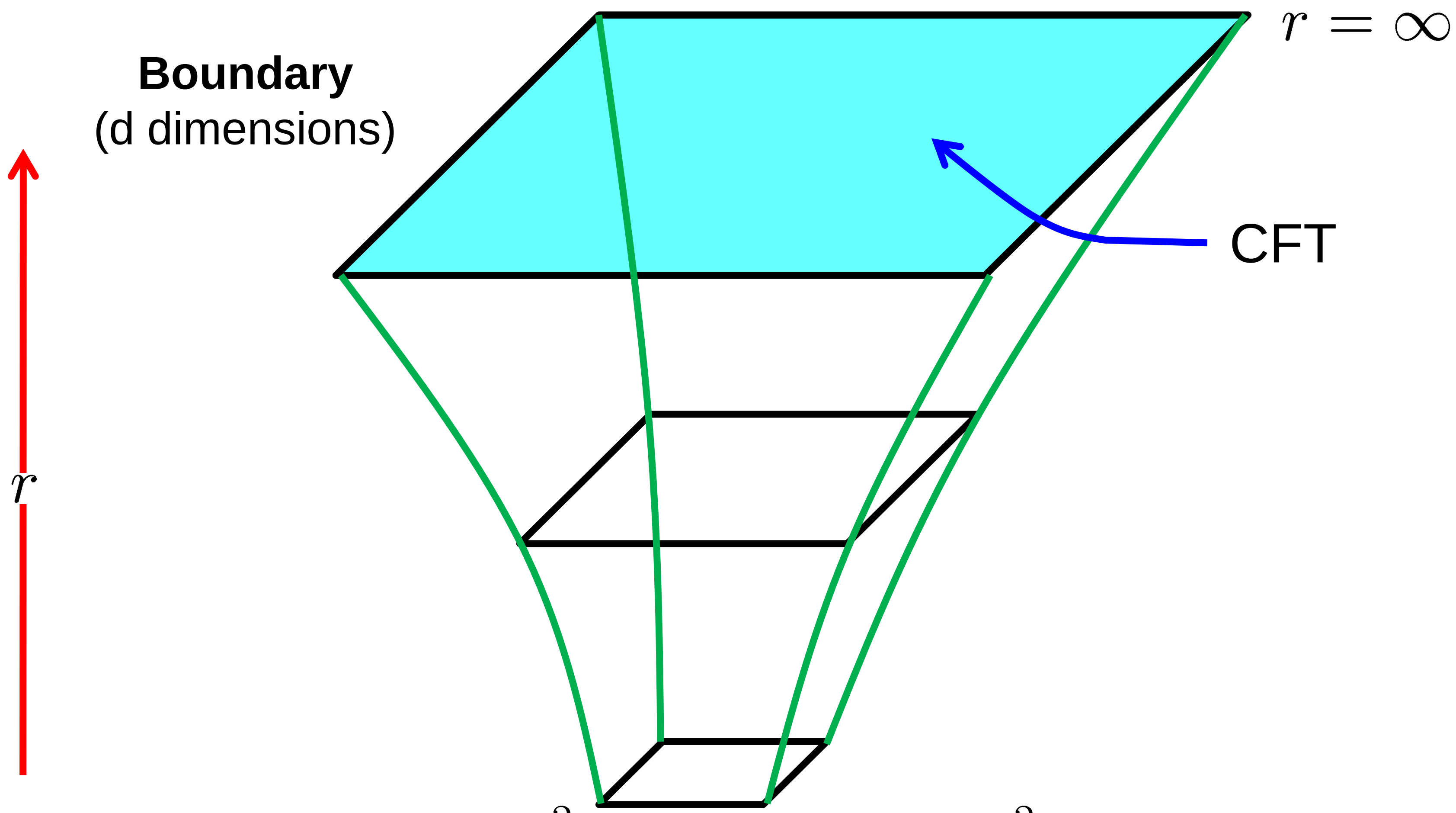

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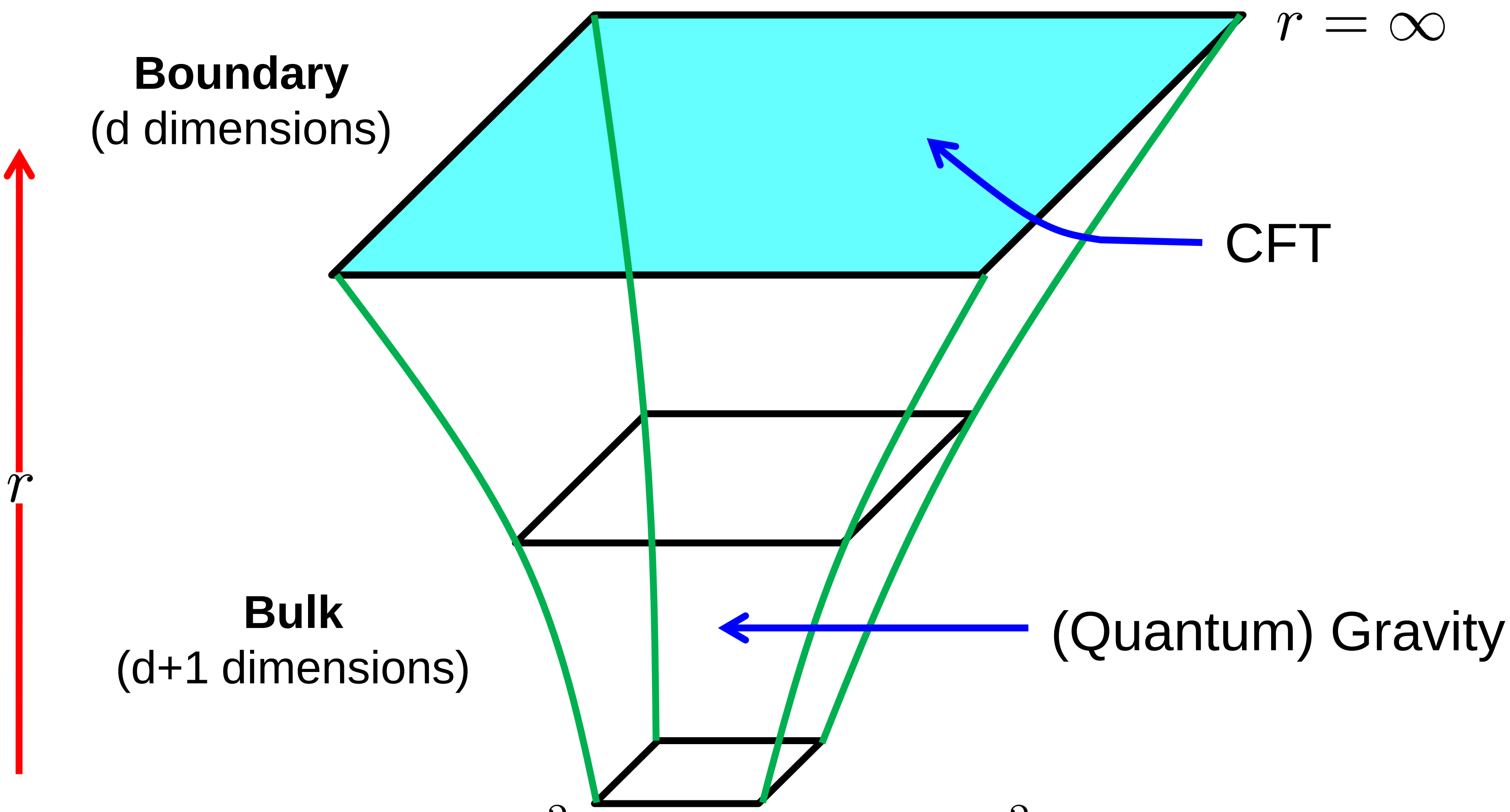

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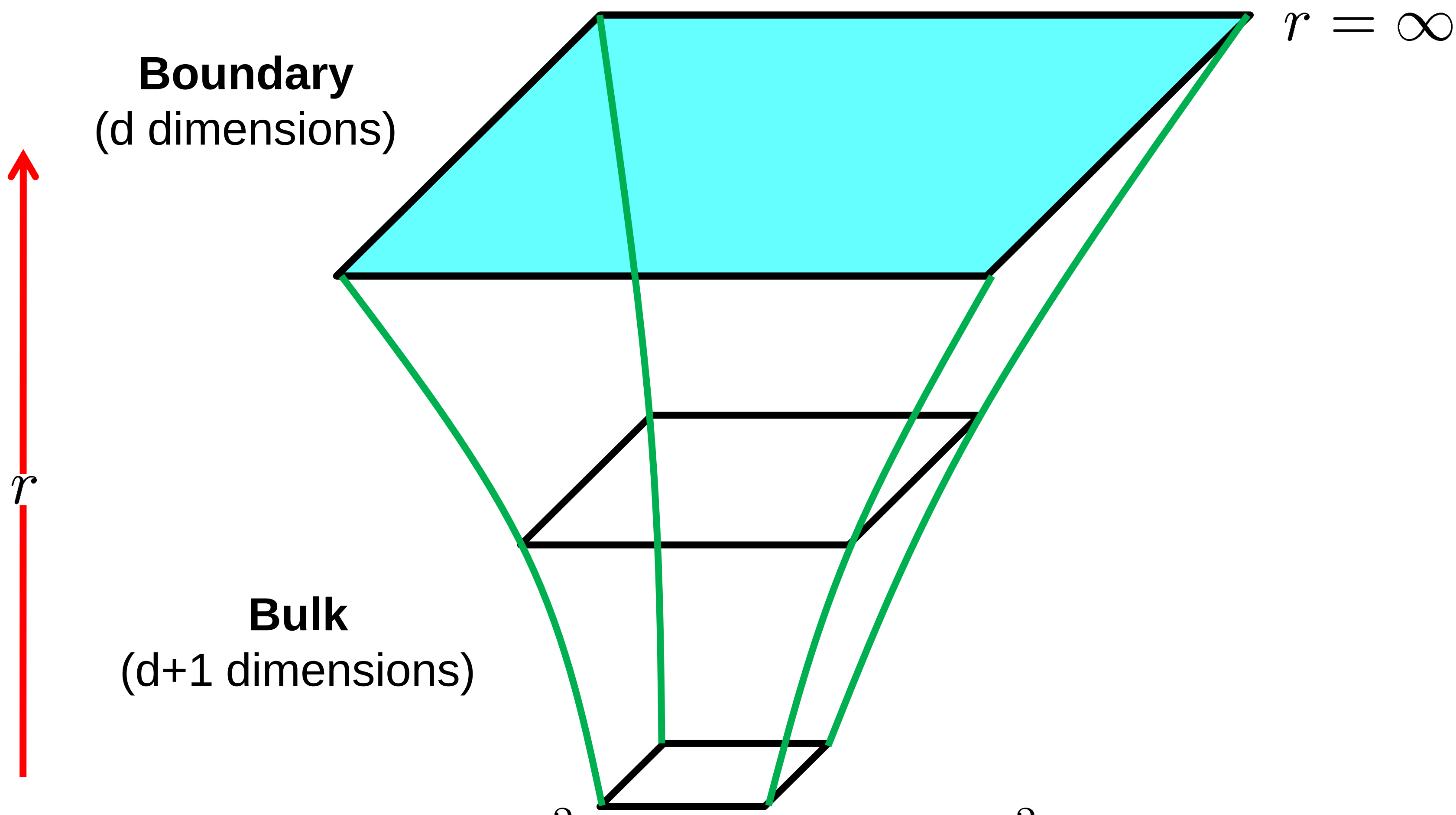

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2}dr^2$$

boundary/CFT metric

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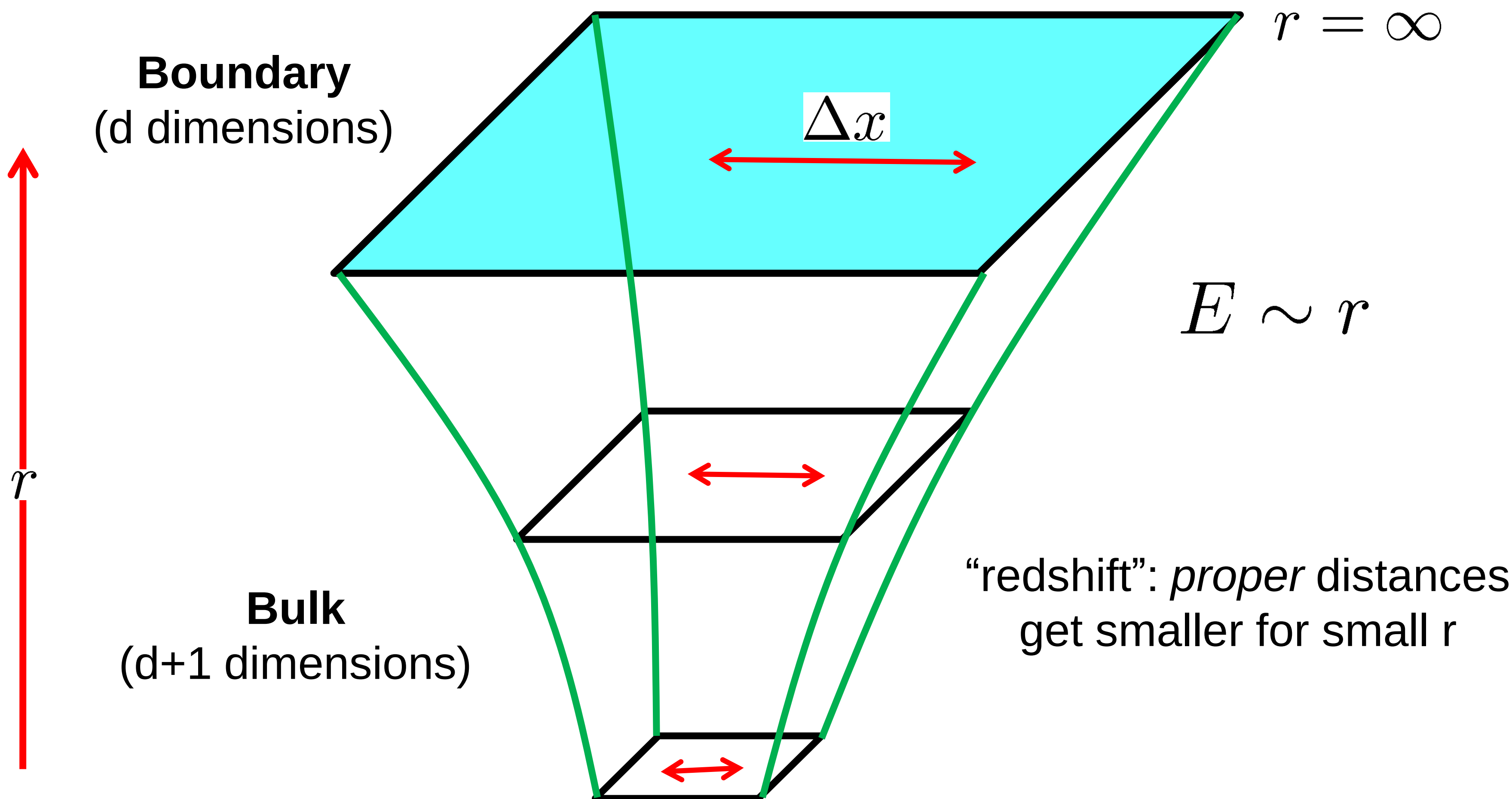

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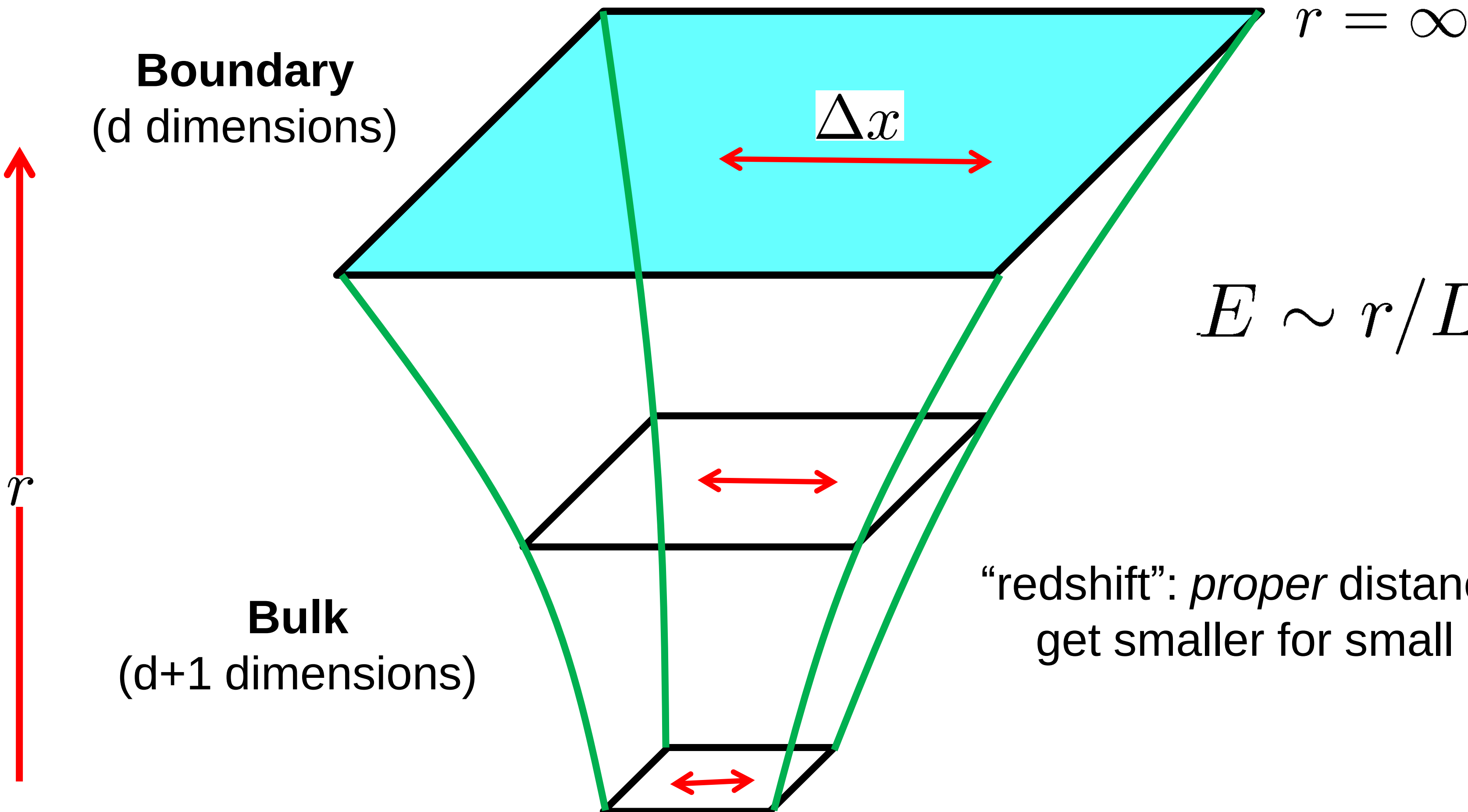

$$E \sim r/L^2$$

"redshift": *proper* distances get smaller for small r

$$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2}dr^2$$

boundary/CFT metric

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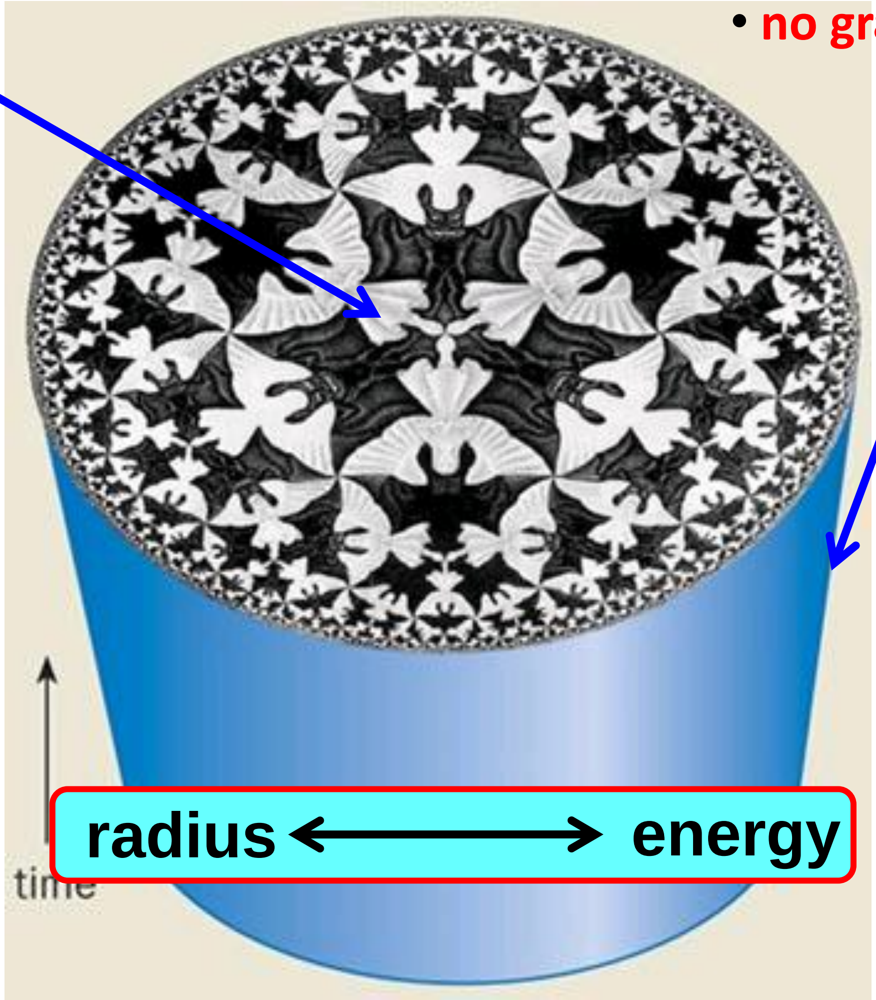

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Entanglement Entropy

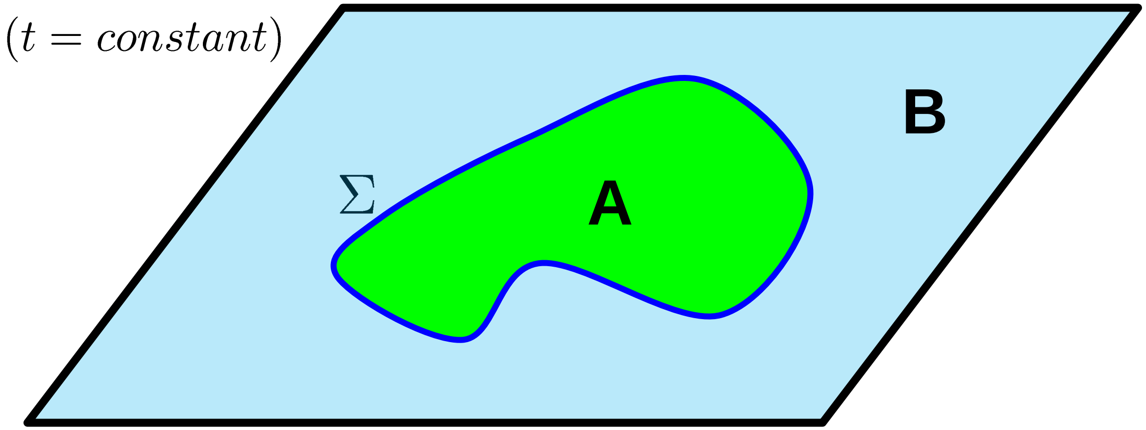
- general tool; divide quantum system into two parts and use entropy as measure of correlations between subsystems

- in QFT, typically introduce a (smooth) boundary or entangling surface Σ which divides the space into two separate regions

- integrate out degrees of freedom in "outside" region

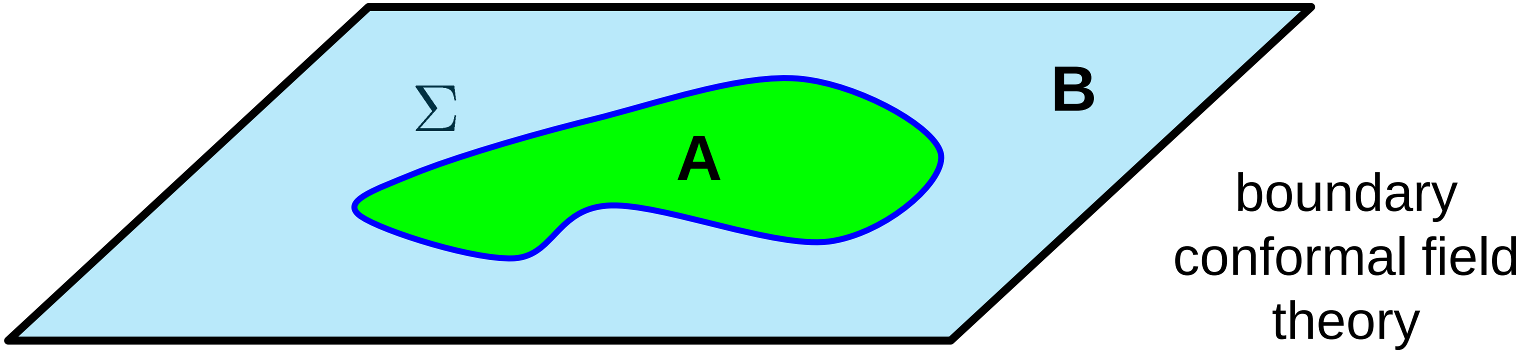
- remaining dof are described by a density matrix ρ_A

→ calculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A \, \log \rho_A\right]$



(Ryu & Takayanagi `06)

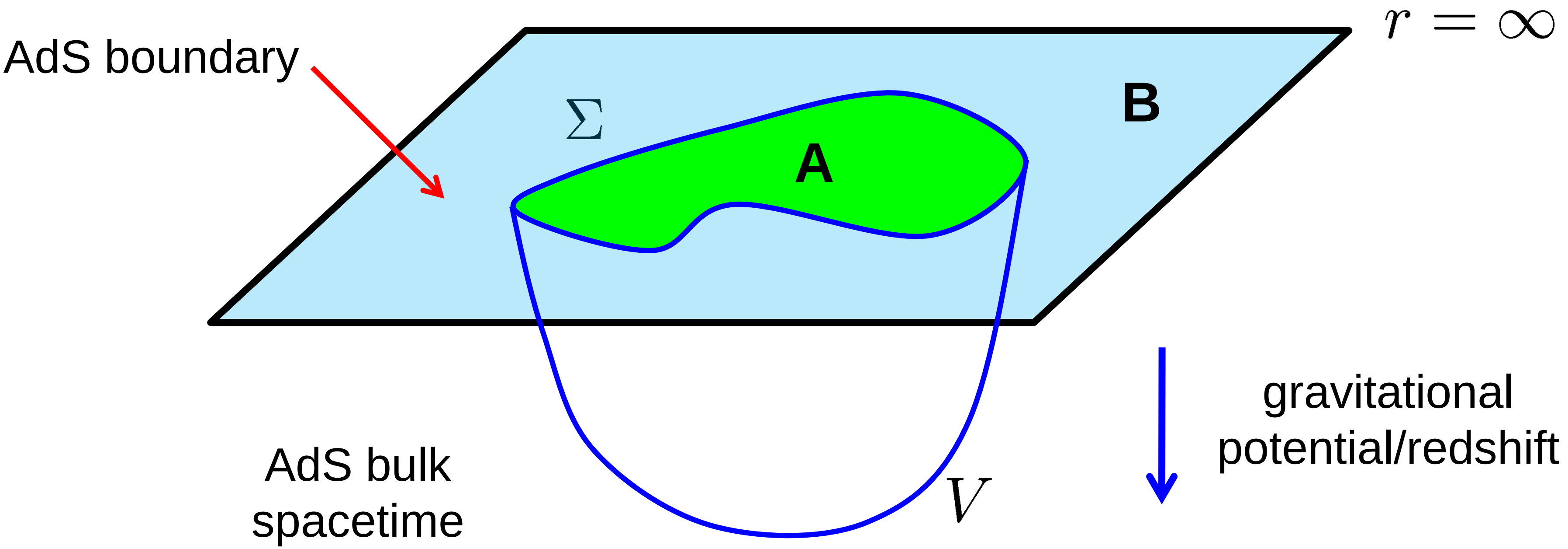
Holographic Entanglement Entropy:



$$S(A) \; = ??$$

(Ryu & Takayanagi `06)

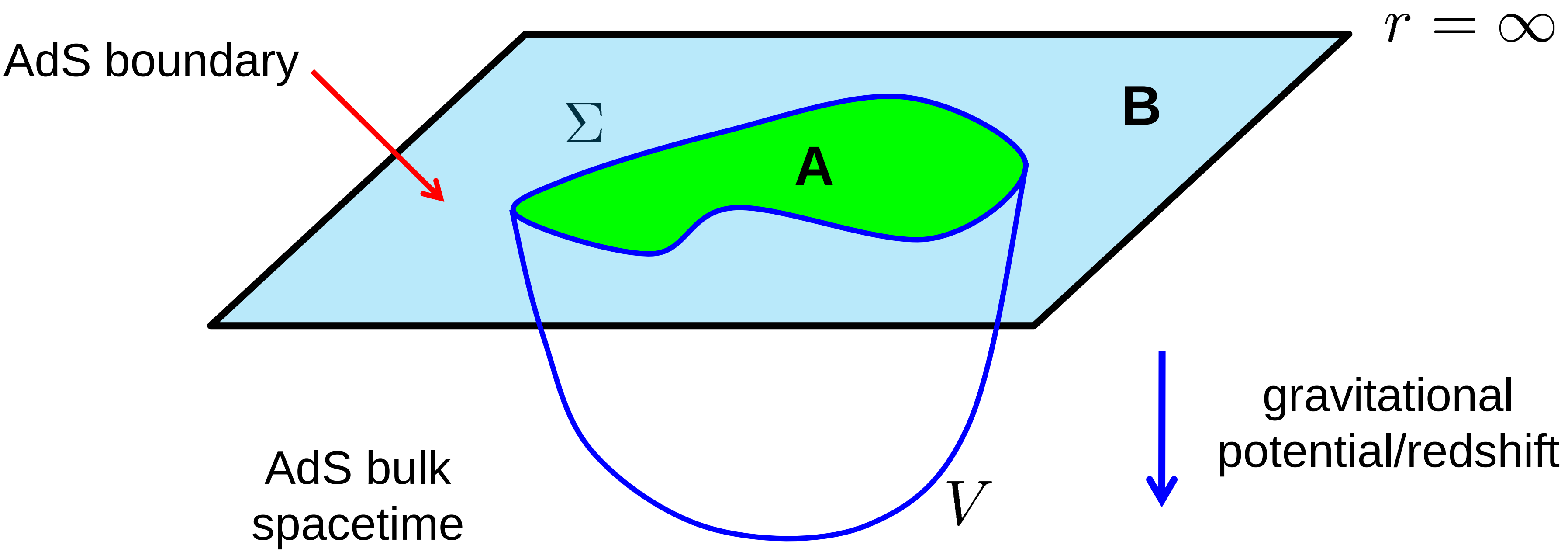
Holographic Entanglement Entropy:



$$S(A) = \min_{\partial V = \Sigma} \frac{A_V}{4G_N}$$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



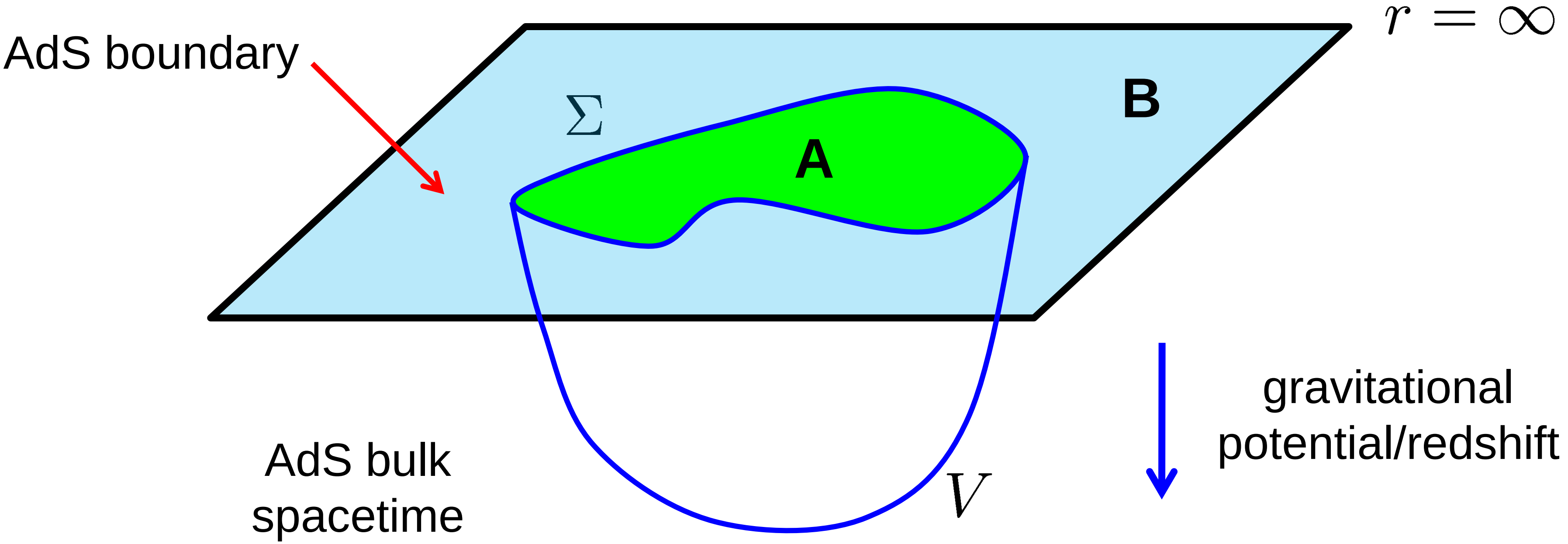
$(d-1)$ dimensional

$$S(A) = \underset{V \sim A}{\mathrm{ext}} \frac{A_V}{4G_N}$$

looks like BH entropy!

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



$$S(A) = \underset{V \sim A}{\text{ext}} \frac{A_V}{4G_N} = \infty!!$$

(Ryu & Takayanagi `06)

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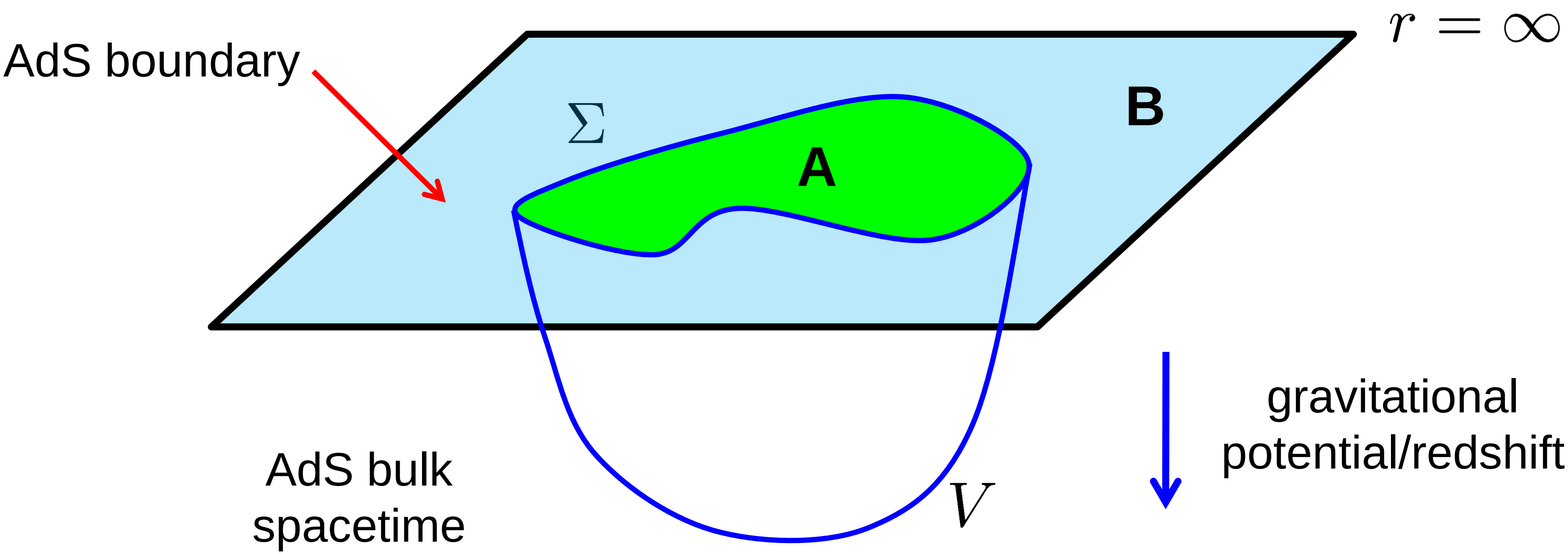


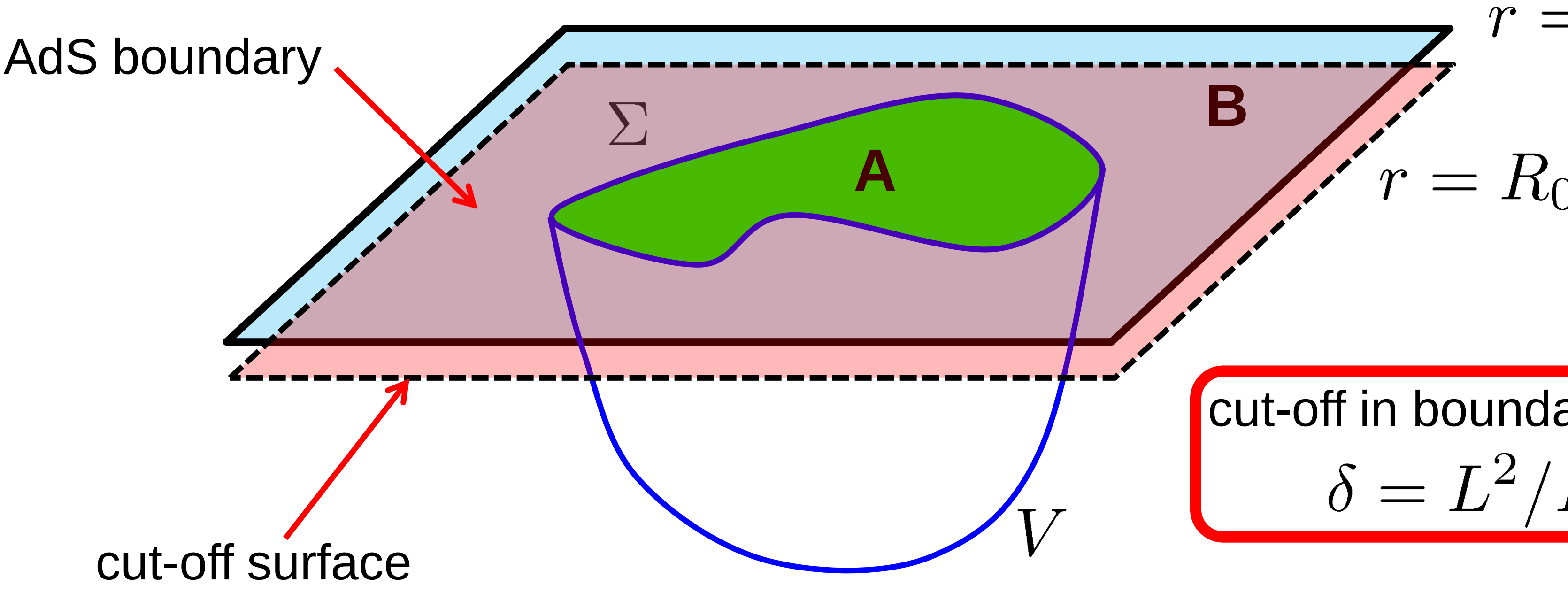
$$S(A) \; = \; \underset{V \sim A}{\text{ext}} \; \frac{A_V}{4G_N} \; = \; \infty!!$$

- "UV divergence" because area integral extends to $r = \infty$

(a feature! **not** a bug!)

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



cut-off in boundary CFT:

$$\delta = L^2/R_0$$

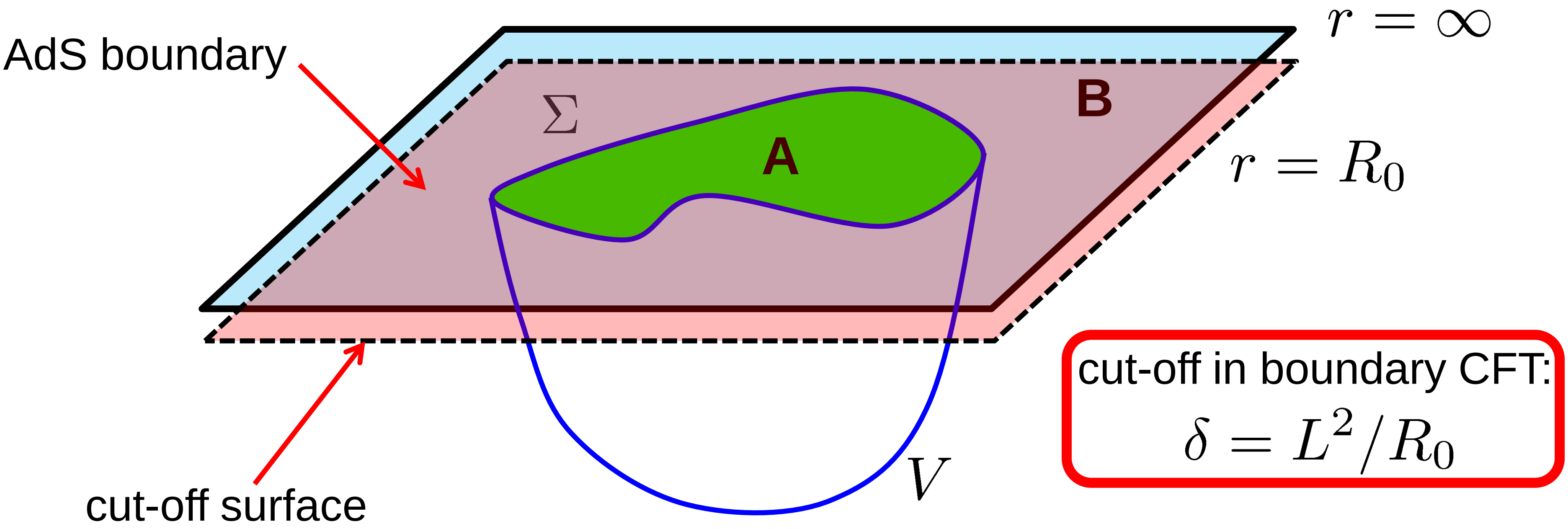
$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N}$$

- "UV divergence" because area integral extends to $r = \infty$
- finite result by stopping radial integral at large radius: $r = R_0$

→ short-distance cut-off in boundary theory: $\delta = L^2/R_0$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



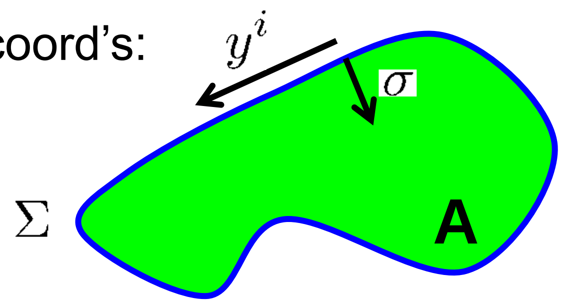
$$S(A) = \underset{\partial V = \Sigma}{\mathrm{ext}} \frac{A_V}{4G_N} \simeq \frac{L^{d-1}}{G_N} \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$$

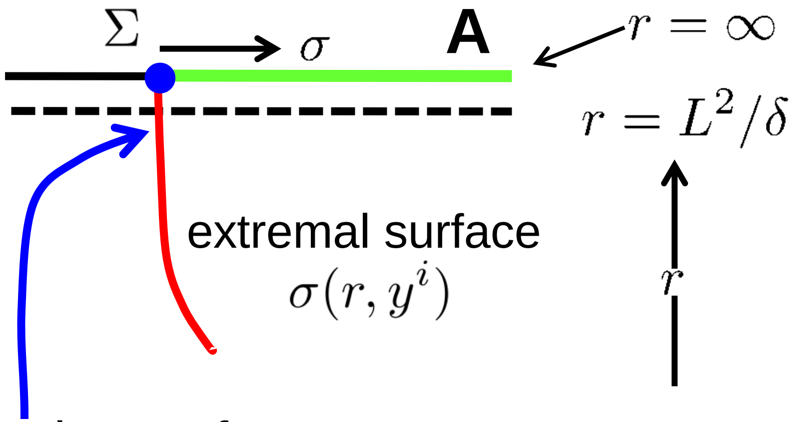
"Area Law"

Area law contribution:

recall AdS metric: $$ds^2 = \frac{r^2}{L^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{L^2}{r^2}dr^2$$

bdry coord's:





to leading order, surface falls straight down, ie, $\sigma \simeq 0$

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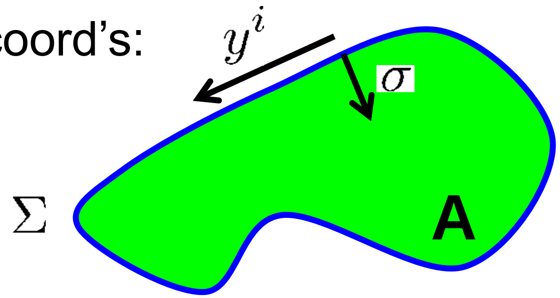

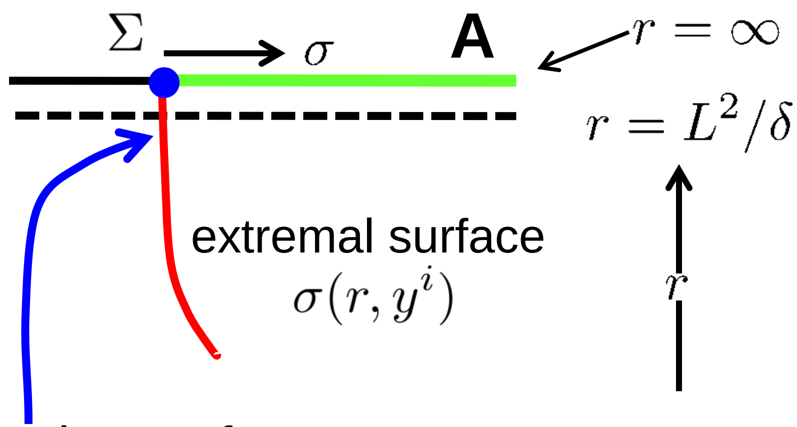

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$$S(A) = \frac{A_V}{4G_N} = \frac{1}{4G_N}\int d^{d-1}\sigma\sqrt{h}$$

$$= \frac{1}{4G_N}\int^{R_0} dr\left(\frac{L}{r}\right) \times \underline{\int d^{d-2}y\sqrt{h_y}}\left(\frac{r}{L}\right)^{d-2}$$

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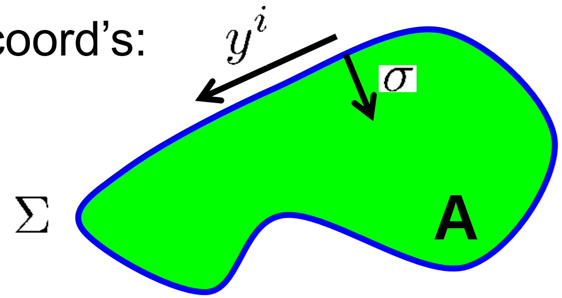

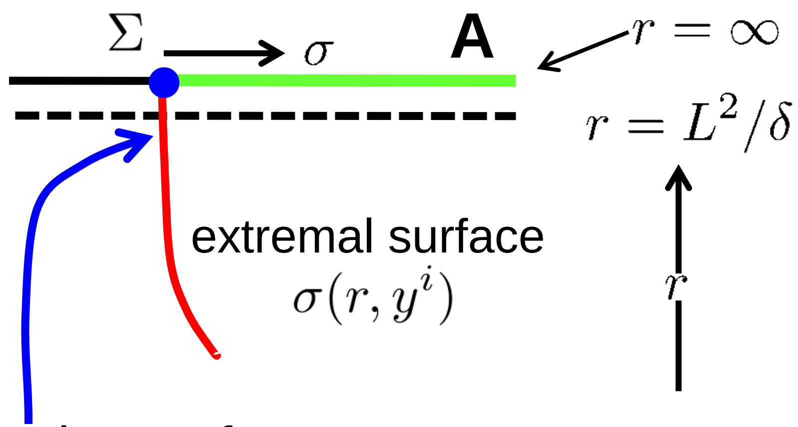

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$$S(A) \;=\; \frac{\mathcal{A}_V}{4G_N} = \frac{1}{4G_N}\int d^{d-1}\sigma\sqrt{h}$$

$$= \frac{1}{4G_N}\mathcal{A}_\Sigma \int^{L^2/\delta} dr\left(\frac{r}{L}\right)^{d-3} \simeq \frac{L^{d-1}}{4(d-2)G_N}\;\frac{\mathcal{A}_\Sigma}{\delta^{d-2}}$$

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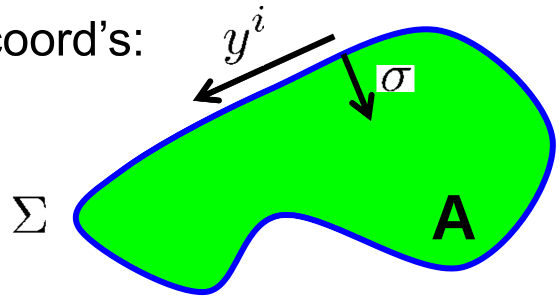

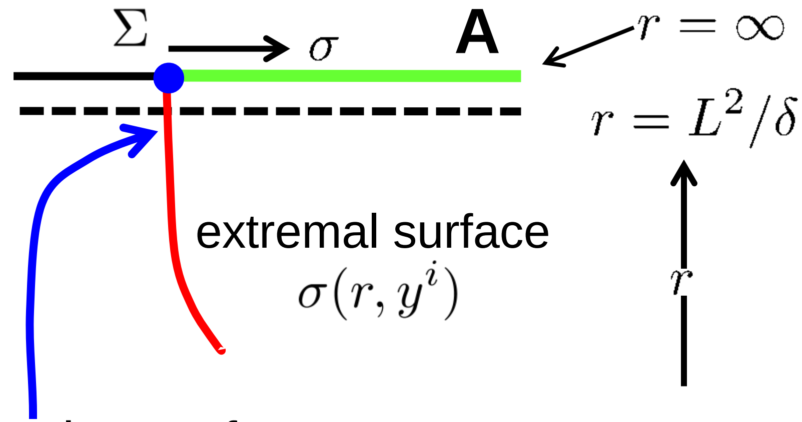

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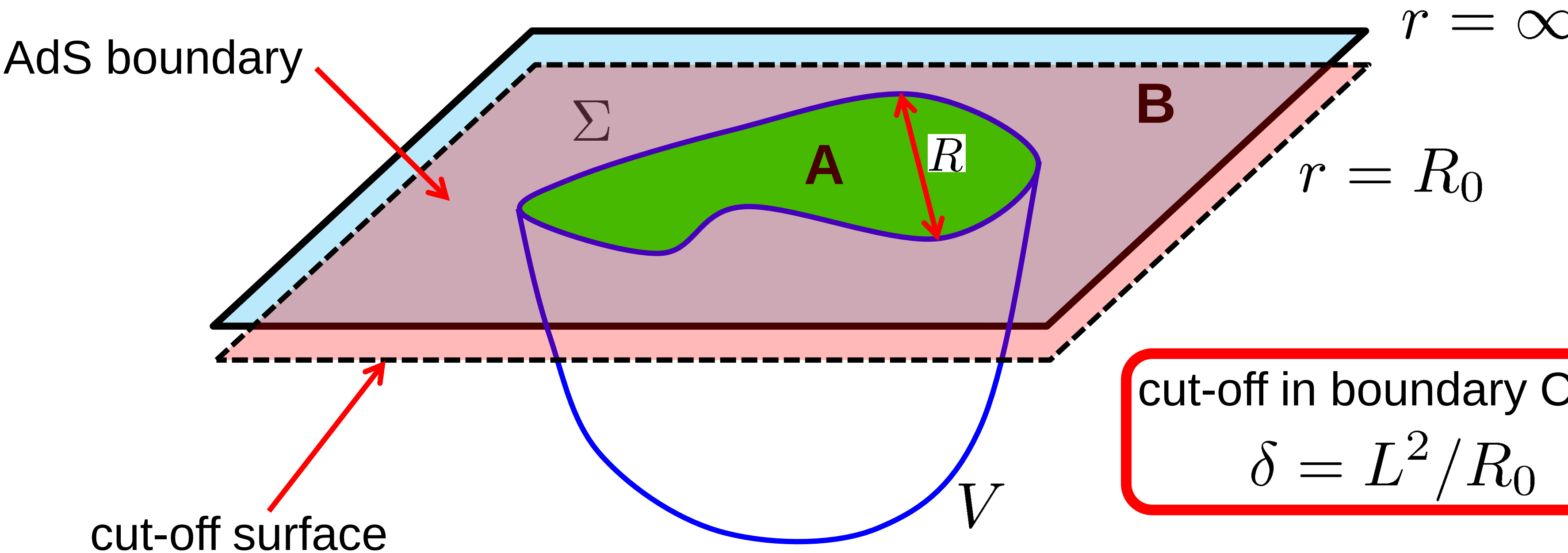
$$S(A) = \frac{A_V}{4G_N} = \frac{1}{4G_N}\int d^{d-1}\sigma\sqrt{h}$$

central charge (counts dof)

$$= \frac{1}{4G_N}\mathcal{A}_\Sigma \int^{L^2/\delta} dr \left(\frac{r}{L}\right)^{d-3} \sim \frac{L^{d-1}}{\ell_{Planck}^{d-1}}\,\frac{\mathcal{A}_\Sigma}{\delta^{d-2}}$$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



cut-off in boundary CFT:
$$\delta = L^2/R_0$$

general expression (as desired):

$$S(A) \;\simeq\; c_0(R/\delta)^{d-2} + c_1(R/\delta)^{d-4} + \cdots$$

$$\begin{cases} +c_{d-2}\log(R/\delta) + \cdots & \text{(d even)} \\ +\, c_{d-2} \, + \cdots & \text{(d odd)} \end{cases}$$

universal contributions

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \underset{\partial V = \Sigma}{\mathrm{ext}} \frac{A_V}{4G_N}$$

<u>appeared as conjecture!</u>

(Ryu & Takayanagi `06)

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Extensive consistency tests:

1) leading contribution yields "area law"

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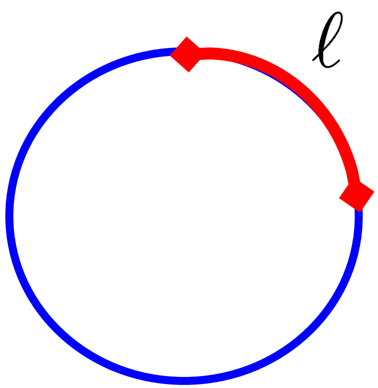
Extensive consistency tests:

1) leading contribution yields "area law"

$$S \simeq \frac{L^{d-1}}{G_N} \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$$

2) recover known results of Calabrese & Cardy for d=2 CFT

$$S = \frac{c}{3} \log\left(\frac{C}{\pi\,\delta} \sin \frac{\pi \ell}{C} \right)$$



$C = \text{circumference}$

(also result for thermal ensemble)

(Ryu & Takayanagi `06)

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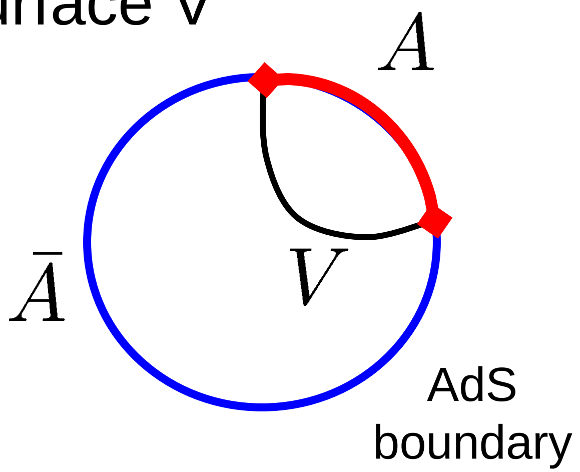
3) $S(A) = S(\bar{A})$ in a pure state

→ A and $\bar{A}$ both yield same bulk surface V



(Ryu & Takayanagi `06)

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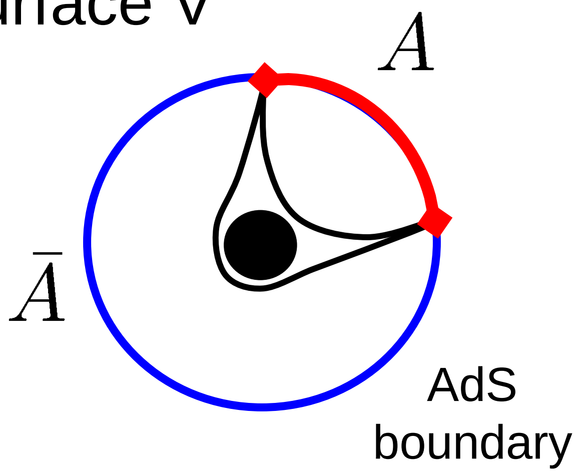

cf: thermal ensemble ≠ pure state

horizon in bulk → $S(A) \neq S(\bar{A})$

AdS/CFT Dictionary: thermal bath ⟷ black hole

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N}$$

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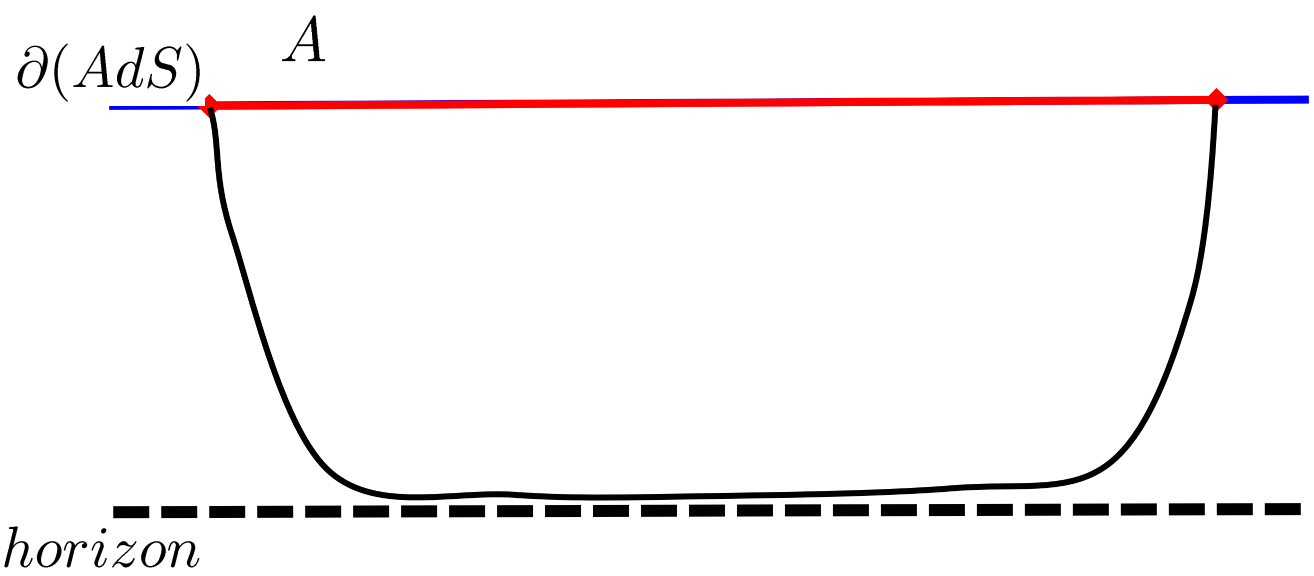
Extensive consistency tests:

4) for thermal bath: $S(A) \supset S_{therm} = \alpha\, T^{d-1} \times volume$

$R \gg 1/T$



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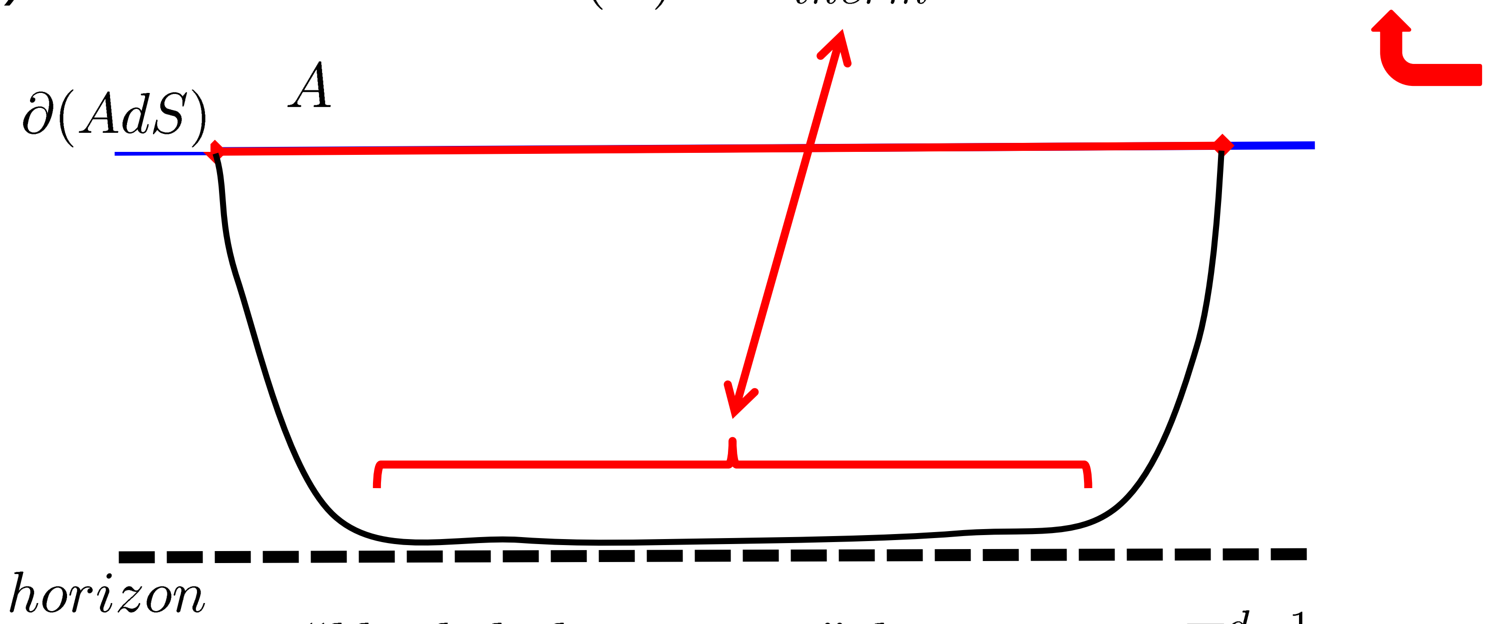

$$\text{“}black\ hole\ entropy\text{”}\ density = \alpha\, T^{d-1}$$

(Ryu & Takayanagi `06)

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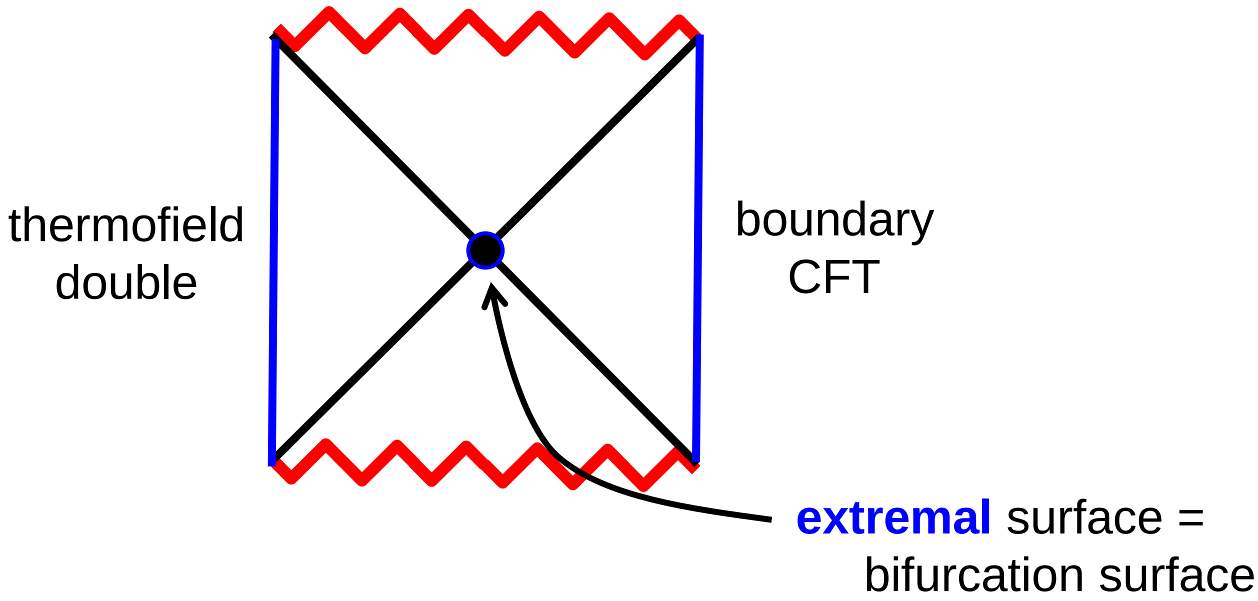
Extensive consistency tests:

4b) Entropy of eternal black hole =
entanglement entropy of boundary CFT & thermofield double

(Headrick)



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$$S(A) = \underset{\partial V = \Sigma}{\text{ext}} \frac{A_V}{4G_N}$$

appeared as conjecture!

Extensive consistency tests:

5) strong sub-additivity: $S(A) + S(B) \geq S(A \cup B) + S(A \cap B)$

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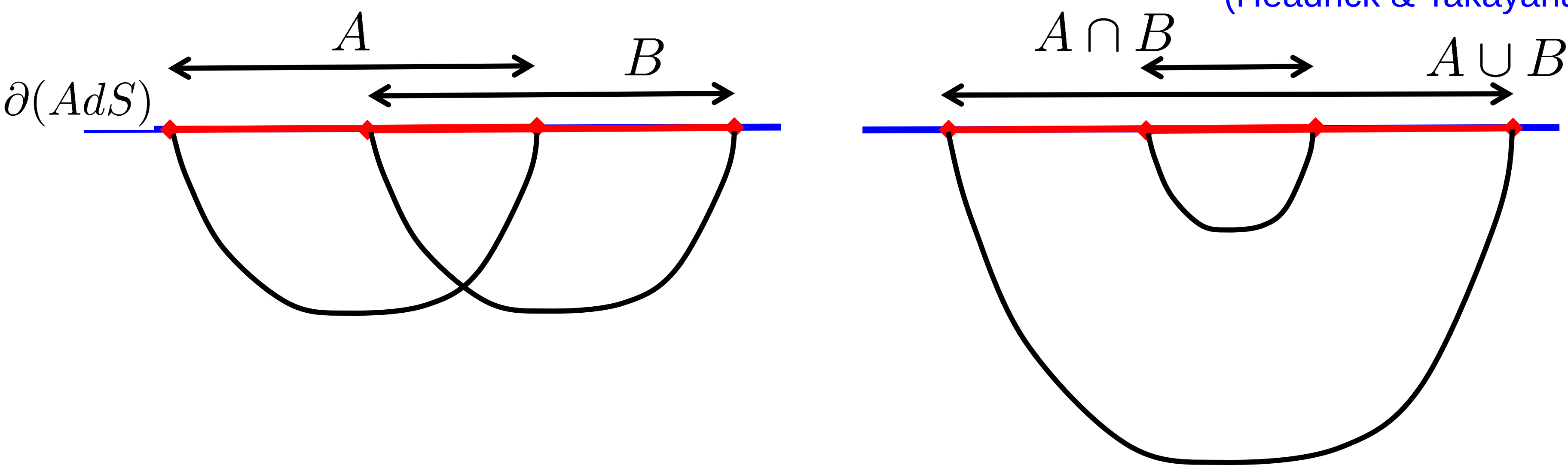

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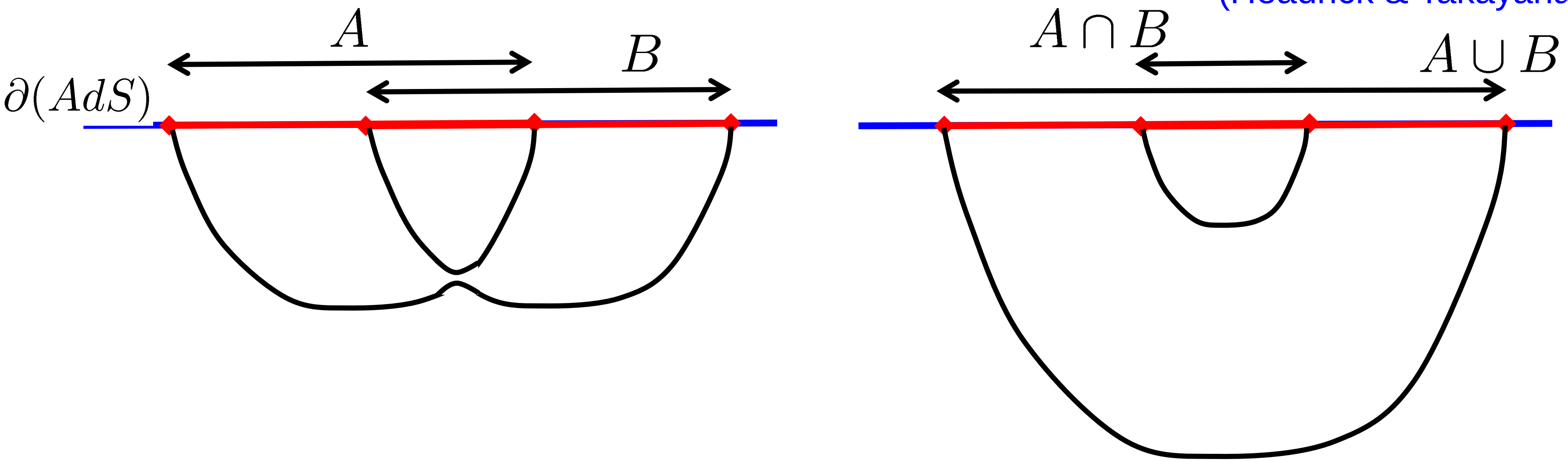

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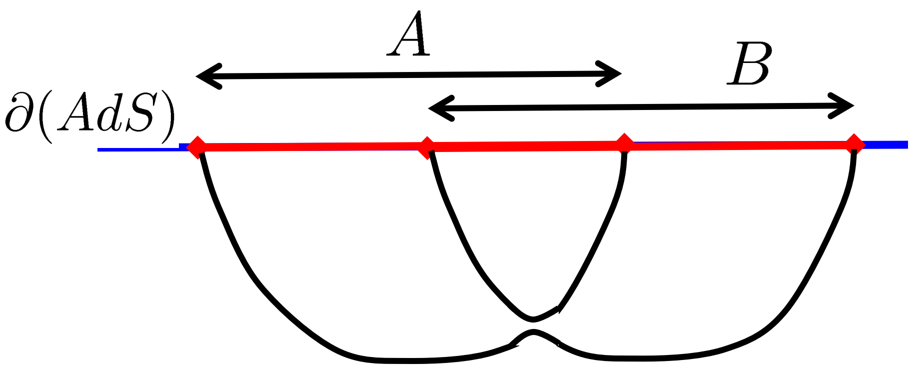

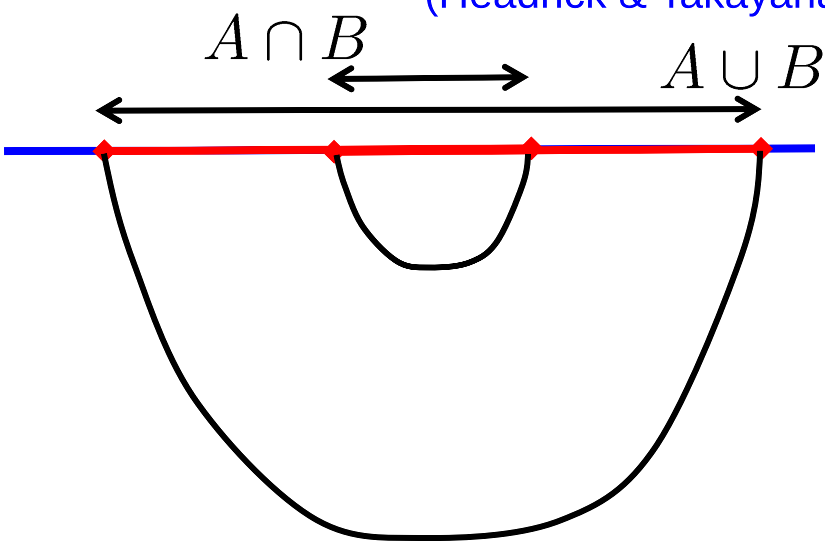

[extended to dynamical setting: Wall]

[further monogamy relations: Hayden, Headrick & Maloney]

(Ryu & Takayanagi `06)

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Extensive consistency tests:

6) for even d, connection of universal/logarithmic contribution in S_{EE} to central charges of boundary CFT, eg, in d=4

$$S_{uni} = \log(R/\delta)\,\frac{1}{2\pi}\int_\Sigma d^2x\sqrt{h}\left[\mathbf{c}\left(C^{ijkl}\,\tilde{g}^\perp_{ik}\,\tilde{g}^\perp_{jl} - K^{i\,b}_a K^{i\,a}_b + \frac{1}{2}K^{i\,a}_a K^{i\,b}_b\right) - \mathbf{a}\,\mathcal{R}\right]$$

(Hung, RM & Smolkin)

7) derivation of holographic EE for spherical entangling surfaces

(Casini, Huerta & RM, RM & Sinha)

(Ryu & Takayanagi `06)

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Extensive consistency tests: ⟶ **recent proof!!!**

(Lewkowycz & Maldacena)

- generalization of Euclidean path integral calc's for S_{BH}, extended to "periodic" bulk solutions without Killing vector
- for AdS/CFT, translates replica trick for boundary CFT to bulk

$$\Delta\tau = 2\pi \to 2\pi n \quad \longrightarrow \quad \log Z(n) = \log \mathrm{Tr}\left[\rho^n\right] = -I_{grav}(n)$$

$$\longrightarrow \quad S = -n\partial_n \left[\log Z(n) - n \log Z(1)\right]\Big|_{n=1}$$

- at $n \sim 1$, linearized gravity eom demand: $K^\alpha = h^{ij}\,K^\alpha_{ij} = 0$

⟶ τ shrinks to zero on an extremal surface in bulk

- evaluating Einstein action yields $S = A/4G_N$ for extremal surface

Overview:

1) Entanglement entropy in QFT

2) Primer on AdS/CFT correspondence

3) Introduction to Holographic Entanglement Entropy

4) Derivation of Holographic Entanglement Entropy (for spherical entangling surfaces)

5) Conclusions

Why is deriving Holographic Entanglement Entropy hard?

$$S_{EE} = -Tr\left[\rho_A \, \log \rho_A\right]$$

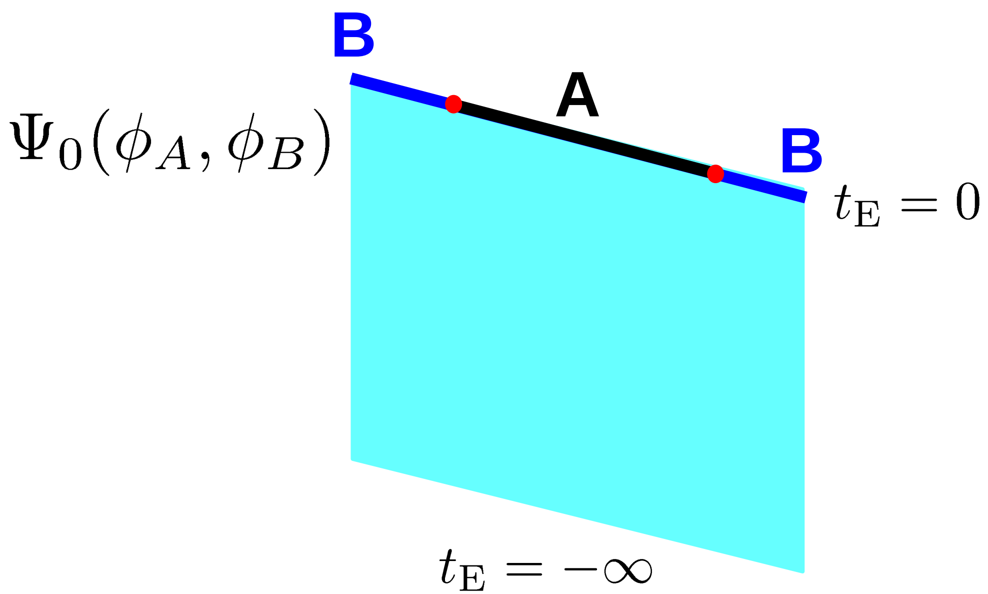
- a “standard” approach relies on replica trick, first calculating Renyi entropy and then taking n $\longrightarrow$ 1 limit

$$S_n = \frac{1}{1-n} \log Tr\left[\rho_A^n\right] \qquad\qquad S_{EE} = \lim_{n\to 1} S_n$$

Replica trick:

0. analytically continue: $t_{\mathrm{E}} = i\,t$

1. path integral representation
   of ground state wave function



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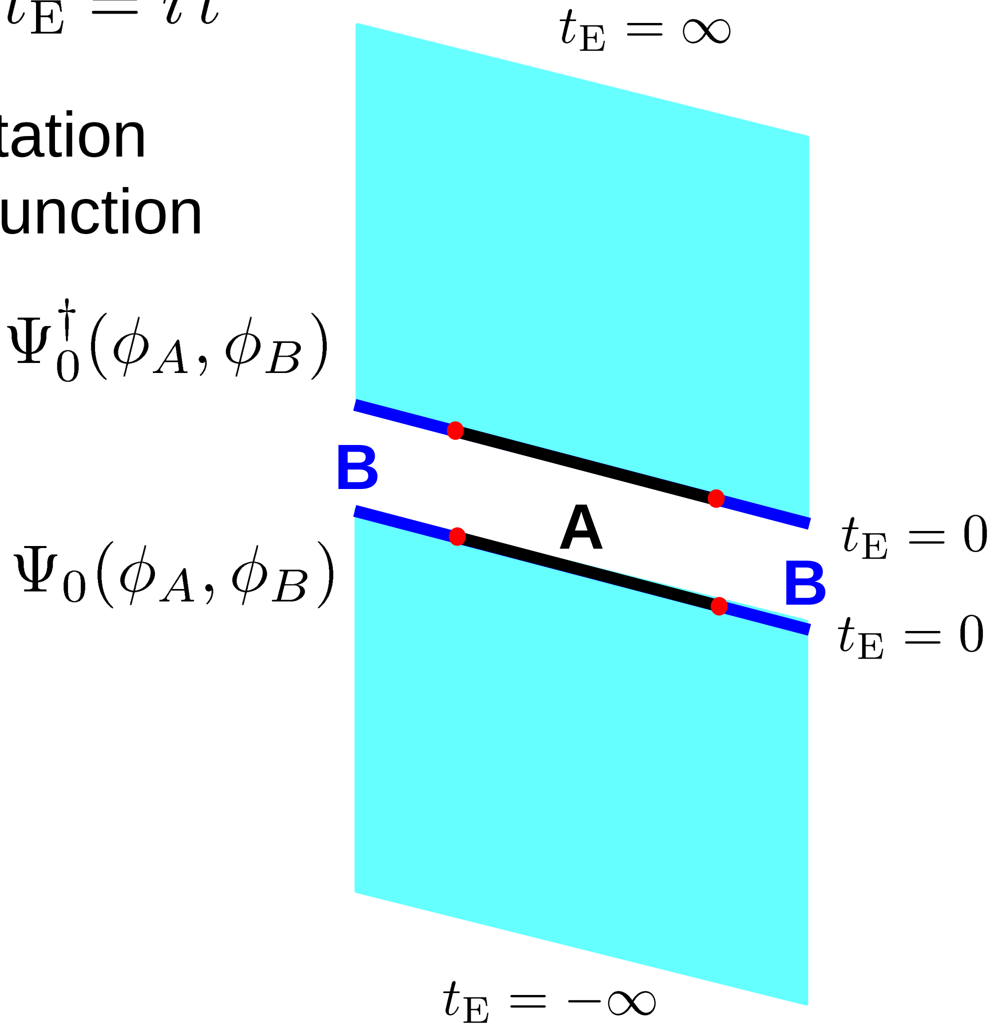

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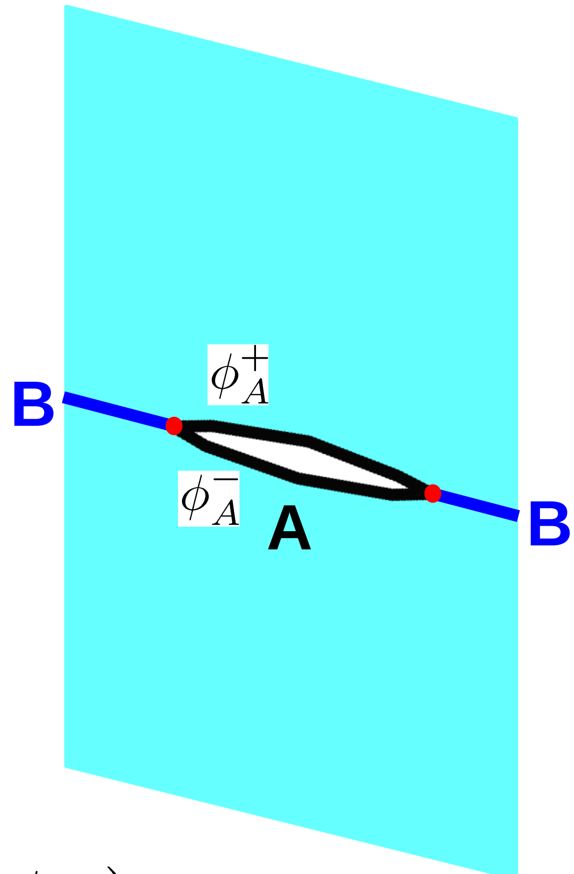

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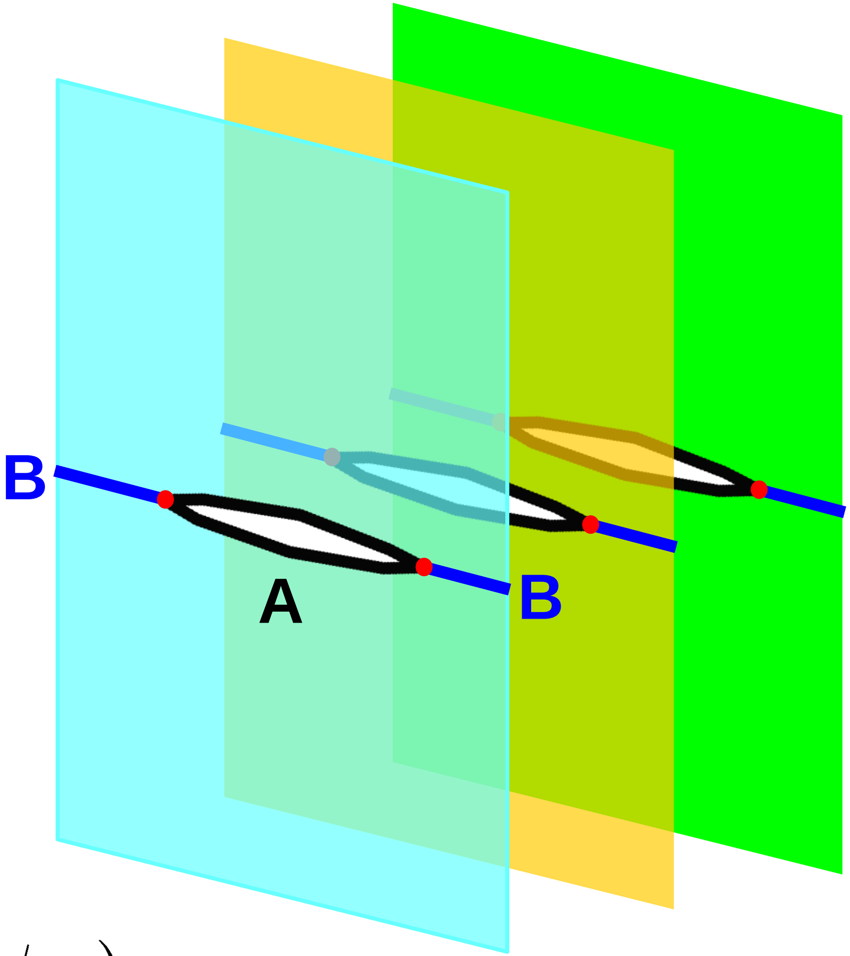

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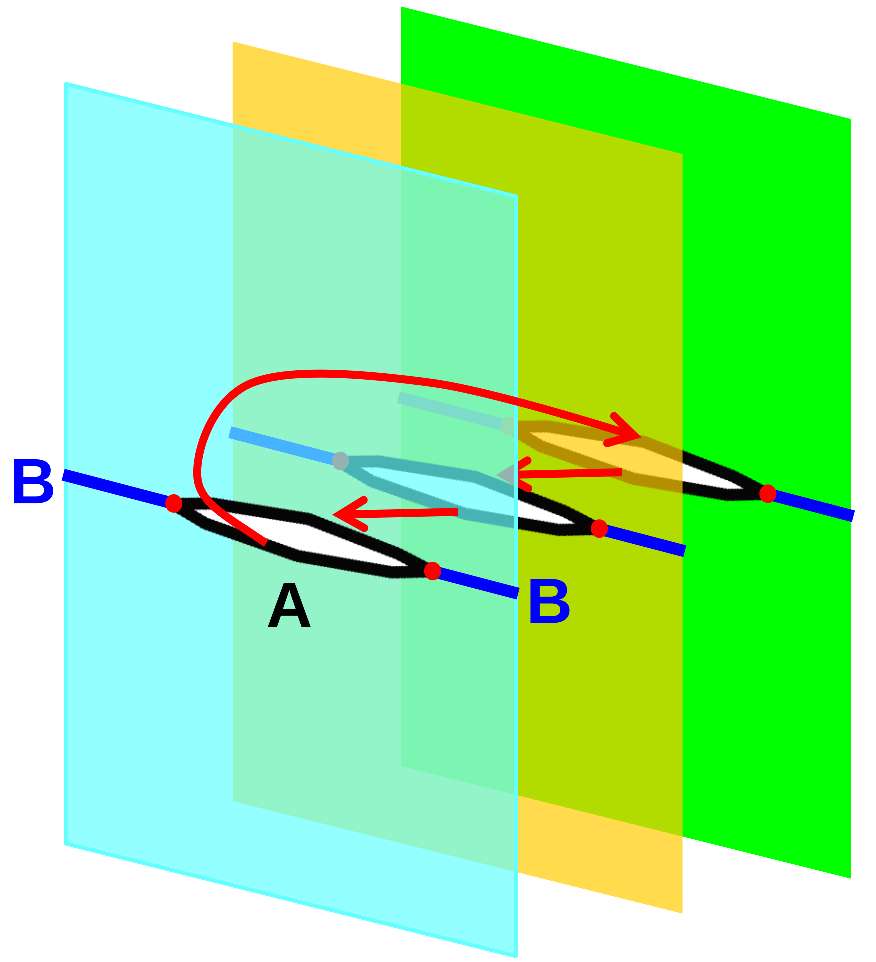

$$\mathrm{Tr}(\rho_A^n) = \mathrm{Tr}_{\phi_A^i}\left[\rho_A(\phi_A^1, \phi_A^{n-1}) \, \cdots \, \rho_A(\phi_A^3, \phi_A^2)\, \rho_A(\phi_A^2, \phi_A^1)\right]$$

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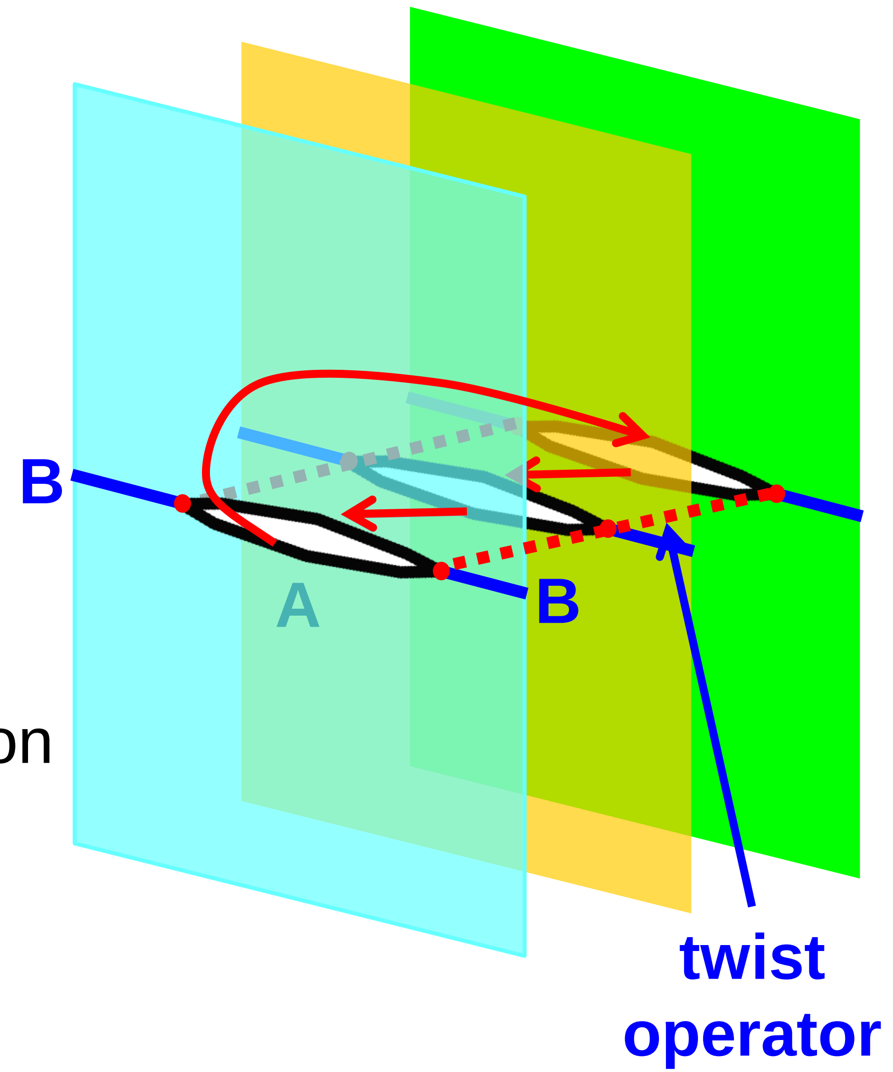

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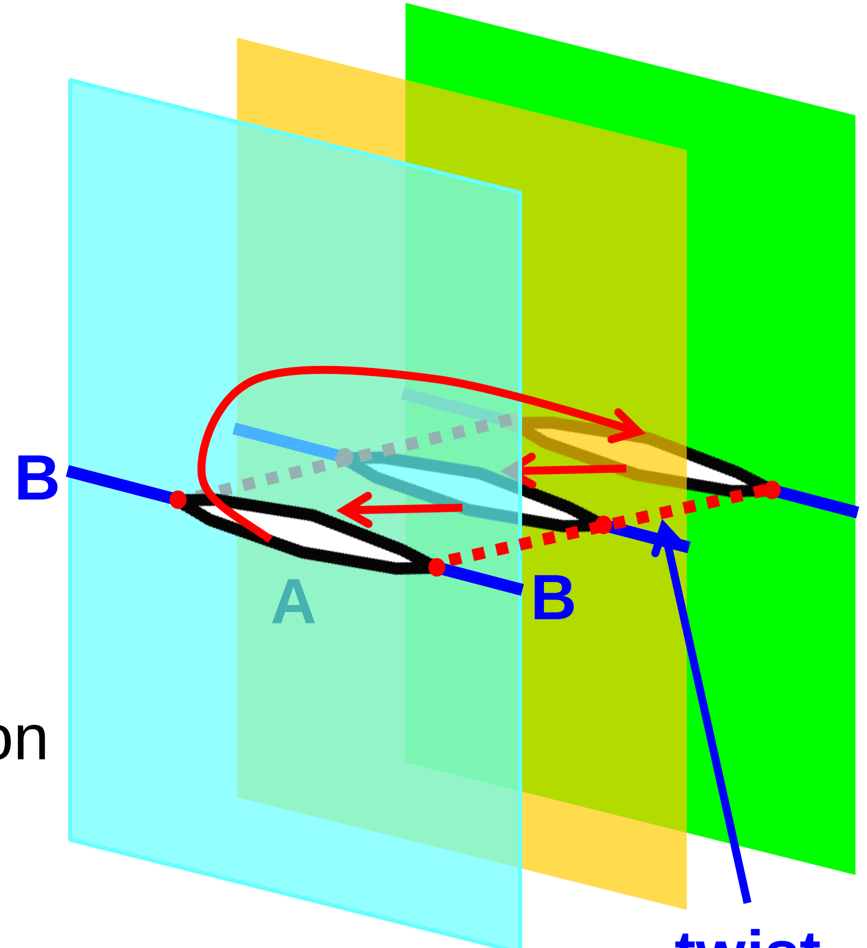
➡

evaluate euclidean partition function
on n-fold cover of original space

or

evaluate euclidean partition function for
n copies of field theory (on single copy of space) with
twist operator inserted at boundary of region **A**



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- "smooth it out!" $\longrightarrow$ find smooth bulk solution for new boundary metric; use conformal symmetry to "unwrap" singularity; (particularly "simple" for d=2: all boundary metrics locally conformally flat, all bulk sol's locally AdS_3)

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(Casini, Huerta & RM)

- need another calculation with simpler holographic translation*

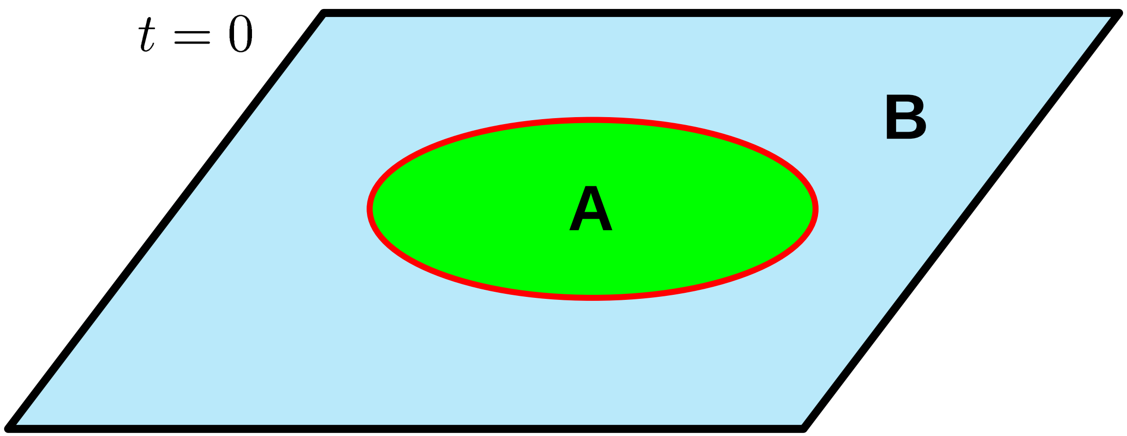
(*realizing "smooth it out!" strategy in disguise)

Calculating Entanglement Entropy:

(Casini, Huerta & RCM)

- take CFT in d-dim. flat space and choose $\Sigma = S^{d-2}$ with radius R

→ entanglement entropy: $S_{EE} = -Tr\left[\rho_A \, \log \rho_A\right]$



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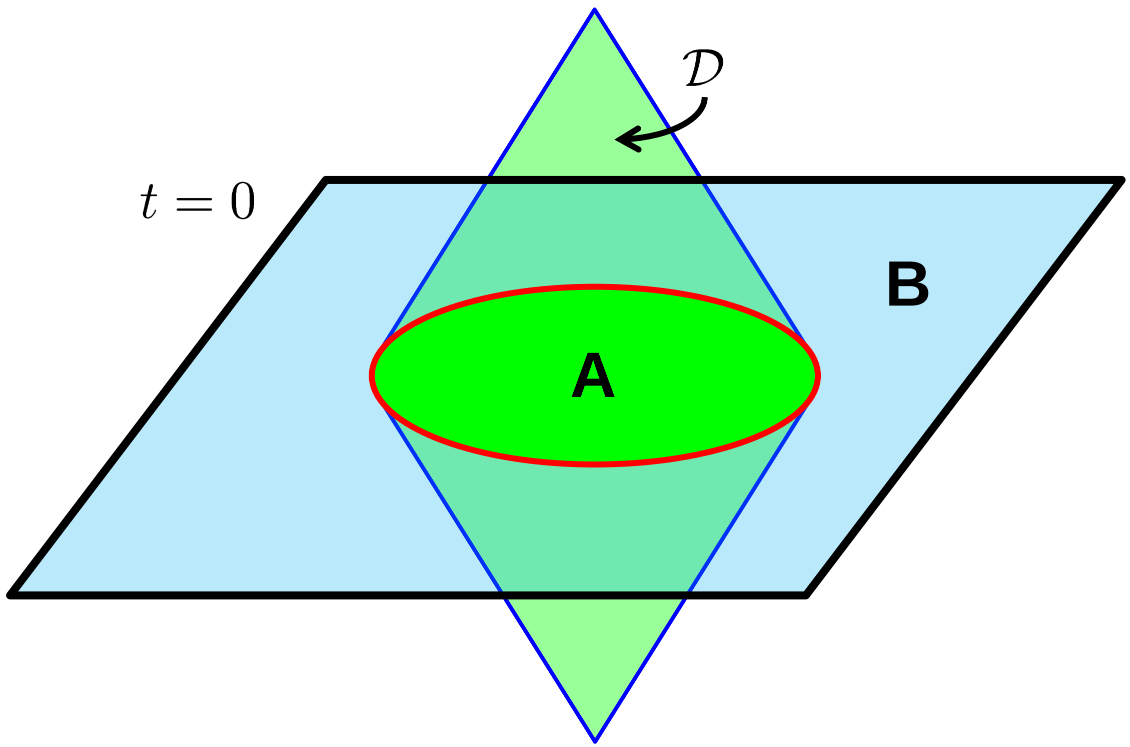

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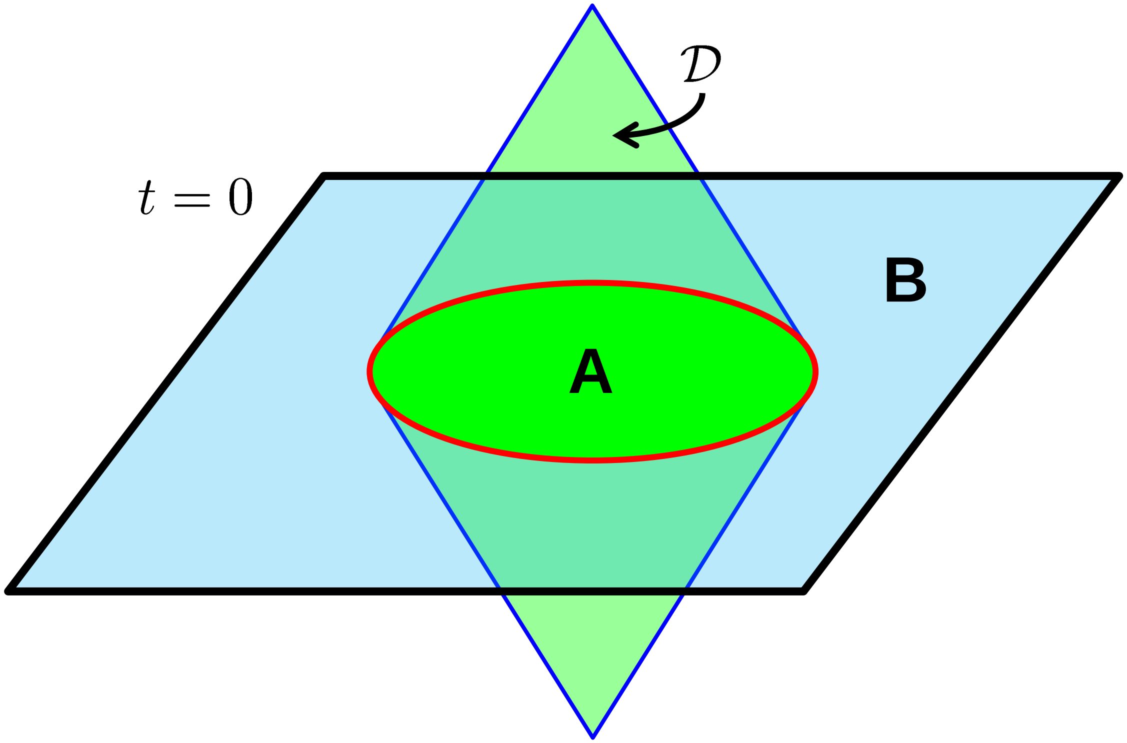

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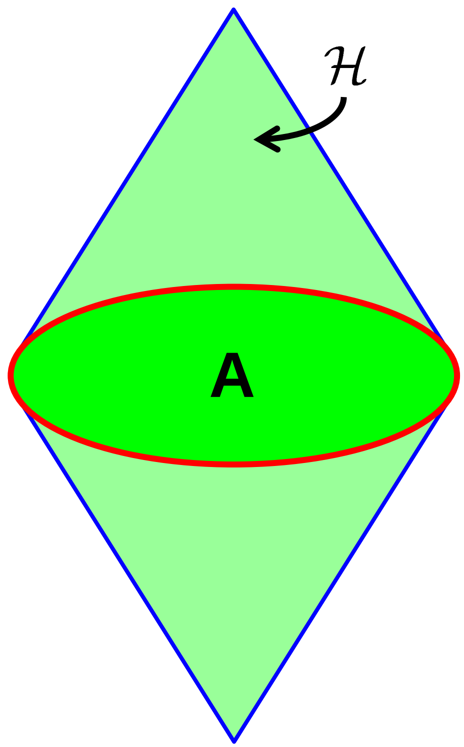

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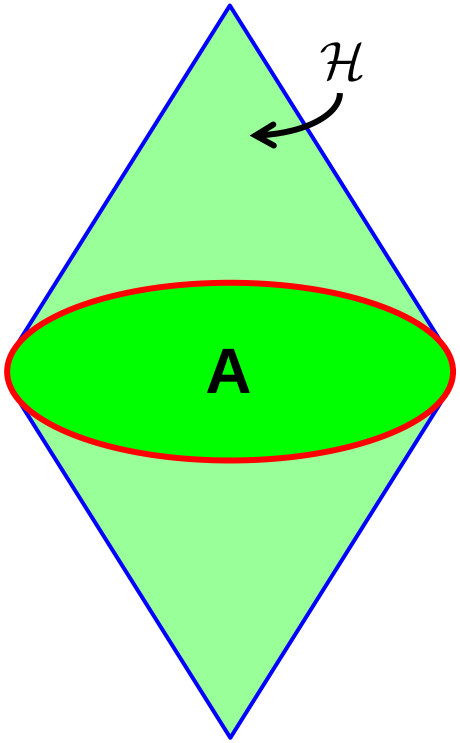

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- for CFT: $\rho_{thermal} = U\,\rho_A\,U^{-1}$ → $S_{EE} = S_{thermal}$

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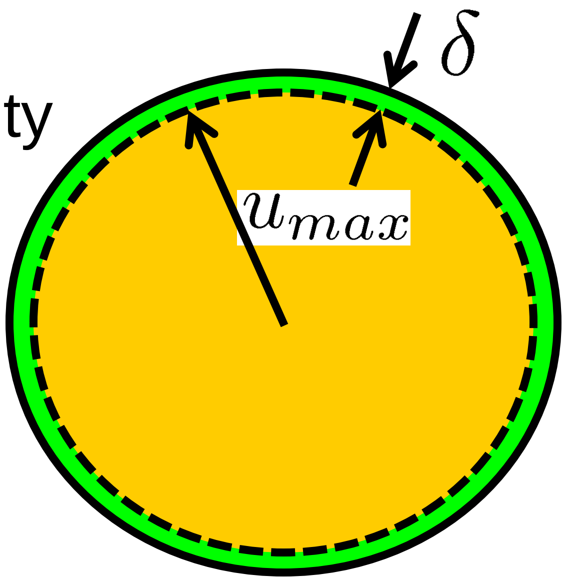
$$\boxed{S_{EE} = S_{thermal}}$$

- note both sides of equality are divergent

→ $S_{thermal}$ sums constant entropy density over infinite volume

- conformal map takes original UV cut-off to IR cut-off on H^{d-1}

$$u_{max} \;\simeq\; R/\delta$$



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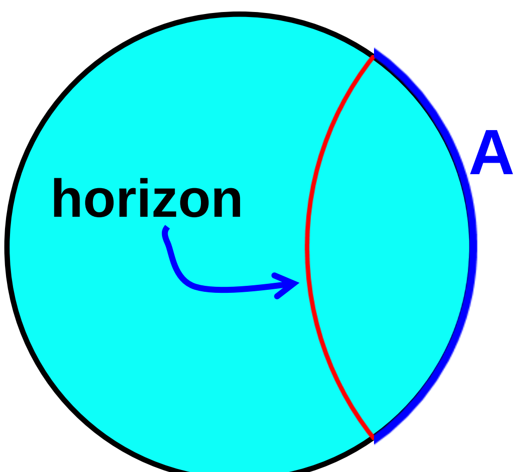

- only need to find appropriate black hole

→ topological BH with hyperbolic horizon which intersects ∂A on AdS boundary

(Aminneborg et al; Emparan; Mann; . . .)

$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$

- desired "black hole" is a hyperbolic foliation of empty AdS space

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 \; - dt^2 + d\vec{x}^2\right)$$

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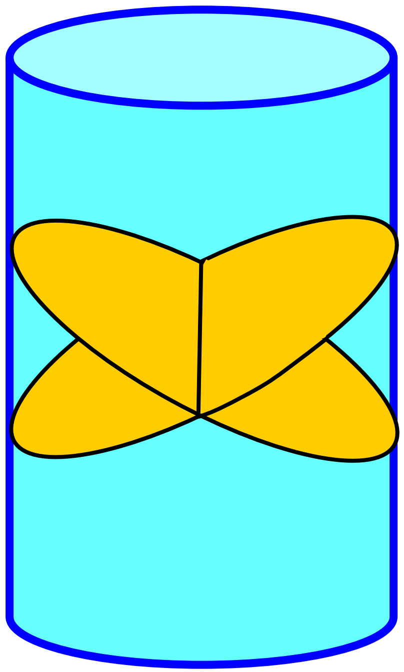
• Rindler coordinates of AdS space:

"Rinder-AdS coordinates"



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where a_d^* contains all of the couplings from the gravity theory

eg, $a_d^* = \dfrac{\pi^{d/2}}{\Gamma\left(d/2\right)} \dfrac{L^{d-1}}{\ell_{\mathrm{P}}^{d-1}}$ for Einstein gravity

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where a_d^* = central charge for “A-type trace anomaly”

for even d

= entanglement entropy defines effective central charge

for odd d

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intersection with standard regulator surface: $z_{min} = \delta$

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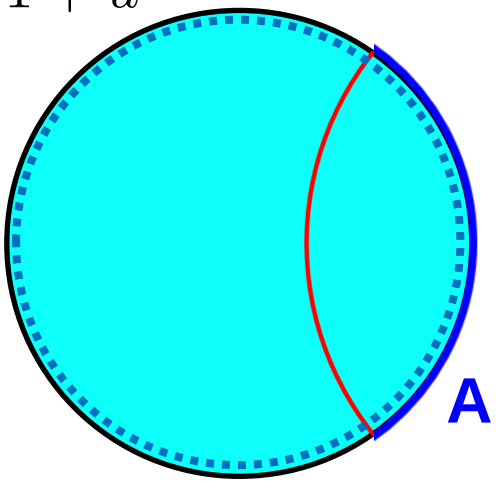

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$$ds^2 = \frac{du^2}{1+u^2} + u^2\, d\Omega_2^{d-2}$$

intersection with standard regulator surface: $z_{min} = \delta$

$$S = a_d^* \frac{4\pi^{\frac{d-3}{2}}}{(d-2)\Gamma\left(\frac{d-1}{2}\right)} \left(\frac{R}{\delta}\right)^{d-2} + \cdots$$

“area law” for d-dimensional CFT

$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$

- desired "black hole" is a hyperbolic foliation of empty AdS space

$$ds^2 = \frac{L^2\, d\rho^2}{(\rho^2 - L^2)} - \frac{\rho^2 - L^2}{R^2}\, d\tau^2 + \rho^2\, d\Sigma_2^{d-1} \quad \longrightarrow \quad T = \frac{1}{2\pi R}$$

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universal contributions:

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\, a_d^* \, \log\left(2R/\delta\right) + \cdots \quad \text{for even d}$$

$$\cdots + (-)^{\frac{d-1}{2}} 2\pi\, a_d^* + \cdots \quad \text{for odd d}$$

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- discussion extends to case with background: $R^{1,d-1} \to R \times S^{d-1}$
- for Einstein gravity, coincides with Ryu & Takayanagi result and horizon (bifurcation surface) coincides with R&T surface

→ no extremization procedure?!? but bifur. surface is extremal

- applies for classical bulk theories beyond Einstein gravity
- can imagine calculating “quantum” corrections (eg, Hawking rad)

"smooth it out":

- consider Euclidean version of previous calculation

- conformal mapping for spherical entangling surface

→ Euclidean version gives one-to-one map: $R^d \leftrightarrow S^1 \times H^{d-1}$

→ with $\Delta\tau_E = n/T_0 = 2\pi R\, n\ (n \in \mathbb{Z})$ get n-fold cover of R^d

$$S^1 \times H^{d-1}: \qquad ds^2 = d\tau_E^2 + R^2\,\left(du^2 + \sinh^2\! u\, d\Omega_{d-2}^2\right)$$

coord. transformation: $\exp\left(-u - i\tau_E/R\right) = \dfrac{R - r - it_E}{R + r + it_E}$

$$\left[R^d\right]_n: \qquad ds^2 = \Omega^2\,\left[\,dt_E^2 + dr^2 + r^2\, d\Omega_{d-2}^2\,\right]$$

↑ *n*-fold cover

$$\Omega^2 = \frac{4R^4}{(R^2 - r^2 + t_E^2)^2 + 4r^2 t_E^2}$$

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- (Euclidean) hyperbolic black holes provide smooth bulk metric for any n!!
- special case: symmetry emerges in certain conformal frame

Conclusions:

- AdS/CFT correspondence (gauge/gravity duality) has proven an excellent tool to study strongly coupled gauge theories

- holographic entanglement entropy is part of an interesting dialogue has opened between string theorists and physicists in a variety of fields (eg, condensed matter, nuclear physics, . . .)

- potential to learn lessons about issues in boundary theory
  eg, readily calculate Renyi entropies for wide class
  of theories in higher dimensions

- potential to learn lessons about issues in bulk gravity theory
  eg, holographic entanglement entropy may give new
  insight into quantum gravity or emergent spacetime

Lots to explore!