Deriving the entangling surface in higher curvature duals

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Based on JHEP10(2014)130 (Arxiv:-1405.3511)
with Menika Sharma
Also partially based on JHEP 1308 (2013) 012 and JHEP 1401 (2014) 021 with Kaviraj,Sinha, Sharma.

Outline

– Introduction

– Lewkowycz-Maldacena Interpretation- Generalized Entropy

– High curvature duals

– First principle derivation of the entangling surface equation

– The “Mismatch”

– Cosmic String interpretation

– Conclusion

Introduction

– In General Relativity black hole entropy (for Einstein Gravity) is given by the famous "Area Law".

– For a D+2 dimensional space-time the horizon entropy is:

$$S_{BH} = \int d^D x \sqrt{h}$$

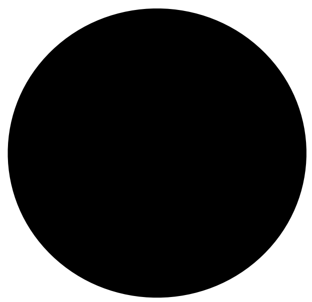
– Ryu-Takayanagi (2006) (time independent) :
For D dimensional entangling region at the boundary of D+2 dimensional bulk space-time.

$$\text{AdS} \longleftrightarrow \text{CFT}$$

$$\boxed{\frac{(\text{Area})_{\text{D}}}{4\,G}} \longleftrightarrow \boxed{-Tr(\rho \ln \rho)}$$

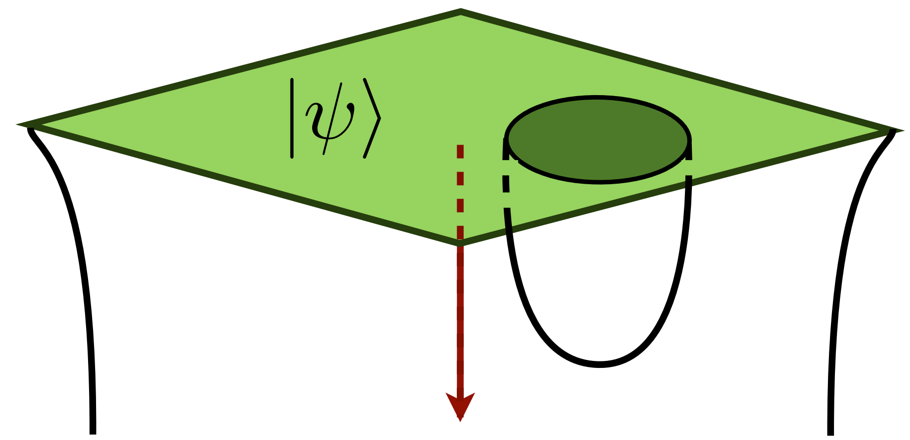
Lewkowycz -Maldacena Interpretation

(JHEP 1308(2013)090)



$$S = \frac{(\text{Area})_{D-2}}{4G}$$

Bekenstein,
Hawking



Ryu, Takayanagi

Maldacena, Lewkowycz

(Camps Mainz '14)

(Bombelli, Koul, Lee, Sorkin, Binachi, Myers …. and many more exploring the relation between black hole entropy with the entanglement entropy)

– Concept of black hole entropy is being generalized for the cases with no **time-like killing symmetry**.

– For U(1) symmetry,

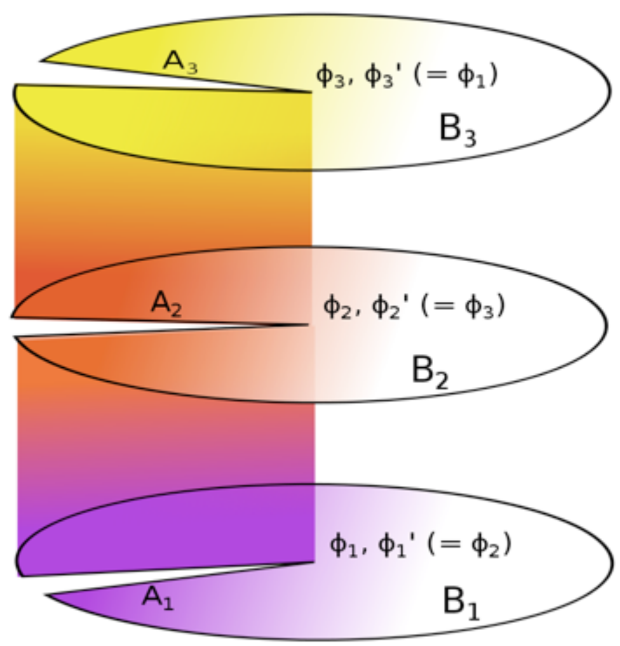
$$Log(\mathcal{Z}(\beta)) = -S_{gravity}$$

- When there is no U(1) symmetry in time direction,

$$Log(Tr\,\hat{\rho}) = -S_{gravity}$$

- Replica trick tells us,

$$Tr(\rho\, Log\, \rho) = \lim_{n\to 1} \frac{Log(Tr\,\rho^n)}{n-1}$$



(Covered in details by Justin and Myers)

Generalized Gravitational Entropy

– Entanglement entropy from Replica method : $S = -\partial_n \left\{ Log \, \frac{tr(\rho^n)}{tr(\rho)^n} \right\}|_{n=1}$

– Holographic Input : $\frac{tr \rho^n}{tr(\rho)^n} \approx \frac{\mathcal{Z}_n(M_n)}{\mathcal{Z}_1(M_1)^n} \approx \frac{e^{-I_n(B_n)}}{e^{-nI_1(B_1)}}$

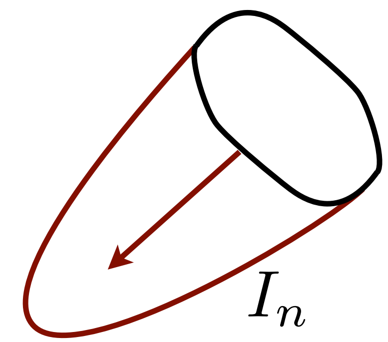
– Finally we get the Generalized Entropy : $S = \partial_n \left(I_n - n \, I_1 \right)|_{n=1}$

I_n corresponds to a dual geometry with a time period $2\pi n$

and

$$\Psi(0) = \Psi(2\,\pi)$$

- I_n corresponds to a regular geometry



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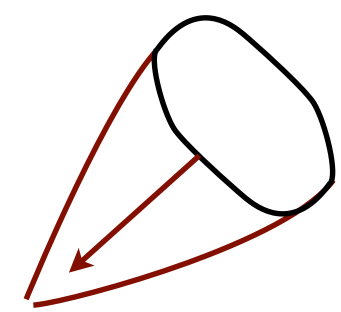
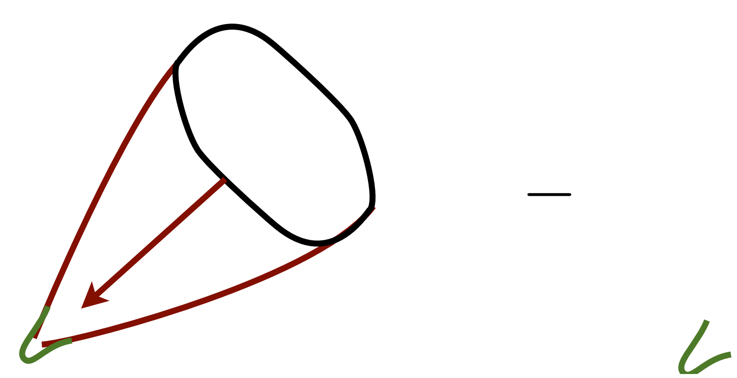
$n\, I_1$ corresponds to a conical geometry
Can be thought of as "n=1" solution
but with a periodicity $2\pi n$

$$\int_0^{2\,\pi\, n} d\tau = n \int_0^{2\,\pi} d\tau$$



- But the "tip" cut off - Think of it as a regularized cone.



– Conical singularity appears (artifact of extending the replica symmetry into the bulk) but they are "mild" singularity Gravitational Action remains finite.

– Further claim:- Bulk equation motion has to be satisfied near the conical singularity. Demanding this we get the correct "extremal surface" equation.

– Once we get the extremal surface equation we can solve it and evaluate the entropy formula on that surface and Voilà !!!

One of the acid tests for the holographic entropy functionals will be that-extremal surface equation coming from minimizing them should match with what comes from the "Generalized Entropy" method. Rest of the talk will be focused on exploring this issue in some details.

Parametrizing the "Cone"

We start with a most general form of the metric:-

$$ds^2 = e^{2\rho(z,\bar{z})}\{dz\, d\bar{z} + e^{2\rho(z,\bar{z})}\Omega(\bar{z}\, dz - z\, d\bar{z})^2 + (g_{ij} + \mathcal{K}_{rij}x^r + \mathcal{Q}_{rsij}x^r x^s)dy^i dy^j$$
$$2e^{2\rho(z,\bar{z})}(\mathcal{A}_i + \mathcal{B}_{ri}x^r)(\bar{z}dz - zd\bar{z})dy^i + \cdots$$

– where,
$$\rho = -\frac{\epsilon}{2}Log(z\bar{z}), \quad \epsilon = n - 1$$
$$r, s = 1, 2 \;\&\; x^1 = z\,, x^2 = \bar{z}$$

In cartesian co-ordinate, $z = w_1 + i\, w_2$, $\bar{z} = w_1 - i\, w_2$

This bulk space-time has $\mathcal{Z}_n$. Orbifolding by this symmetry leads to a fixed point - a codimension-2 surface with a conical deficit.

Entangling surface can be identified with this codimension-2 surface in $n \to 1$ limit characterized by $w_1 = w_2 = 0.$

(LM,Dong,Camps '13, AB Sharma '14)

Einstein Gravity:- As a Warm up

– We consider the Einstein gravity first. Demanding that the bulk equation of motion is satisfied near the conical singularity we will see that one can obtain correct extremal surface equation.

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + g_{\alpha\beta}\Lambda = 0$$

– Solve it around the tip of the cone.
And in $\epsilon \to 0 \,, z(\bar{z}) \to 0$ we extract the singular piece,

$$\boxed{R_{zz} - \frac{1}{2} g_{zz} \Rightarrow \frac{\epsilon}{z} \mathcal{K}_z}$$

(same for $\bar{z}$ component)

– Here we have taken z=0 limit first. (heavily dependent on the limiting procedure, will be crucial for higher derivative case)

- This matches with what comes from minimizing the area functional for Einstein case.

$$S_{EE} = \int d^D x \sqrt{h}$$

$$h_{ij} = e_i^{\mu} e_j^{\nu} g_{\mu\nu}$$

- "g" is the bulk metric and "h" is the induce metric on the surface.
- Variation of the induce metric is encoded in the variation of the tangent vector.

$$\delta e_i^{\mu} = n_s^{\mu} \nabla_i \zeta^s + e_j^{\mu} \mathcal{K}_{si}^{j} \zeta^s$$

- Finally we get, $\delta S_{EE} = \int d^D x \, \mathcal{K}_s \, \zeta^s$
- This gives the well known equation for the extremal surface for Einstein gravity

$$\boxed{\mathcal{K}_s = 0}$$

- **Matches perfectly with the "Generalized Entropy" result.**

Higher derivative Gravity

- We will consider R^2 theory. This is sufficient for capturing all the main issues .
- Entanglement area functionals for higher derivative theories containing only polynomials of curvature tensors are first proposed by Dong and Camps `13.

$$S_{EE} = 2\pi \int d^d y \sqrt{h} \left\{ \frac{\partial \mathcal{L}}{\partial R_{z\bar{z}z\bar{z}}} + \sum_\alpha \left(\frac{\partial^2 \mathcal{L}}{\partial R_{zizj} \partial R_{\bar{z}k\bar{z}l}} \right)_\alpha \frac{8 \mathcal{K}_{zij} \mathcal{K}_{\bar{z}kl}}{q_\alpha + 1} \right\}$$

(Recently it has been extended for theories containing derivative of curvature tensors in ArXiv:1411.5579 by Miao and Guo)

- To use this formula, perform the functional differentiation,

$$\begin{aligned} R_{pqij} &= \tilde{R}_{pqij} + \mathcal{K}_{pjk}\mathcal{K}^k_{qi} - \mathcal{K}_{pik}\mathcal{K}^k_{qj} \\ R_{piqj} &= \tilde{R}_{piqj} + \mathcal{K}_{pjk}\mathcal{K}^k_{qi} - \mathcal{Q}_{pqij} \\ R_{ijkl} &= \mathcal{R}_{ijkl} + \mathcal{K}_{pil}\mathcal{K}pjk - \mathcal{K}_{pij}\mathcal{K}_{pkl} \end{aligned} \qquad \mathcal{Q} = \frac{1}{2}\partial_p \partial_q g_{ij}|_\Sigma$$

and then use the above relations

- q_α defined as number of $\mathcal{Q}_{zzij}$, $\mathcal{Q}_{\bar{z}\bar{z}ij}$ plus one half of $\mathcal{K}_{pqij}$, R_{pqri} R_{pijk} .

Entropy functional for Higher Curvature theory- Continue

$$\frac{\partial \mathcal{L}}{\partial R_{z\bar{z}z\bar{z}}} \longrightarrow \text{Wald Term}$$

$$\left(\frac{\partial^2 \mathcal{L}}{\partial R_{zizj}\,\partial R_{\bar{z}k\bar{z}l}}\right)\mathcal{K}_{ij}\mathcal{K}_{kl} \longrightarrow \text{“Anomalous term”}$$

$$\longrightarrow \text{Vanishes on Bifurcation surface}$$

For R^2 theory :- $S_{R^2} = \frac{1}{\ell_p^3}\int d^5x\sqrt{g}\left[R + \frac{12}{L^2} + \frac{L^2}{2}(\lambda_1\, R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + \lambda_2\, R_{\alpha\beta}R^{\alpha\beta} + \lambda_3 R^2)\right]$

We get the corresponding area functional :-

$$\boxed{S_{EE} = \frac{2\,\pi}{\ell_p^3}\int d^3x\sqrt{h}\left[1 + \frac{L^2}{2}\Big(2\lambda_1\,\{R_{\alpha\beta\gamma\delta}n_r^{\alpha}n_s^{\beta}n_r^{\gamma}n_s^{\delta} - \mathcal{K}_{sij}\mathcal{K}_s^{ij}\} + \lambda_2\,\{R_{\alpha\beta}n_s^{\alpha}n_s^{\beta} - \frac{1}{2}\mathcal{K}_s\mathcal{K}_s\} + 2\lambda_3 R\Big)\right]}$$

(fursaev, patrushev, solodukhin,’13, Dong-Camps’13)

Gauss-Bonnet Gravity

- For this we put , $\lambda_1 = \lambda_3 = \lambda \,,\; \lambda_2 = -4\,\lambda$

-Formula simplifies remarkably because of the Gauss-Codazzi identity.

$$\mathcal{R} = R_{\alpha\beta\gamma\delta} n_r^{\alpha} n_s^{\beta} n_r^{\gamma} n_s^{\delta} - \mathcal{K}_{sij}\mathcal{K}_s^{ij} = 2\,R_{\alpha\beta} n_s^{\alpha} n_s^{\beta} + \mathcal{K}_s \mathcal{K}_s + R$$

-We get the famous Jacobson-Myers entropy functional.

$$\boxed{S_{EE} = \frac{2\,\pi}{\ell_p^3} \int d^3x \sqrt{h} \left(1 + \lambda L^2 \mathcal{R}\right)}$$

(Jacobson-Myers'95, Hung,Myers, Smolkin '10)

-By varying this we get the following extreme surface equation,

$$\mathcal{K} + \lambda L^2 (\mathcal{R}\mathcal{K} - 2\,\mathcal{R}_{ij}\mathcal{K}^{ij}) = 0$$

(AB, Kaviraj,Sinha '13)

- Let's see if we can derive this from "generalized entropy" method.

Extremal Surface equation For Gauss-Bonnet Theory

- We evaluate Gauss-Bonnet equation (a two derivative one) of motion near the conical singularity and extract all the divergences.

- zz-component :-

$$\boxed{\frac{\epsilon}{z}\left[\lambda(\mathcal{R}\mathcal{K}-2\mathcal{K}_{ij}\mathcal{R}^{ij})\right]+\frac{\epsilon}{z}\left[e^{-\rho(z,\bar{z})}\lambda\{-\mathcal{K}^3+3\mathcal{K}_2\mathcal{K}-2\mathcal{K}_3\}\right]}$$

- zi-component :-

$$\boxed{\begin{aligned}&\frac{\epsilon}{z}\Big[e^{-2\rho(z,\bar{z})}\lambda\Big\{2\mathcal{K}\nabla_i\mathcal{K}-2\mathcal{K}\nabla_j\mathcal{K}_i^j-2\mathcal{K}_i^j\nabla_j\mathcal{K}+2\mathcal{K}_{ij}\nabla_k\mathcal{K}^{kj}-\\&\qquad 2\mathcal{K}_{kj}\nabla_i\mathcal{K}^{kj}+2\mathcal{K}_{jk}\nabla^j\mathcal{K}_i^k\Big\}\Big].\end{aligned}}$$

- ij-component :-

$$\boxed{\begin{aligned}&\frac{4\epsilon}{z}\Big[e^{-4\rho(z,\bar{z})}\lambda\Big\{2\mathcal{K}_{ik}\mathcal{K}^{kl}\mathcal{K}_{lj}+h_{ij}\mathcal{K}\mathcal{K}_2-\mathcal{K}_{ij}\mathcal{K}_2-h_{ij}\mathcal{K}_3-\mathcal{K}\mathcal{K}_{ik}\mathcal{K}_j^k-4h_{ij}\mathcal{K}\mathcal{Q}_{zz}\\&\qquad+4h_{ij}\mathcal{K}_{kl}\mathcal{Q}_{zz}^{kl}-8\mathcal{K}_{ki}\mathcal{Q}_{zzj}^k+4\mathcal{K}_{ij}\mathcal{Q}_{zz}+4\mathcal{K}\mathcal{Q}_{zzij}\Big\}\Big]+\\&\frac{2\epsilon^2}{z^2}\Big[e^{-4\rho(z,\bar{z})}\lambda\Big\{2\mathcal{K}_{ij}\mathcal{K}-2\mathcal{K}_{ik}\mathcal{K}_j^k-h_{ij}\mathcal{K}^2+h_{ij}\mathcal{K}_2\Big\}\Big]\end{aligned}}$$

- Now we have to set all the divergences in all the components to zero. This will prove quite tricky in this case.

(AB, Sharma '14)

Limiting procedure and the "ambiguity"

- First we try to take the limit as in the Einstein case i,e z=0 limit first.
- We specialize to the AdS background. So "Codazzi-Mainardi" relation gives us,

$$\nabla_k \mathcal{K}_{rij} = \nabla_i \mathcal{K}_{rkj}$$

$$\frac{\epsilon}{z}\Big[e^{-2\rho(z,\bar{z})}\lambda\Big\{2\mathcal{K}\nabla_i\mathcal{K} - 2\mathcal{K}\nabla_j\mathcal{K}^j_i - 2\mathcal{K}^j_i\nabla_j\mathcal{K} + 2\mathcal{K}_{ij}\nabla_k\mathcal{K}^{kj} - 2\mathcal{K}_{kj}\nabla_i\mathcal{K}^{kj} + 2\mathcal{K}_{jk}\nabla^j\mathcal{K}^k_i\Big\}\Big].$$

- Also we get, $\mathcal{Q}_{zzij} = \frac{1}{4}\mathcal{K}_{ik}\mathcal{K}^k_j$ this kills a portion of divergences in "ij" component.

$$\frac{4\epsilon}{z}\Big[e^{-4\rho(z,\bar{z})}\lambda\Big\{2\mathcal{K}_{ik}\mathcal{K}^{kl}\mathcal{K}_{lj} + h_{ij}\mathcal{K}\mathcal{K}_2 - \mathcal{K}_{ij}\mathcal{K}_2 - h_{ij}\mathcal{K}_3 - \mathcal{K}\mathcal{K}_{ik}\mathcal{K}^k_j - 4h_{ij}\mathcal{K}\mathcal{Q}_{zz} + 4h_{ij}\mathcal{K}_{kl}\mathcal{Q}^{kl}_{zz} - 8\mathcal{K}_{ki}\mathcal{Q}^k_{zzj} + 4\mathcal{K}_{ij}\mathcal{Q}_{zz} + 4\mathcal{K}\mathcal{Q}_{zzij}\Big\}\Big] +$$
$$\frac{2\epsilon^2}{z^2}\Big[e^{-4\rho(z,\bar{z})}\lambda\Big\{2\mathcal{K}_{ij}\mathcal{K} - 2\mathcal{K}_{ik}\mathcal{K}^k_j - h_{ij}\mathcal{K}^2 + h_{ij}\mathcal{K}_2\Big\}\Big]$$

– Remaning divergences are:-

- zz-component :- $\boxed{\frac{\epsilon}{z}\Big[\lambda(\mathcal{R}\mathcal{K}-2\mathcal{K}_{ij}\mathcal{R}^{ij})\Big]+\frac{\epsilon}{z}\Big[e^{-\rho(z,\bar{z})}\lambda\{-\mathcal{K}^3+3\mathcal{K}_2\mathcal{K}-2\mathcal{K}_3\}\Big]}$

– and

-ij-component :- $\boxed{\frac{2\,\epsilon^2\,\lambda\,e^{-4\rho(z,\bar{z})}}{z^2}\Big[2\mathcal{K}_{ij}\mathcal{K}-2\mathcal{K}_{ik}\mathcal{K}^k_j-h_{ij}\mathcal{K}^2+h_{ij}\mathcal{K}_2\Big]}$

– Setting them simultaneously to zero (and of-course adding the Einstein piece) we get, two condition,

$$\boxed{\mathcal{K}+L^2\lambda(\mathcal{R}\mathcal{K}-2\mathcal{K}^{ij}\mathcal{R}_{ij})=0}$$

– Agrees with Jacobson-Myers functional. But wait, we get another "constraint" from the subleading divergences !!!

$$\boxed{\lambda(-\mathcal{K}^3+3\mathcal{K}_2\mathcal{K}-2\mathcal{K}_3)=0}$$

– System becomes over constrained. To satisfy both,

$$\boxed{(1-2f_\infty\lambda)\,\mathcal{K}+\alpha\lambda L^2(\mathcal{K}^3-3\mathcal{K}\mathcal{K}_2+2\mathcal{K}_3)\;=0.}$$

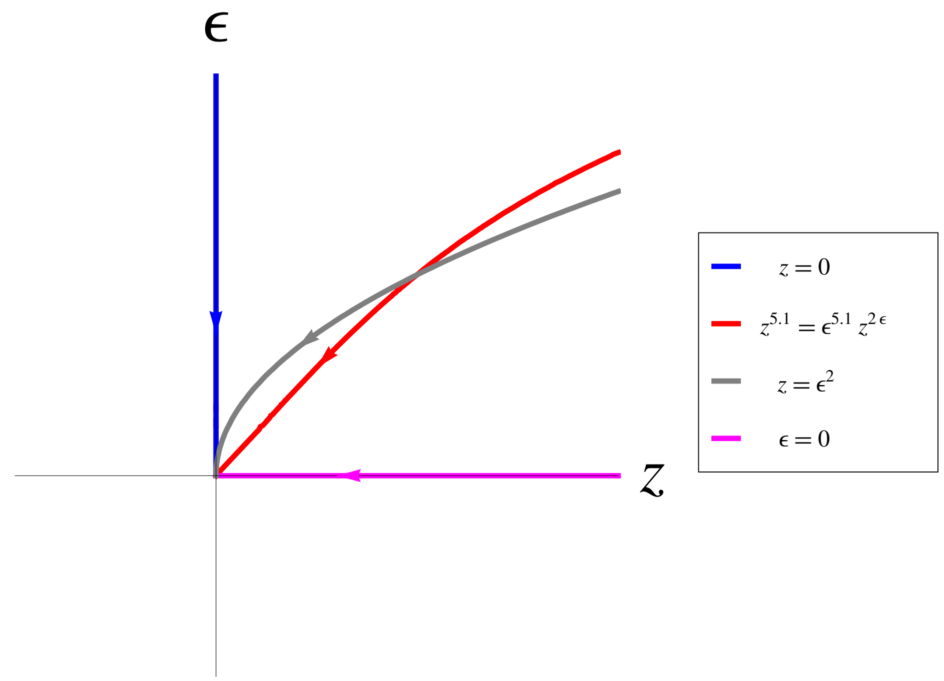
(AB, Sharma '14, Chen et al '13)

– α is a variable which can take any value.

- Now the point is that, can we do better than this?

- There are infinite choice. So there lies the "ambiguity"

- One can think of taking the limit as choosing a specific path in the ϵ, z plane.



- For Gauss-Bonnet gravity we are lucky , there exist a specific path in which the limit has to be taken. Then we will get the correct result.

- If we demand $\frac{\epsilon}{z} = \frac{1}{\hat{\epsilon}}$ and $z^{2\epsilon} = \left(\hat{\epsilon}\right)^{1+v}$

- From this two equation we get, ("v" is small)

$$z^{2\epsilon} \approx \left(\frac{z}{\epsilon}\right)^{1+v} \qquad v > 0$$

$$\frac{\epsilon}{z} e^{-2\rho(z,\bar{z})} \Longrightarrow \left(\frac{z}{\epsilon}\right)^{v}$$

(AB, Sinha, Sharma'13, AB, Sharma '14)

$$\frac{\epsilon}{z} e^{-4\rho(z,\bar{z})} \Longrightarrow \left(\frac{z}{\epsilon}\right)^{1+v}$$

- Now in the $z \to 0$ limit all this divergences vanish leaving only the $\frac{\epsilon}{z}$ part. Setting it to zero we get the correct surface equation.

$$\mathcal{K} + \lambda L^2 (\mathcal{R}\mathcal{K} - 2\mathcal{R}_{ij}\mathcal{K}^{ij})$$

- So for Gauss-Bonnet we have a way out.

General R^2 Theory

- Now let's test this limiting procedure for general R^2 theory

- Surface equation derived by minimizing the entropy functional,

$$\boxed{S_{EE} = \frac{2\pi}{\ell_p^3}\int d^3x\sqrt{h}\left[1+\frac{L^2}{2}\left(2\lambda_1\left\{R_{\alpha\beta\gamma\delta}n_r^\alpha n_s^\beta n_r^\gamma n_s^\delta - \mathcal{K}_{sij}\mathcal{K}_s^{ij}\right\}+\lambda_2\left\{R_{\alpha\beta}n_s^\alpha n_s^\beta - \frac{1}{2}\mathcal{K}_s\mathcal{K}_s\right\}+2\lambda_3 R\right)\right]}$$

$$\begin{aligned}\mathcal{K} + L^2\{\lambda_3(\mathcal{R}\mathcal{K} - 2\mathcal{R}^{ij}\mathcal{K}_{ij} - \mathcal{K}^3 + 3\mathcal{K}\mathcal{K}_2 - 2\mathcal{K}_3 - \tfrac{18\,f_\infty}{L^2}\mathcal{K})+\\ \lambda_2(\tfrac{1}{2}\nabla^2\mathcal{K} - \tfrac{1}{4}\mathcal{K}^3 + \tfrac{1}{2}\mathcal{K}\mathcal{K}_2 - \tfrac{11\,f_\infty}{2\,L^2}\mathcal{K})+\\ \lambda_1(2\nabla^2\mathcal{K} - \mathcal{K}\mathcal{K}_2 + 2\mathcal{K}_3 - \tfrac{4\,f_\infty}{L^2}\mathcal{K})\} = 0\end{aligned}$$

(AB, Sharma '14)

Now let's see what we get from the "Generalized Entropy" method using the limiting procedure discussed before. This will be our next goal.

- This surface equation straightforwardly reduces to the Jacobson-Myers case if we put the Gauss-Bonnet values for couplings after using Gauss-Codazzi relations.

Surface equation from Generalized Entropy method for R^2 theory

- Bulk equation of motion for this theory is not any more a two derivative one.

- We need third terms $\mathcal{O}(z^3, \bar{z}^3)$ in the metric.

$$ds^2 = e^{4\rho(z,\bar{z})}\Delta_{pqrst}x^p x^q x^r dx^s dx^t + \mathcal{W}_{rspij}x^r x^s x^p dy^i dy^j + 2e^{2\rho(z,\bar{z})}\mathcal{C}_{rsi}x^r x^s(\bar{z}dz - zd\bar{z})dy^i$$

- Metric coefficients can be found by calculating various curvature quantities. For eg. $e^{4\rho(z,\bar{z})}\Delta_{pqrst} \equiv -1/6\partial_p(R_{\mu\nu\rho\sigma}n_q^\mu n_r^\nu n_s^\rho n_t^\sigma)$

- Also, $R_{zi\bar{z}j}\Big|_{(z=0,\bar{z}=0)} = \frac{1}{2}e^{2\rho(z,\bar{z})}\mathcal{F}_{ij} - 2e^{2\rho(z,\bar{z})}\mathcal{A}_i\mathcal{A}_j + \frac{1}{4}\mathcal{K}_{zik}\mathcal{K}^k_{\bar{z}j} - \frac{1}{2}\mathcal{Q}_{z\bar{z}ij}$

- Using AdS background we get from this, $\mathcal{A}_i\mathcal{A}_j\mathcal{K}^{ij} = \frac{f_\infty \mathcal{K}}{4\,L^2} + \cdots$ and $\mathcal{A}_i\mathcal{A}^i\mathcal{K} = \frac{3f_\infty \mathcal{K}}{4\,L^2} + \cdots$

- Again, $\partial_z R_{\bar{z}\bar{z}}\Big|_{(z=0,\bar{z}=0)} = -\mathcal{W} + 2e^{2\rho(z,\bar{z})}\mathcal{K}^{ij}\mathcal{A}_i\mathcal{A}_j - 2e^{2\rho(z,\bar{z})}\Omega\mathcal{K} + \cdots$

- Using AdS background we finally get,

$$\boxed{\mathcal{W} = \frac{2e^{2\rho(z,\bar{z})}f_\infty\,\mathcal{K}}{3\,L^2} + \cdots\,.}$$

Continue……

– Now we evaluate the bulk equation of motion. And list all possible divergences.

- zz-component :-

$$\frac{\epsilon}{z}\Big[-\tfrac{1}{2}(\lambda_2+4\lambda_3)\nabla^2\mathcal{K}+(2\lambda_1+\lambda_2+2\lambda_3)\nabla^i\nabla^j\mathcal{K}_{ij}+\lambda_3(\mathcal{R}\mathcal{K}-2\mathcal{K}^{ij}\mathcal{R}_{ij})\ +$$
$$4(-2\lambda_1+3\lambda_2+14\lambda_3)\mathcal{K}_{ij}\mathcal{A}^i\mathcal{A}^j-6(\lambda_2+4\lambda_3)\mathcal{K}\mathcal{A}^i\mathcal{A}_i\ +$$
$$8(3\lambda_1+2\lambda_2+5\lambda_3)\mathcal{K}\Omega\Big]\ -$$
$$\boxed{\frac{\epsilon}{z^2}\Big[e^{-2\rho(z,\bar z)}\Big\{(2\lambda_1-\lambda_2-6\lambda_3)\mathcal{K}_2+\tfrac{1}{2}(\lambda_2+4\lambda_3)\mathcal{K}^2+2(\lambda_2+4\lambda_3)\mathcal{Q}\Big\}\Big]}+$$
$$\frac{\epsilon}{z}\Big[e^{-2\rho(z,\bar z)}\Big\{-\lambda_3\mathcal{K}^3+(\lambda_2+7\lambda_3)\mathcal{K}\mathcal{K}_2-2(3\lambda_1+2\lambda_2+6\lambda_3)\mathcal{K}_3\ +$$
$$(6\lambda_1+5\lambda_2+14\lambda_3)\mathcal{K}_{ij}\mathcal{Q}^{ij}-\tfrac{3}{2}(\lambda_2+4\lambda_3)\mathcal{K}\mathcal{Q}\ -$$
$$4(\lambda_2+4\lambda_3)\mathcal{W}\Big\}\Big].$$

Becomes unsuppressed

Appearance of a new type of divergence.

- $z\bar{z}$ -component :- $\frac{2\epsilon}{z}\Big[e^{-2\rho(z,\bar z)}\Big\{(\lambda_3+\tfrac{1}{4}\lambda_2)\mathcal{K}^3+(\lambda_1-\tfrac{3}{2}\lambda_2-7\lambda_3)\mathcal{K}\mathcal{K}_2\ +$

$$2(-2\lambda_1+\lambda_2+6\lambda_3)\mathcal{K}_3+(2\lambda_1-3\lambda_2-14\lambda_3)\mathcal{K}_{kl}\mathcal{Q}^{kl}\ +$$
$$\tfrac{3}{2}(\lambda_2+4\lambda_3)\mathcal{K}\mathcal{Q}+8(\lambda_2+4\lambda_3)\mathcal{W}\Big\}\Big]$$

previously not there

- zi-component :-

$$
\begin{aligned}
\frac{2\epsilon}{z}\Big[e^{-2\rho(z,\bar{z})}\Big\{ & -\tfrac{1}{2}(2\lambda_1+\lambda_2)\mathcal{K}_i^k\nabla_k\mathcal{K}-(3\lambda_1-\lambda_2-6\lambda_3)\mathcal{K}^{kl}\nabla_i\mathcal{K}_{kl}\ - \\
& \tfrac{1}{4}(3\lambda_2+8\lambda_3)\mathcal{K}\nabla_i\mathcal{K}+(5\lambda_1+\lambda_2)\mathcal{K}_i^k\nabla_l\mathcal{K}_k^l-\lambda_1\mathcal{K}\nabla_k\mathcal{K}_i^k\ + \\
& (9\lambda_1+2\lambda_2)\mathcal{K}^{kj}\nabla_k\mathcal{K}_{ji}-(\lambda_2+4\lambda_3)\nabla_i\mathcal{Q}-(4\lambda_1+\lambda_2)\nabla_k\mathcal{Q}_i^k\ - \\
& (10\lambda_1-2\lambda_2-18\lambda_3)\mathcal{A}_i\mathcal{K}_2-\tfrac{1}{2}(3\lambda_2+12\lambda_3)\mathcal{A}_i\mathcal{K}^2\ + \\
& 8(4\lambda_1+\lambda_2)\mathcal{K}_{ij}\mathcal{K}^{jk}\mathcal{A}_k-2(\lambda_2+4\lambda_3)\mathcal{A}_i\mathcal{Q}\Big\}\Big]
\end{aligned}
$$

- ij-component :-

$$
\begin{aligned}
\frac{4\epsilon}{z}\Big[e^{-4\rho(z,\bar{z})}\Big\{ & (\tfrac{1}{3}\lambda_1+\tfrac{1}{4}\lambda_2+\tfrac{2}{3}\lambda_3)h_{ij}\mathcal{K}^3-(7\lambda_1+2\lambda_2+2\lambda_3)\mathcal{K}\mathcal{K}_{ik}\mathcal{K}_j^k\ + \\
& 2(16\lambda_1+4\lambda_2+\lambda_3)\mathcal{K}_{ik}\mathcal{K}^{kl}\mathcal{K}_{lj}-(\lambda_1+3\lambda_2+10\lambda_3)h_{ij}\mathcal{K}\mathcal{K}_2\ - \\
& (3\lambda_1-2\lambda_3)\mathcal{K}_{ij}\mathcal{K}_2-\tfrac{1}{3}(\lambda_1-18\lambda_2-70\lambda_3)h_{ij}\mathcal{K}_3\ + \\
& 2(4\lambda_1+\lambda_2)\mathcal{Q}_{ij}\mathcal{K}+2(\lambda_2+4\lambda_3)h_{ij}\mathcal{K}\mathcal{Q}-8(4\lambda_1+\lambda_2)\mathcal{K}_{ik}\mathcal{Q}_j^k\ - \\
& (\lambda_2+4\lambda_3)\mathcal{K}_{ij}\mathcal{Q}-7(\lambda_2+4\lambda_3)h_{ij}\mathcal{K}_{kl}\mathcal{Q}^{kl}+32(4\lambda_1+\lambda_2)\mathcal{W}_{ij}+ \\
& \boxed{32(\lambda_2+4\lambda_3)h_{ij}\mathcal{W}\Big\}\Big]}
\end{aligned}
$$

effectively has a $e^{-2\rho(z,\bar{z})}$ factor

- We take the limit as before i.e $\quad z^{2\epsilon}=\left(\frac{z}{\epsilon}\right)^{1+v}$
- This removes all the divergences , except …….

(AB, Sharma '14)

The “Mismatch”

- zz-component :- We are left with

$$\frac{\epsilon}{z}\Big[-\tfrac{1}{2}(\lambda_2+4\lambda_3)\nabla^2\mathcal{K}+(2\lambda_1+\lambda_2+2\lambda_3)\nabla^i\nabla^j\mathcal{K}_{ij}+\lambda_3(\mathcal{R}\mathcal{K}-2\mathcal{K}^{ij}\mathcal{R}_{ij})\ +$$
$$4(-2\lambda_1+3\lambda_2+14\lambda_3)\mathcal{K}_{ij}\mathcal{A}^i\mathcal{A}^j-6(\lambda_2+4\lambda_3)\mathcal{K}\mathcal{A}^i\mathcal{A}_i\ +$$
$$8(3\lambda_1+2\lambda_2+5\lambda_3)\mathcal{K}\Omega\Big]\ -$$
$$\frac{\epsilon}{z^2}\Big[e^{-2\rho(z,\bar{z})}\Big\{(2\lambda_1-\lambda_2-6\lambda_3)\mathcal{K}_2+\tfrac{1}{2}(\lambda_2+4\lambda_3)\mathcal{K}^2+2(\lambda_2+4\lambda_3)\mathcal{Q}\Big\}\Big]$$

$$\left(\frac{z}{\epsilon}\right)^{v-1}$$

– Setting $\frac{\epsilon}{z}$ piece to zero and using, $\mathcal{A}_i\mathcal{A}_j\mathcal{K}^{ij}=\dfrac{f_\infty\mathcal{K}}{4\,L^2}+\cdots$ and

$$\mathcal{A}_i\mathcal{A}^i\mathcal{K}=\frac{3f_\infty\mathcal{K}}{4\,L^2}+\cdots$$

$$\Omega=-\frac{f_\infty}{12\,L^2}$$

Adding the Einstein piece

$$\boxed{\mathcal{K}+L^2\{(2\lambda_1+\tfrac{1}{2}\lambda_2)\nabla^2\mathcal{K}+\lambda_3(\mathcal{R}\mathcal{K}-2\mathcal{K}^{ij}\mathcal{R}_{ij})-\mathcal{K}\frac{f_\infty}{L^2}(4\lambda_1-\frac{11}{2}\lambda_2-18\lambda_3)\}=0}$$

Comparing

$$\mathcal{K}+L^2\{\lambda_3(\mathcal{R}\mathcal{K}-2\mathcal{R}^{ij}\mathcal{K}_{ij}-\mathcal{K}^3+3\mathcal{K}\mathcal{K}_2-2\mathcal{K}_3-\tfrac{18\,f_\infty}{L^2}\mathcal{K})+$$
$$\lambda_2(\tfrac{1}{2}\nabla^2\mathcal{K}-\tfrac{1}{4}\mathcal{K}^3+\tfrac{1}{2}\mathcal{K}\mathcal{K}_2-\tfrac{11\,f_\infty}{2\,L^2}\mathcal{K})+$$
$$\lambda_1(2\nabla^2\mathcal{K}-\mathcal{K}\mathcal{K}_2+2\mathcal{K}_3-\tfrac{4\,f_\infty}{L^2}\mathcal{K})\}=0$$

– Everything matches except for these , $\mathcal{K}^3$ terms.

– Setting the subleading divergence to zero we get further constraints,

$$\boxed{(2\lambda_1 - \lambda_2 - 6\lambda_3)\mathcal{K}_2 + \frac{1}{2}(\lambda_2 + 4\lambda_3)\mathcal{K}^2 + 2(\lambda_2 + 4\lambda_3)\mathcal{Q} = 0}$$

- $z\bar{z}$ -component :- $\frac{2\epsilon}{z}\Big[e^{-2\rho(z,\bar{z})}\Big\{(\lambda_3 + \frac{1}{4}\lambda_2)\mathcal{K}^3 + (\lambda_1 - \frac{3}{2}\lambda_2 - 7\lambda_3)\mathcal{K}\mathcal{K}_2 + 2(-2\lambda_1 + \lambda_2 + 6\lambda_3)\mathcal{K}_3 + (2\lambda_1 - 3\lambda_2 - 14\lambda_3)\mathcal{K}_{kl}\mathcal{Q}^{kl} + \frac{3}{2}(\lambda_2 + 4\lambda_3)\mathcal{K}\mathcal{Q} + 8(\lambda_2 + 4\lambda_3)\mathcal{W}\Big\}\Big]$

Divergent

– $\mathcal{K}^3$ terms mismatch.

– There are some problematic divergences in $z\bar{z}$ component

– extra $\frac{1}{z^2}$ divergences.

So there is a "mismatch" and seems that these two methods are not consistent with each other.

Cosmic String (Brane) Interpretation

- Looking for a new way to interpret the result.
- It was first observed by Lewkowycz-Maldacena that equation of motion of a cosmic string is same as the extremal surface equation for Einstein Gravity.
- It produces a space time with a conical defect. For Einstein case the action is just "Nambu Goto" action. So we get

$$\boxed{\mathcal{K} = 0}$$

- The argument has been extended for higher derivative Gravity case (Gauss-Bonnet or more generally Lovelock gravity) by Dong '13. Referred to as "Cosmic Brane".
- Like comic string case it leads to singular space time (ϵ).
- Now suppose consider a brane localized at $z = \bar{z} = 0$ and extends in y^i

Key Idea

Main idea is that the cosmic brane will be a solution of the bulk equation of motion with a non vanishing stress tensor in "ij" directions. As the brane is a localized object , stress tensor will contain delta functions. So by solving the bulk equation of motion in the conical background and identifying the delta functions, one can identify the stress tensor for the brane.

$$\boxed{T_{ij} = \frac{2}{\sqrt{h}} \frac{\delta S}{\delta h_{ij}}}$$

and hence the action which is expected to be the "Entropy Functional" .

– Let us know look at the Gauss-Bonnet theory first.

Cosmic Brane to Entropy Functional

- By the evaluating the Bulk equation of motion in the singular background we identify the following expression as the stress-tensor for Gauss-Bonnet gravity.

$$\delta(z,\bar{z}) = e^{-2\rho(z,\bar{z})}\partial_{\bar{z}}\partial_z\rho(z,\bar{z})$$

$$T_{ij} = \delta(z,\bar{z})\Big\{ -4\,\lambda\boxed{(h_{ij}\mathcal{R} - 2\mathcal{R}_{ij})}+ -2\,\lambda\boxed{e^{-2\rho(z,\bar{z})}(h_{ij}\mathcal{K}_2 - h_{ij}\mathcal{K}^2 + 2\mathcal{K}_{ij}\mathcal{K} - 2\mathcal{K}_{ik}\mathcal{K}^k_j)}\Big\}$$

This term goes away because of our limit taking procedure

it can be easily seen that this term is coming from Jacobson-Myers functional

$$T_{zz} = T_{\bar{z}\bar{z}} = T_{zi} = T_{\bar{z}i} = 0$$

- This justifies Jacobson-Myers functional. But now what about general R^2 theory?

Cosmic Brane to Entropy Functional-Continue…

– We repeat the same game for the R^2 theory.

$$
\begin{aligned}
T_{ij} \;=\; \delta(z,\bar z)\Big[& -4\,\lambda_3\,(h_{ij}\mathcal{R} - 2\mathcal{R}_{ij}) - 16(6\lambda_1 + 11\lambda_2 + 38\lambda_3)h_{ij}\Omega \,+ \\
& e^{-2\rho(z,\bar z)}\Big\{ -(\lambda_2 + 2\lambda_3)h_{ij}\mathcal{K}^2 + 2(\lambda_2 + 3\lambda_3)h_{ij}\mathcal{K}_2 - \\
& \qquad 2(12\lambda_1 + 4\lambda_2 + 4\lambda_3)\mathcal{Q}_{ij} - 2(\lambda_2 + 4\lambda_3)\mathcal{Q}h_{ij} - \\
& \qquad 2(4\lambda_1 + \lambda_2 + 2\lambda_3)\mathcal{K}_{ij}\mathcal{K} + 2(14\lambda_1 + 4\lambda_2 + 4\lambda_3)\mathcal{K}_{kj}\mathcal{K}^k_j) \,+ \\
& \qquad 16(20\lambda_1 + 11\lambda_2 + 24\lambda_3)\mathcal{A}_i\mathcal{A}_j\Big\}\Big] + \\
& e^{-2\rho(z,\bar z)}\Big\{\partial_z\delta(z,\bar z) + \partial_{\bar z}\delta(z,\bar z)\Big\}\Big\{-2(2\lambda_1 + \lambda_2 + 2\lambda_3)\mathcal{K}_{ij} + (4\lambda_3 + \lambda_2)h_{ij}\mathcal{K}\Big\} - \\
& 4e^{-2\rho(z,\bar z)}\,\partial_z\partial_{\bar z}\delta(z,\bar z)(\lambda_2 + 4\lambda_3)
\end{aligned}
$$

Suppressed

Correctly corresponds to the R^2 area functional

– So far so good. But wait……

– We get additional divergences in other components

$$\boxed{T_{zz} = -4\partial_z^2\delta(z,\bar{z})(2\lambda_1+\lambda_2+2\lambda_3) - 2\partial_z\delta(z,\bar{z})(4\lambda_1+\lambda_2)\mathcal{K}}$$

and
$$\boxed{\begin{aligned} T_{z\bar{z}} = & -2\big\{\partial_z\delta(z,\bar{z})+\partial_{\bar{z}}\delta(z,\bar{z})\big\}\big(2\lambda_1+\lambda_2+2\lambda_3\big)\mathcal{K} + \\ & 4\,\partial_z\partial_{\bar{z}}\delta(z,\bar{z})(2\lambda_1+\lambda_2+2\lambda_3)\,. \end{aligned}}$$

– For Gauss-Bonnet case obviously they vanish. But in this case these divergences stand.

Any attempt to use this method to show that the proposed entropy functional is the correct entropy functional for R^2 theory should be able to account for these extra delta divergences.

Conclusions

- For Gauss-Bonnet theory (more generally for Lovelock theory) proposed area functional and Generalized entropy method are in accord with each other. (though the limit taking procedure is quite artificial)
- For more general theories, Generalized entropy method does not produce the extremal surface which follows from minimizing the Dong's(camps) functional.
- Maybe some subleading terms are missing in the area functional which can cure this problem.-Ambiguity is there.
- Strong subadditivity and second law may serve as independent validity check. (Work in progress with Menika)
- Modification of Lewkowycz-Maldacena method ?

Lot more to explore !!!