Colour-kinematic duality

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work with R. Monteiro; N.E.J. Bjerrum-Bohr, P. Damgaard & R. Monteiro; R. Boels, R.S. Isermann, & R. Monteiro

Scattering Without Spacetime, Bengaluru, September 26 2012

Structure in amplitudes

- Colour-ordered amplitudes are simple. Eg, Parke-Taylor:

$$A(1,2,\ldots,n) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

- Obtain full colour-dressed amplitude from ordered amplitudes

$$\mathcal{A}(1,2,\ldots,n) = \sum_\sigma A(1,\sigma_2,\ldots,\sigma_n)\ \mathrm{tr}\, T^{a_1}T^{a_{\sigma_2}}\cdots T^{a_{\sigma_n}}$$

- Ordered amplitudes have lots of structure!
  - Dual coordinates, dual conformal properties, Yangian, Grassmannians, ...

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- Can directly compute colour-dressed amplitude
  - Eg Feynman diagrams!

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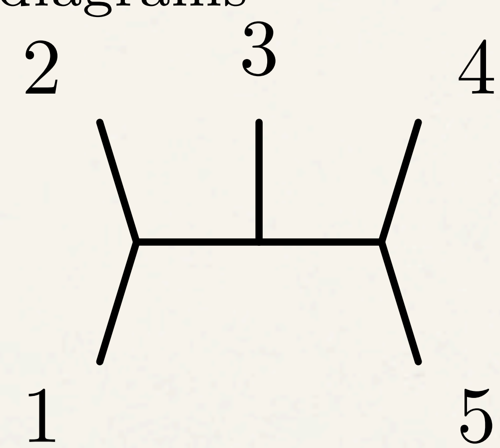
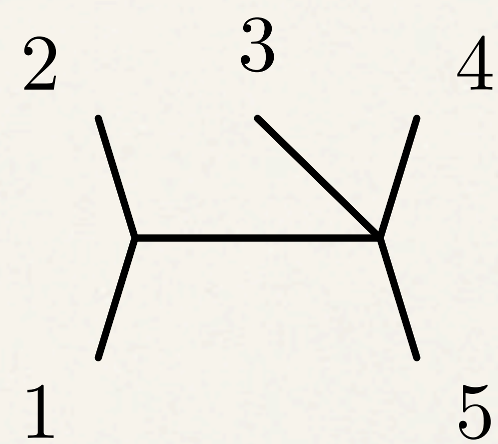
$=$ 

$+$ 

$+\cdots$

- Lots of diagrams, complicated

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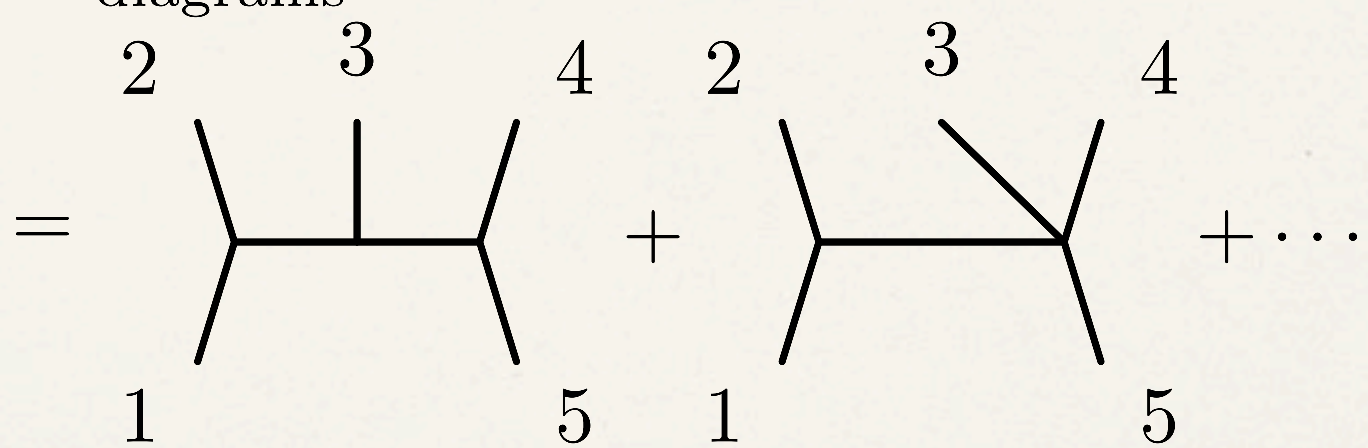
Colour factors

$$\mathcal{A}(1,2,\ldots,n) = \sum_{\text{diagrams}} \frac{c_i n_i}{D_i}$$



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Propagators

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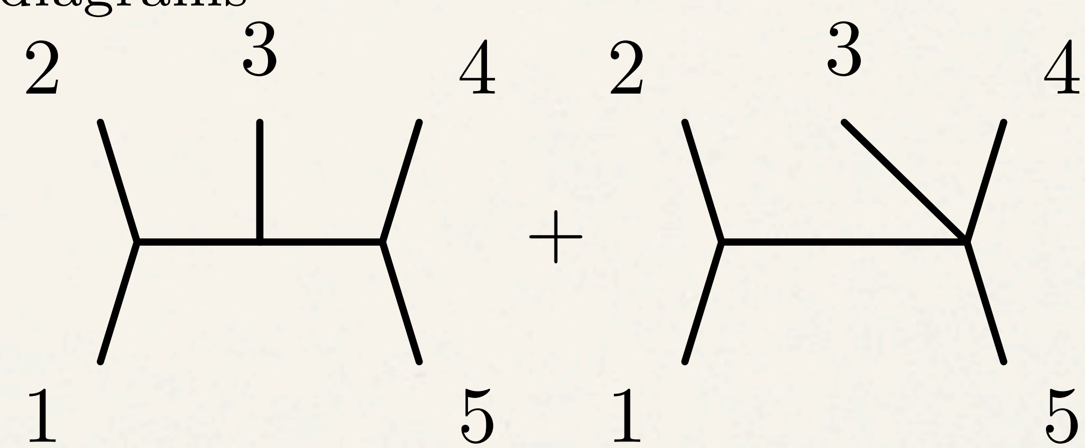

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Everything else

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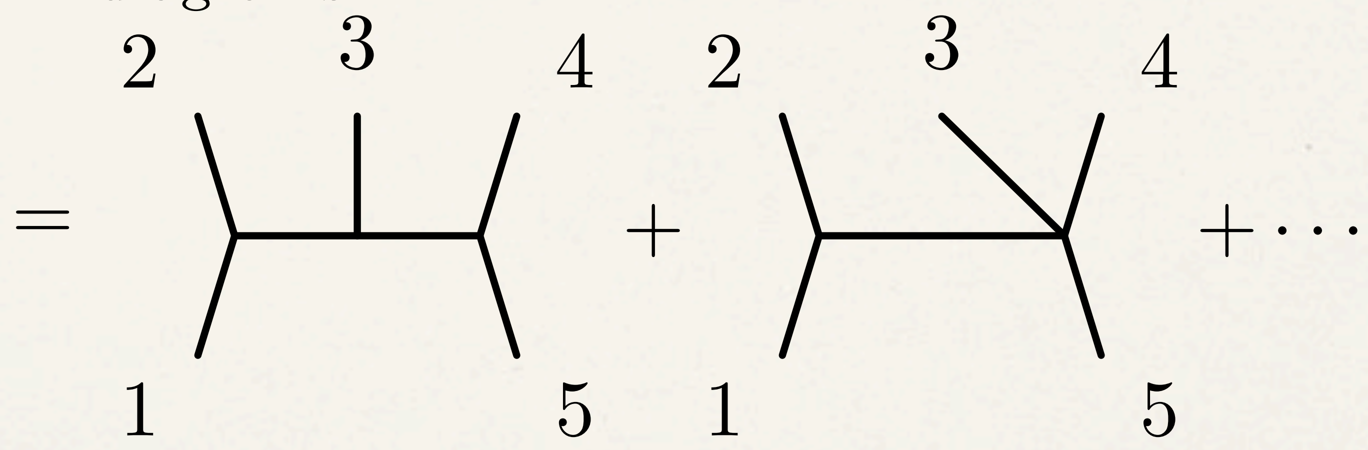

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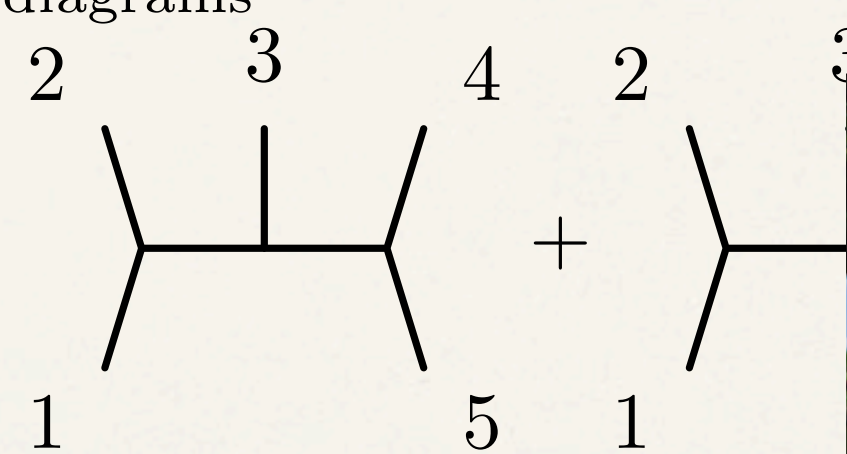

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Structure in amplitudes

- But the colour-dressed amplitude itself has interesting structure

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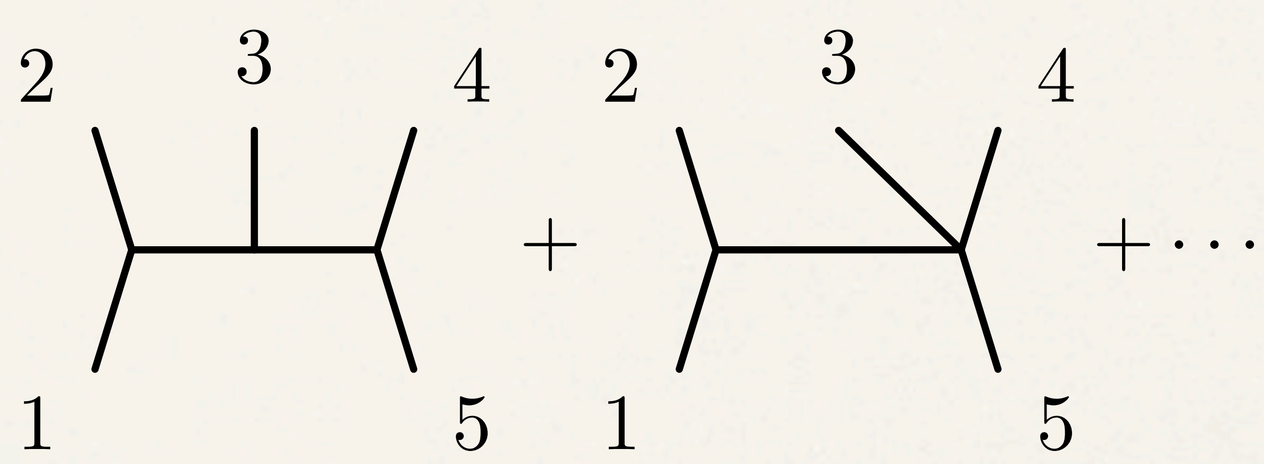

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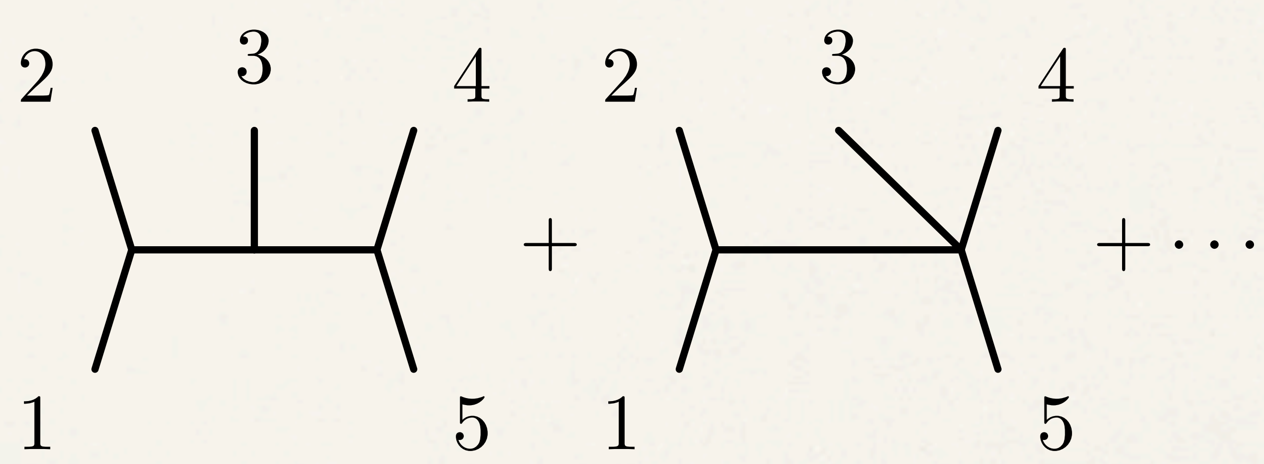

$$c = f^{a_1 a_2 b_1} f^{b_1 a_3 b_2} f^{b_2 a_4 a_5}$$

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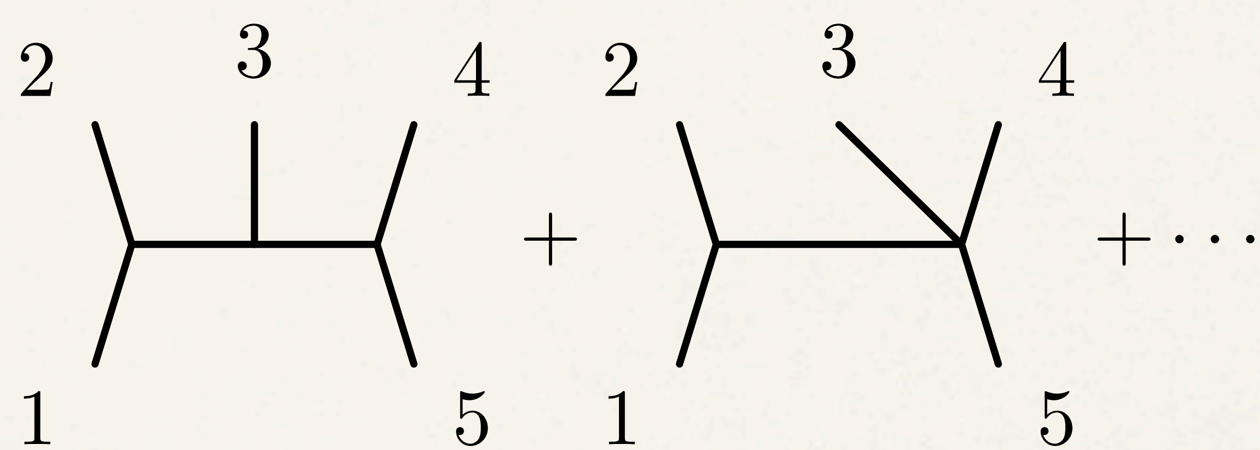

$$c = \begin{cases} f^{a_1 a_2 b_1}\, f^{b_1 a_3 b_2}\, f^{b_2 a_4 a_5} \\ f^{a_1 a_2 b_1}\, f^{b_1 a_4 b_2}\, f^{b_2 a_5 a_3} \\ f^{a_1 a_2 b_1}\, f^{b_1 a_5 b_2}\, f^{b_2 a_3 a_4} \end{cases}$$

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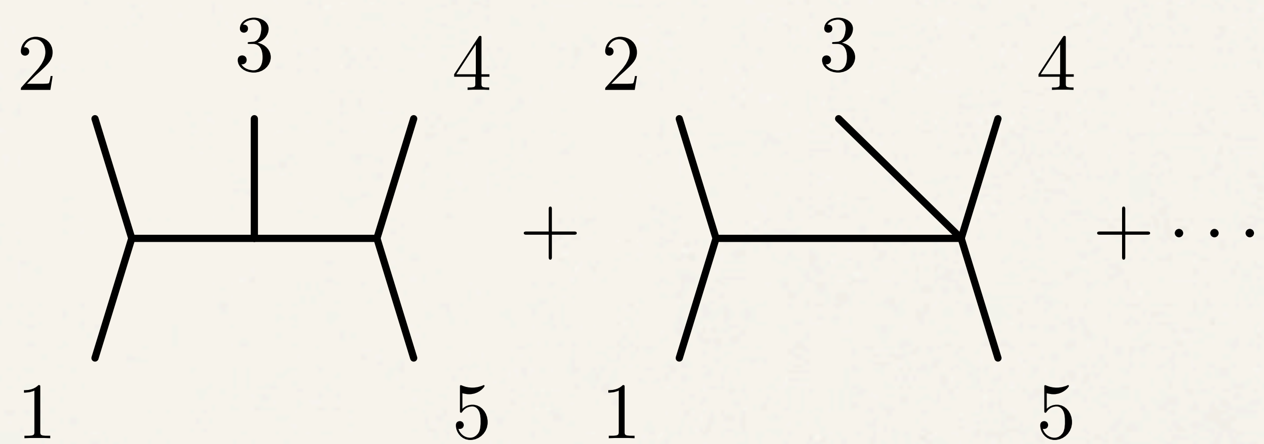

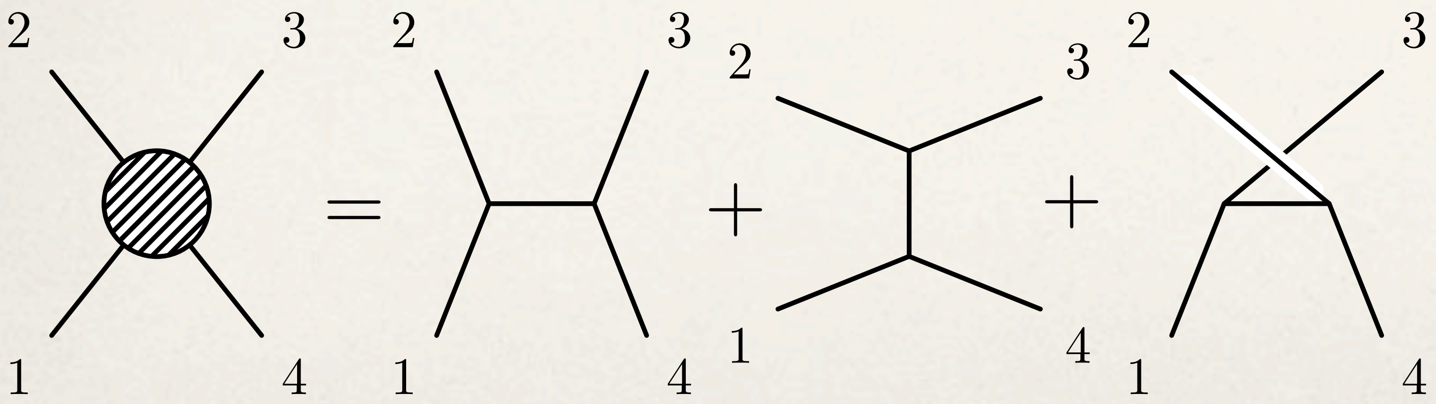

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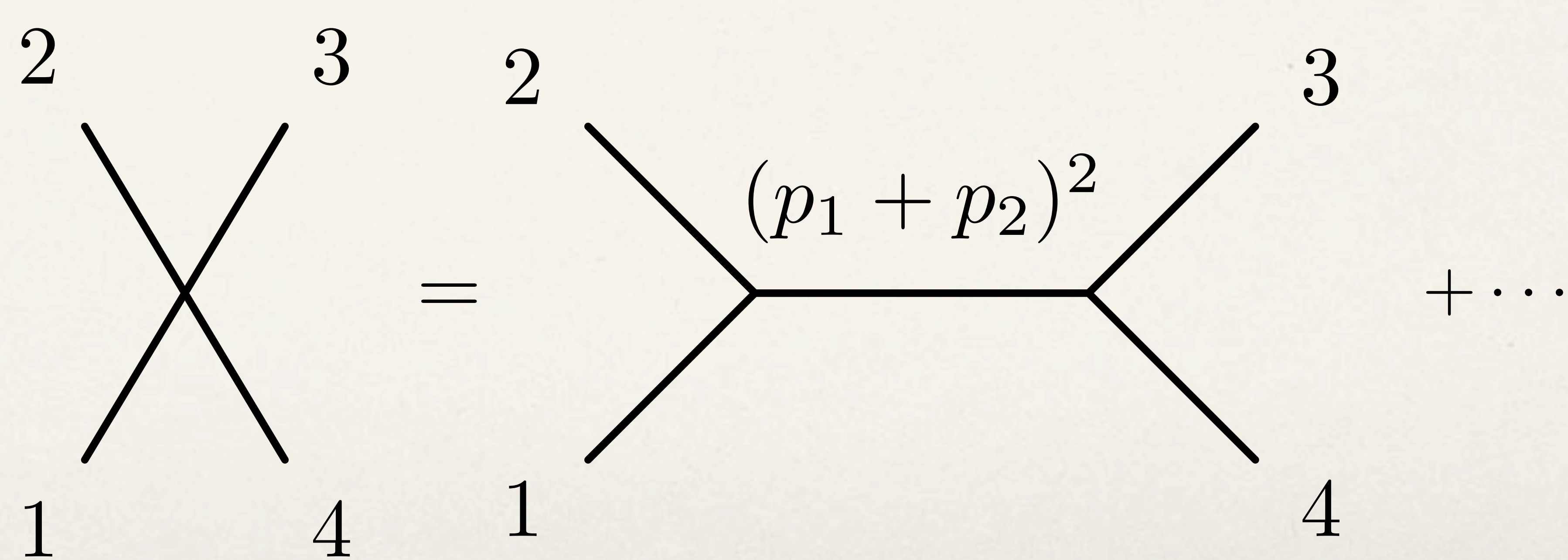
Structure in amplitudes

- Can remove quartic vertex
  - Assign terms coming from quartic to three point vertices according to colour factor
  - Not unique



Structure in amplitudes

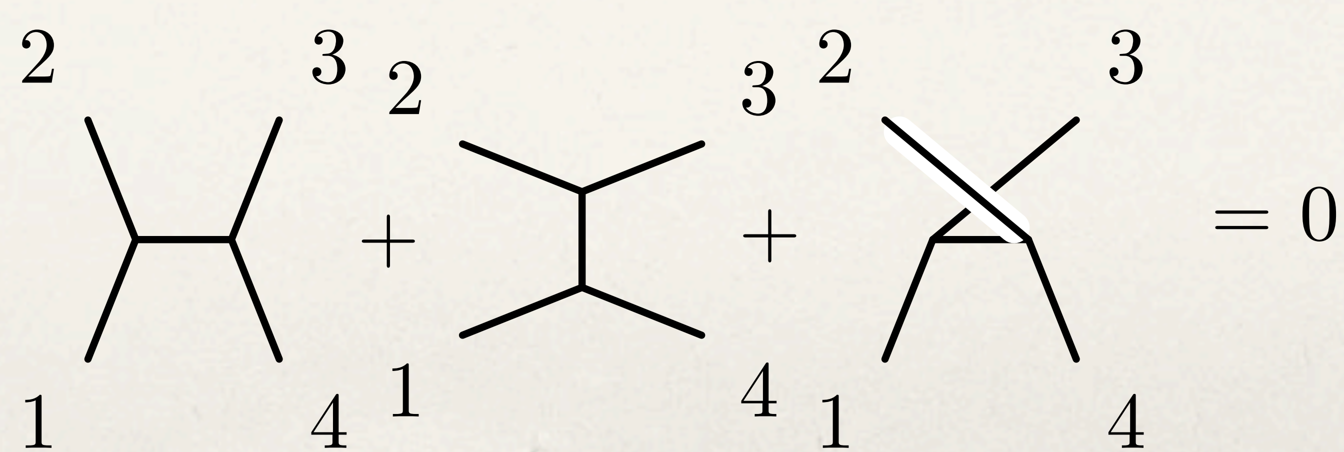
- Amplitude a sum over cubic diagrams

Bern, Carrasco, Johansson (BCJ)

$$\mathcal{A}_{YM} = \sum_{i \in \text{cubic}} \frac{n_i c_i}{D_i}$$

$$c_i + c_j + c_k = 0$$

Jacobi identity, group theory



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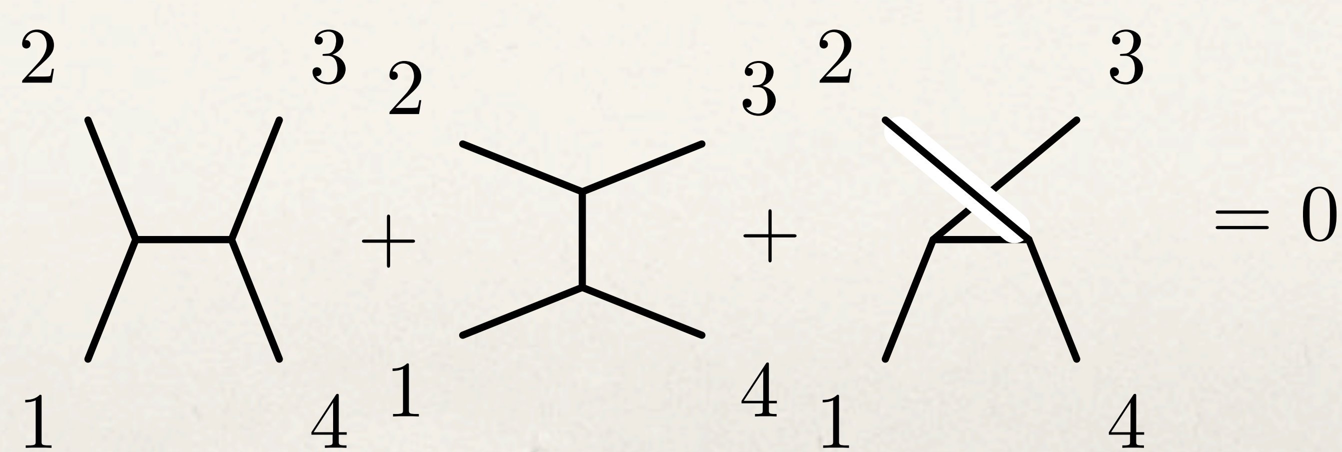

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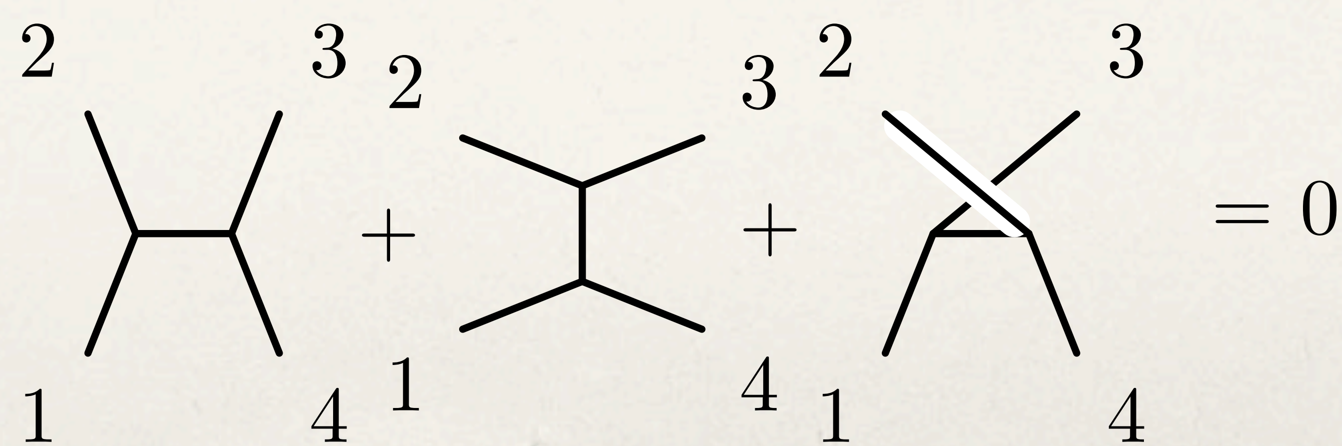
$c_i + c_j + c_k = 0$ Jacobi identity, group theory

$n_i + n_j + n_k = 0$ Is there a kinematic group?



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Colour-kinematic duality

$$\begin{cases} c_i + c_j + c_k = 0 \\ n_i + n_j + n_k = 0 \end{cases}$$

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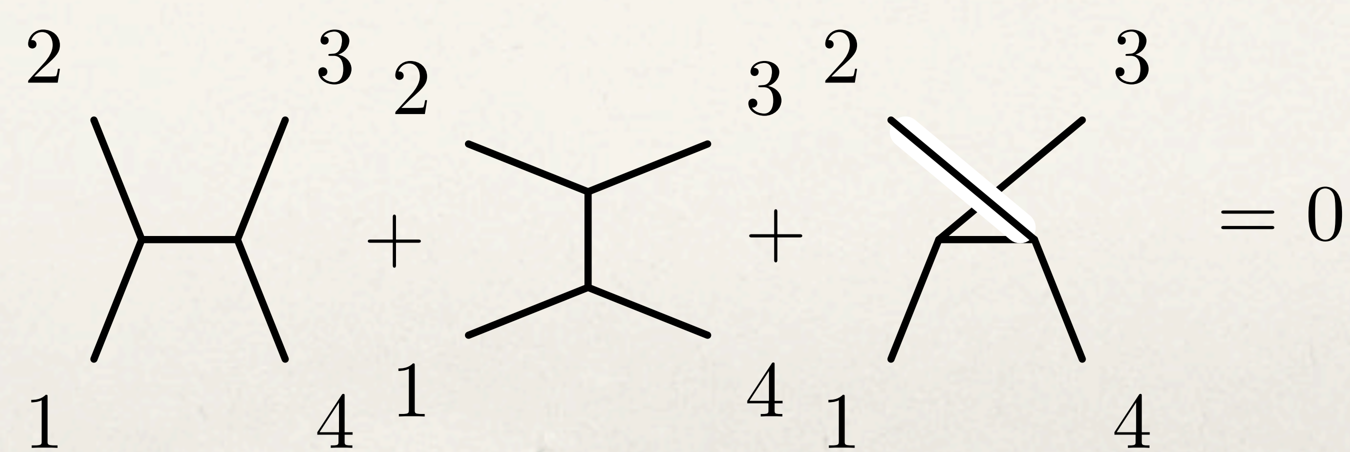

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Colour-kinematic duality
$$\left\{ \begin{array}{l} c_i + c_j + c_k = 0 \\ n_i + n_j + n_k = 0 \end{array} \right.$$

Jacobi identity, group theory

Is there a kinematic group?

- Loops

$$\mathcal{A}_{\text{YM}} = \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \sum_{i \in \text{cubic}} \frac{1}{S_i} \frac{n_i c_i}{D_i}$$

Some consequences

1. BCJ relations for colour-ordered amplitudes

$$s_{35}A_5(1,2,4,3,5) - (s_{13}+s_{23})A_5(1,2,3,4,5) - s_{13}A_5(1,3,2,4,5) = 0$$

- Basis of independent colour-ordered *n*-point amplitudes *(n - 3)!*

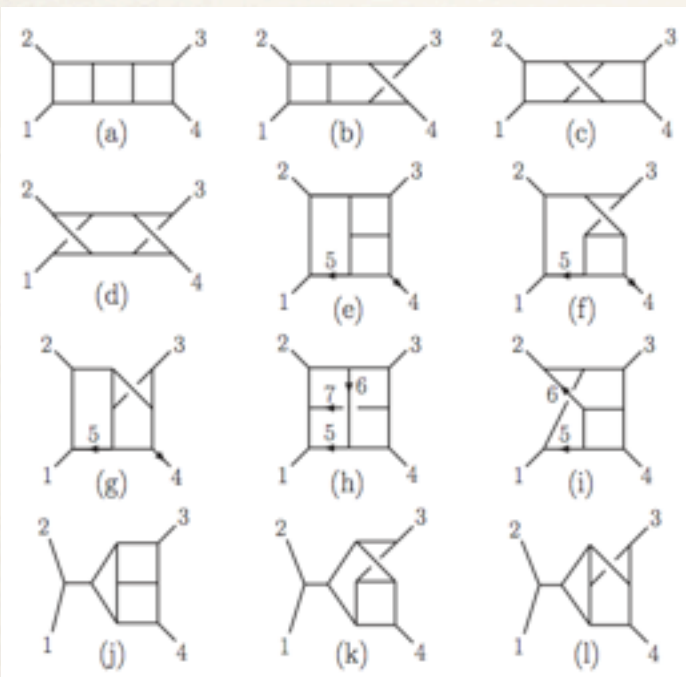
2. New expression for gravity amplitude (the double-copy formula)

$$\mathcal{A}_{YM} = \sum_{i\in\text{cubic}} \frac{n_i c_i}{D_i} \qquad \mathcal{M} = \sum_{i\in\text{cubic}} \frac{n_i \tilde{n}_i}{D_i}$$

An example: 3 loops

- Complete 3 loop maximal SYM/SUGRA amplitude

BCJ



(a)–(d)	s^2
(e)–(g)	$\big(s\,(-\tau_{35}+\tau_{45}+t) - t\,(\tau_{25}+\tau_{45}) + u\,(\tau_{25}+\tau_{35}) - s^2\big)/3$
(h)	$\big(s\,(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)$ $+t\,(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17})+s^2\big)/3$
(i)	$\big(s\,(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)$ $+t\,(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46})+u\,\tau_{25}+s^2\big)/3$
(j)-(l)	$s(t-u)/3$

$$\tau_{ij} = 2l_i \cdot p_j$$

- Jacobi relations relate planar and non-planar diagrams
  - Simplicity beyond the planar theory; simplicity in gravity

But…

- This is great!
- How do we compute BCJ numerators?
- We don't know
- Make a guess, solve for undetermined coefficients, repeat

Current status

- Three main parts aspects of the BCJ story

1. Existence of the numerators
2. BCJ relations for amplitudes
3. Double copy formula

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Bjerrum-Bohr, Damgaard, Vanhove, Stieberger, Feng, Huang, Jia, Cachazo, Mafra, Schlotterer, Monteiro, DOC, Bern, Carrasco, Johansson, Dennen, Huang, Kiermaier

Manifest duality in loops?

- So what do we know about loops?

	1 loop	2 loop	3 loop	4 loop	L loop
4 pt	✓	✓	✓	✓	?
5 pt	✓	✓	✓*	?	?
6 pt	?	?	?	?	?
n pt	?	?	?	?	?

- Maximal supersymmetry only

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Bern,
Carrasco,
Dixon,
Johansson,
Roiban

- Maximal supersymmetry only

Outline

1. Manifest colour-kinematic duality in self-dual Yang-Mills
2. Algebra- and superalgebra-based scalar theories as bases for gauge amplitudes

Manifest duality

SD Yang-Mills and Gravity

- Colour-kinematic duality is manifest in the self-dual sector

R. Monteiro & DOC

$$\partial^2\Phi + ig[\partial_w\Phi, \partial_u\Phi] = 0$$

YM: Leznov & Mukhtarov, Parkes, Bardeen, Chalmers & Siegel, Cangemi

$$\partial^2\phi + \kappa\{\partial_w\phi, \partial_u\phi\} = 0$$

GR: Plebanski equation of motion

w and u are spacetime coords, and

$$\{f, g\} = \partial_w f \partial_u g - \partial_u f \partial_w g$$

Kinematic algebra

- Brace generates diffeomorphism algebra

$$\{e^{-ip_1\cdot x}, e^{-ip_2\cdot x}\} = -X(p_1,p_2)e^{-i(p_1+p_2)\cdot x}, \quad X(p_1,p_2) = p_{1w}p_{2u} - p_{2w}p_{1u}$$

- Generators of infinitesimal diffeomorphisms in *u*, *w* plane

$$L_k = e^{-ik\cdot x}(-k_w\partial_u + k_u\partial_w)$$

- Lie algebra $\quad [L_{p_1}, L_{p_2}] = iX(p_1,p_2)L_{p_1+p_2} = iF_{p_1p_2}{}^k L_k$

$$F_{p_1p_2p_3} = (2\pi)^4\delta^{(4)}(p_1+p_2+p_3)X(p_1,p_2)$$

  - Structure constants of area-preserving diffeormorphisms

Kinematic algebra

- Gravity equation of motion

$$k^2\phi(k) = \frac{1}{2}\kappa \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} F_{p_1p_2}{}^{k} X(p_1,p_2)\phi(p_1)\phi(p_2)$$

- Gauge case

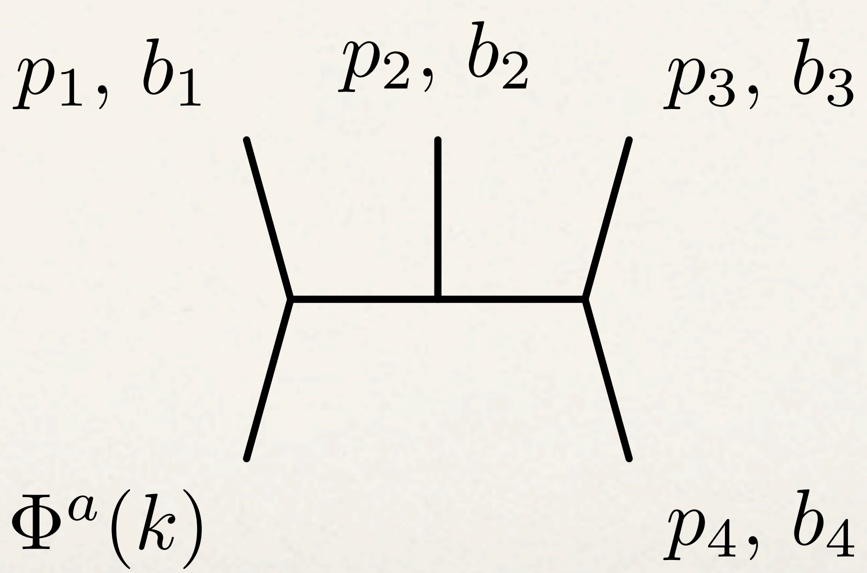
$$k^2\Phi^a(k) = \frac{1}{2}g \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} F_{p_1p_2}{}^{k} f^{b_1b_2a}\Phi^{b_1}(p_1)\Phi^{b_2}(p_2)$$

- Let's see how the duality work in detail!

Kinematic algebra

- Solve equations of motion perturbatively

$$\Phi^a(k) = \cdots + \int \frac{dp_1 dp_2 dp_3 dp_4}{k^2 s_{34} s_{234}} \; F^k{}_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \times$$
$$\times j^{b_1}(p_1) j^{b_2}(p_2) j^{b_3}(p_3) j^{b_4}(p_4) f^a{}_{b_1}{}^{c_1} f_{c_1 b_2}{}^{c_3} f_{c_3 b_3 b_4} + \cdots$$



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- Square numerator?

$$F^k{}_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} = X(-k, p_1) X(p_1 - k, p_2) X(p_3, p_4) \, \delta^{(4)}(p_1 + p_2 + p_3 + p_4 - k)$$

Kinematic algebra

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Double copy

Kinematic algebra

- Side remark, can express solution of gauge theory EOM as

$$\Phi^{(n)}(k) = (ig)^n \int dp_1 dp_2 \ldots dp_{n+1}\, \delta(p_1 + p_2 + \ldots + p_{n+1} - k)$$
$$\times\, j(p_1) j(p_2) \ldots j(p_{n+1}) \frac{1}{Q(p_1, p_2) Q(p_2, p_3) \ldots Q(p_n, p_{n+1})}$$

where

$$Q(p) = \frac{p_{\bar{w}}}{p_u}$$
$$Q(p_1, p_2) = Q(p_1) - Q(p_2)$$

$$X(1,2) = [12]$$
$$Q(1,2) = \langle 12 \rangle$$

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Appears elsewhere in mathematics

$$X(1,2) = [12]$$

$$Q(1,2) = \langle 12 \rangle$$

Kinematic algebra

- Consider space $\mathbb{C}^{n+1}$ coords $Q_i, \quad i = 1, 2, \ldots, n+1$
- Now omit lines where Q's coincide $\{\mathbb{C}^{n+1} | Q_i \neq Q_j\}$ and define

$$\theta_{ij} = \frac{dQ_i - dQ_j}{Q_i - Q_j} = \frac{dQ_i - dQ_j}{Q(i,j)}$$

- Basis for the cohomology group H^{n-1} is

Arnold

$$\theta_{12}\theta_{23} \cdots \theta_{n,n+1} = \frac{(-1)^{n+1} \sum (-1)^i dQ_1 \cdots \widehat{dQ_i} \cdots dQ_n}{Q(1,2)Q(2,3) \cdots Q(n,n+1)}$$

- Space of these objects is the same as space Lie$(n+1)$ of multilinear Lie polynomials in $n+1$ formal variables

Kapranov

Kinematic algebra

- Three point function from gravity equation

$$\partial^2\phi + \kappa\{\partial_w\phi, \partial_u\phi\} = 0 \quad \Rightarrow \mathcal{M}_3 = X(p_1, p_2)^2\delta(p_1 + p_2 + p_3)$$

- "Reverse KLT"

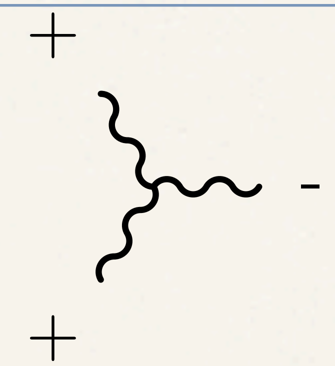
$$\mathcal{A}_3 = X(p_1, p_2)\delta(p_1 + p_2 + p_3)f^{abc} = F_{p_1 p_2 p_3} f^{abc}$$

  - So same algebra appears in YM

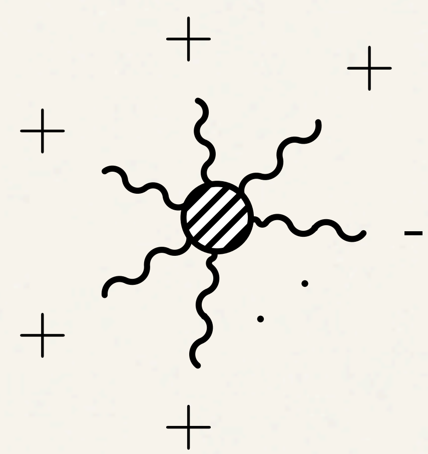
- Physics of colour-kinematics is gravitational

Self-dual amplitudes

- Feynman rules in SD theory



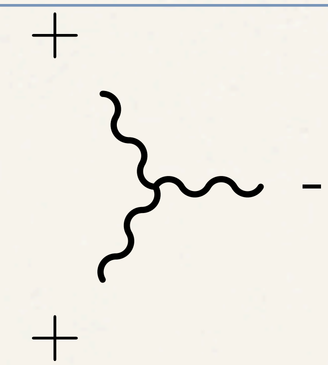
- Tree amplitudes

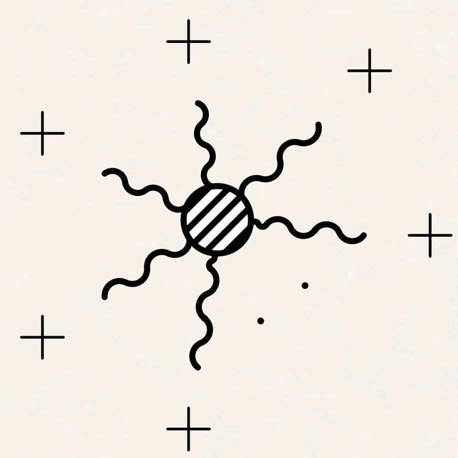


$= 0$

- MHV amplitudes inherit structure of kinematic algebra from SD
  - Choose an appropriate gauge

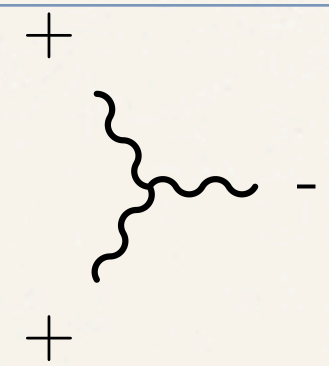
Self-dual amplitudes

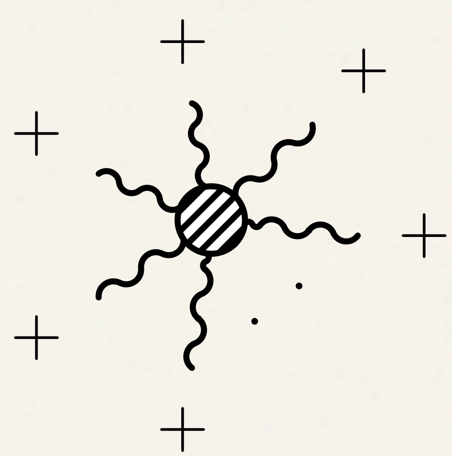
- Only vertex in SD theory is 

- One loop amplitudes 

$\neq 0$

Self-dual amplitudes

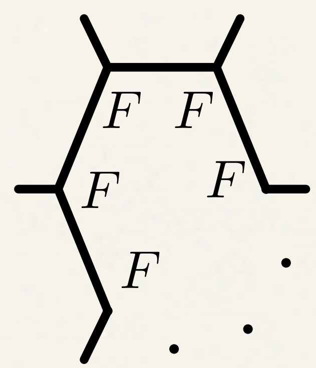
- Only vertex in SD theory is 

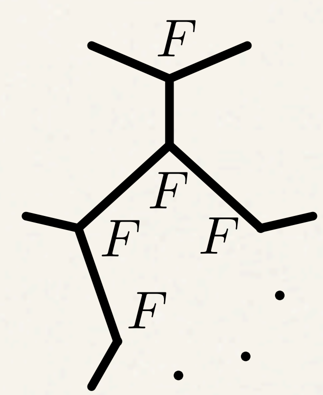
- One loop amplitudes  $\neq 0$

- Same as usual YM all-plus amplitudes

All-plus amplitudes

- All one-loop all-plus amplitudes can be computed in self-dual theory

$\mathcal{A}_{+\cdots+} =$ 

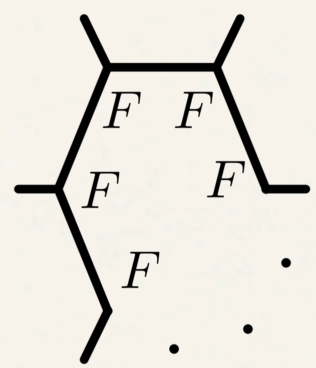
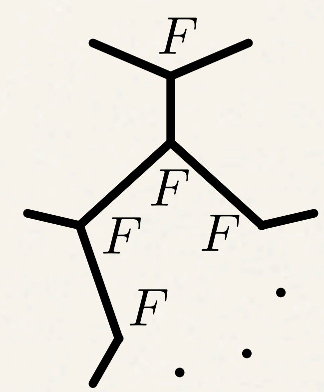
$+$ 

$+ \cdots$

	1 loop	2 loop	3 loop	4 loop	L loop
4 pt	✓	✓	✓	✓	?
5 pt	✓	✓	?	?	?
6 pt	?	?	?	?	?
n pt	?	?	?	?	?

All-plus amplitudes

- All one-loop all-plus amplitudes can be computed in self-dual theory

$\mathcal{A}_{+\cdots+} =$  $+$  $+\cdots$

	1 loop	2 loop	3 loop	4 loop	L loop
4 pt	✓	✓	✓	✓	?
5 pt	✓	✓	?	?	?
6 pt	✓ / ?	?	?	?	?
n pt	✓ / ?	?	?	?	?

A scalar basis using algebras

A linear space

- YM amplitudes satisfy KK, BCJ relations
- These are linear equations
  - There is a linear operator acting on *(n - 1)!* space of amplitudes

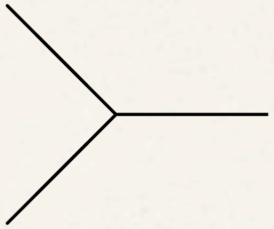
$$\sum_{\beta} \tilde{S}[\alpha_{2,n}|\beta_{2,n}]A(1,\beta_{2,n}) = 0$$

Damgaard, Bjerrum-Bohr, Søndergaard, Vanhove

- Kernel of $\tilde{S}$ has dimensions *(n - 3)!*

Scalar theories

- YM is not only theory with KK, BCJ relations
- Consider cubic scalar theory with two different structure constants at each vertex



$$= g f^{abc} f'^{a'b'c'}$$

Scalar theories

- YM is not only theory with KK, BCJ relations
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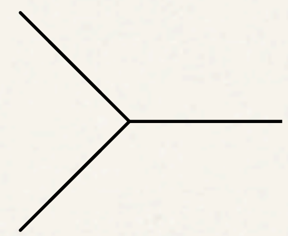
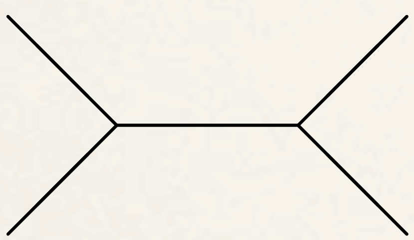
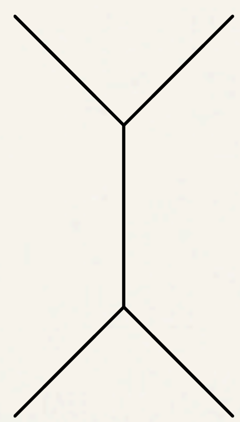
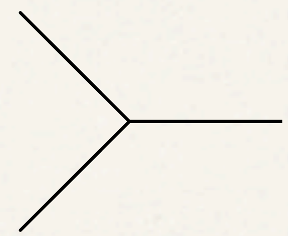
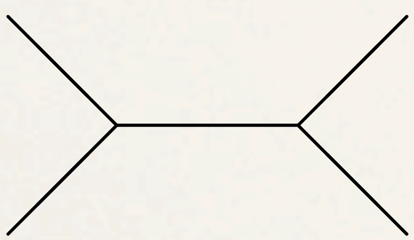
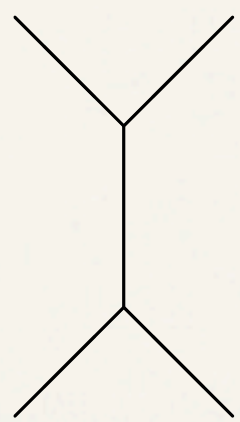
 $= g f^{abc} f'^{a'b'c'}$

 -  = $A(1,2,3,4) = \frac{n_s}{s} - \frac{n_t}{t}$

Scalar theories

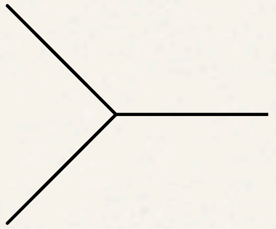
- YM is not only theory with KK, BCJ relations
- Consider cubic scalar theory with two different structure constants at each vertex

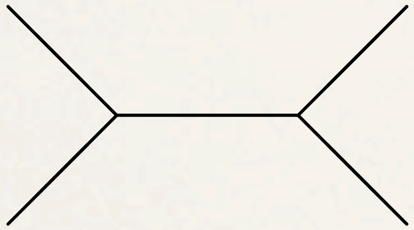
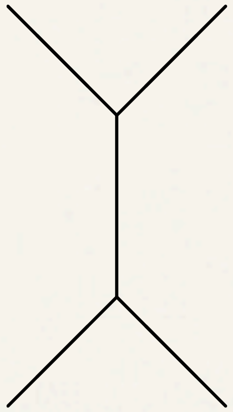
 $= g f^{abc} f'^{a'b'c'}$

 -  $=$ $A(1,2,3,4) = \frac{c_s}{s} - \frac{c_t}{t}$

Scalar theories

- YM is not only theory with KK, BCJ relations
- Consider cubic scalar theory with two different structure constants at each vertex

 $= \; g f^{abc} f'^{a'b'c'}$

 -  $= \quad A(1,2,3,4) = \frac{c_s}{s} - \frac{c_t}{t}$

  - KK true
  - BCJ true (explicit proof by Du, Feng and Fu)

A basis

- YM, scalar theories all in kernel of $\tilde{S}$
- Pick *(n-3)!* independent cubic scalar theories; YM a linear combination

*J*th colour-ordered amplitude of *I*th scalar theory

$$A_{(J)} = \sum_{I=1}^{(n-3)!} \alpha_I \, \theta_{IJ}$$

- When θ is non-singular, unique expansion coefficients α exist

$$\mathcal{A} = \sum_{I=1}^{(n-3)!} \alpha_I \, \Theta_I$$

Colour dressed

Manifest colour-kinematics duality

- Scalar basis has cubic numerators with manifest Jacobi

$$\Theta_I = \sum_k \frac{\tilde{c}_{I,k}\, c_k}{D_k}$$

- So YM numerators represented as

$$n_k^{YM} = \sum_{I=1}^{(n-3)!} \alpha_I\, \tilde{c}_{I,k}$$

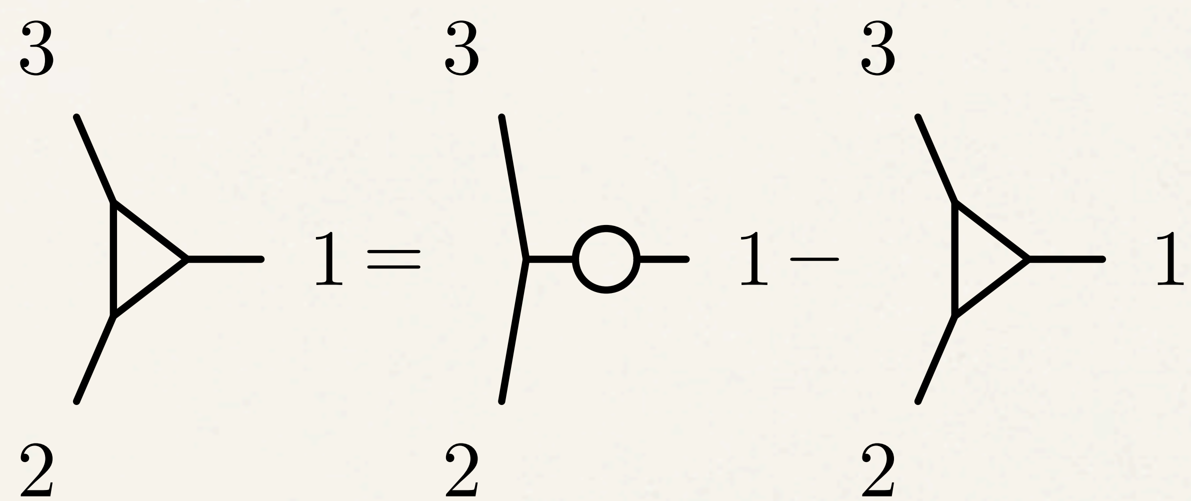
- Jacobi relations follow by linearity

What about loops?

- $\mathcal{N} = 4$ one loop diagrams: no bubbles, no triangles
  - Scalar theories using superalgebras can reproduce this
- Killing form a  $b = (-1)^{|c|} f_{ac}{}^{d} f_{bd}{}^{c} = K_{ab}$
- K vanishes for superalgebras $\mathfrak{psu}(n|n), \ldots$
- Scalar theories based on such algebras: no bubbles
  - Similar ideas in (partially) quenched chiral perturbation theory

Jacobi for Triangle

- Using bosonic external states, Jacobi removes triangle



- So superalgebra-based scalar theories also have no triangles or bubbles

Four point $\mathcal{N} = 4$ amplitude

- Integrand is

$$\mathcal{I} = st\mathcal{A}_{\text{tree}} \frac{1}{l^2(l+p_1)^2(l+p_1+p_2)^2(l-p_4)^2}$$

- Scalar theory

$$\theta = (-1)^{|d|} f^{d1a} f^{a2b} f^{b3c} f^{c4d} \frac{1}{l^2(l+p_1)^2(l+p_1+p_2)^2(l-p_4)^2}$$

- Eg $\mathfrak{psu}(2|2)$

$$(-1)^{|d|} f^{d1a} f^{a2b} f^{b3c} f^{c4d} = 2$$

(completely symmetric in 1,2,3,4)

- So $\mathcal{I} = \alpha\theta, \quad \alpha = \dfrac{st\mathcal{A}_{\text{tree}}}{2}$

Conclusions

- Self-dual sector: manifest colour-kinematics
    - Kinematic algebra is an area-preserving diffeomorphism algebra
    - All one-loop all-plus numerators can be chosen to manifest colour-kinematic duality
    - MHV sector: inherits kinematic algebra from self-dual
- Scalar theories promising route to understand duality at loop level