

Colour-kinematic duality

Donal O'Connell, Niels Bohr International Academy

work with R. Monteiro; N.E.J. Bjerrum-Bohr, P. Damgaard & R. Monteiro; R. Boels, R.S. Isermann, & R. Monteiro

Structure in amplitudes

- ✧ Colour-ordered amplitudes are simple. Eg, Parke-Taylor:

$$A(1, 2, \dots, n) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}$$

- ✧ Obtain full colour-dressed amplitude from ordered amplitudes

$$\mathcal{A}(1, 2, \dots, n) = \sum_{\sigma} A(1, \sigma_2, \dots, \sigma_n) \operatorname{tr} T^{a_1} T^{a_{\sigma_2}} \dots T^{a_{\sigma_n}}$$

- ✧ Ordered amplitudes have lots of structure!
 - ✧ Dual coordinates, dual conformal properties, Yangian, Grassmannians, ...

Structure in amplitudes

- ✧ Colour-ordered amplitudes are simple. Eg, Parke-Taylor:

$$A(1, 2, \dots, n) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

- ✧ Obtain full colour-dressed amplitude from ordered amplitudes

$$\mathcal{A}(1, 2, \dots, n) = \sum_{\sigma} A(1, \sigma_2, \dots, \sigma_n) \text{Tr}(T^{a_1} T^{a_{\sigma_2}} \cdots T^{a_{\sigma_n}})$$

- ✧ Ordered amplitudes have lots of structure

- ✧ Dual coordinates, dual conformal properties
- ✧ Grassmannians, ...



Structure in amplitudes

- ✧ Can directly compute colour-dressed amplitude
 - ✧ Eg Feynman diagrams!

$$\mathcal{A}(1, 2, \dots, n) = \sum_{\text{diagrams}} \frac{c_i n_i}{D_i}$$

=

The first diagram is a box with a vertical line in the middle. The external legs are labeled 1 (bottom-left), 2 (top-left), 3 (top-middle), 4 (top-right), and 5 (bottom-right). The second diagram is a box with a diagonal line from the top-left to the bottom-right. The external legs are labeled 1 (bottom-left), 2 (top-left), 3 (top-middle), 4 (top-right), and 5 (bottom-right). The diagrams are separated by a plus sign, followed by an ellipsis.

- ✧ Lots of diagrams, complicated

Structure in amplitudes

- ✧ Can directly compute colour-dressed amplitude

- ✧ Eg Feynman diagrams!

$$\mathcal{A}(1, 2, \dots, n) = \sum_{\text{diagrams}} \frac{C_i n_i}{D_i}$$

Colour factors

The diagrammatic expansion shows two Feynman diagrams for a 5-point amplitude. The first diagram is a tree-level exchange with external legs labeled 1, 2, 3, 4, 5. The second diagram is a tree-level exchange with an additional internal line, also with external legs labeled 1, 2, 3, 4, 5. The expansion continues with more diagrams, indicated by an ellipsis.

- ✧ Lots of diagrams, complicated

Structure in amplitudes

- ✧ Can directly compute colour-dressed amplitude

- ✧ Eg Feynman diagrams!

$$\mathcal{A}(1, 2, \dots, n) = \sum_{\text{diagrams}} \frac{c_i n_i}{D_i}$$

Propagators

The equation shows the amplitude $\mathcal{A}(1, 2, \dots, n)$ as a sum over diagrams of the form $\frac{c_i n_i}{D_i}$. An orange arrow points from the word "Propagators" to the denominator D_i . Below the equation, two Feynman diagrams are shown, separated by a plus sign and followed by an ellipsis. The first diagram is a box with an internal vertical line, and the second is a box with an internal diagonal line. Both diagrams have external legs labeled 1, 2, 3, 4, 5.

- ✧ Lots of diagrams, complicated

Structure in amplitudes

- ✧ Can directly compute colour-dressed amplitude

- ✧ Eg Feynman diagrams!

$$\mathcal{A}(1, 2, \dots, n) = \sum_{\text{diagrams}} \frac{c_i n_i}{D_i}$$

Everything else
↙

$$= \begin{array}{c} 2 \quad 3 \quad 4 \\ \diagdown \quad | \quad \diagup \\ \text{---} \\ \diagup \quad | \quad \diagdown \\ 1 \quad 5 \end{array} + \begin{array}{c} 2 \quad 3 \quad 4 \\ \diagdown \quad \diagup \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \quad \diagdown \\ 1 \quad 5 \end{array} + \dots$$

- ✧ Lots of diagrams, complicated

Structure in amplitudes

- ❖ Can directly compute colour-dressed amplitude

- ❖ Eg Feynman diagrams!

Everything else

$$\mathcal{A}(1, 2, \dots, n) = \sum_{\text{diagrams}} \frac{c_i n_i}{D_i}$$

$$= \begin{array}{c} \begin{array}{ccccc} & 2 & & 3 & \\ & \diagdown & & | & \diagup \\ & & & & \\ 1 & & & & 5 \\ & \diagup & & & \diagdown \end{array} & + & \begin{array}{ccccc} & 2 & & 3 & \\ & \diagdown & & | & \diagup \\ & & & & \\ 1 & & & & 4 \end{array} \end{array}$$

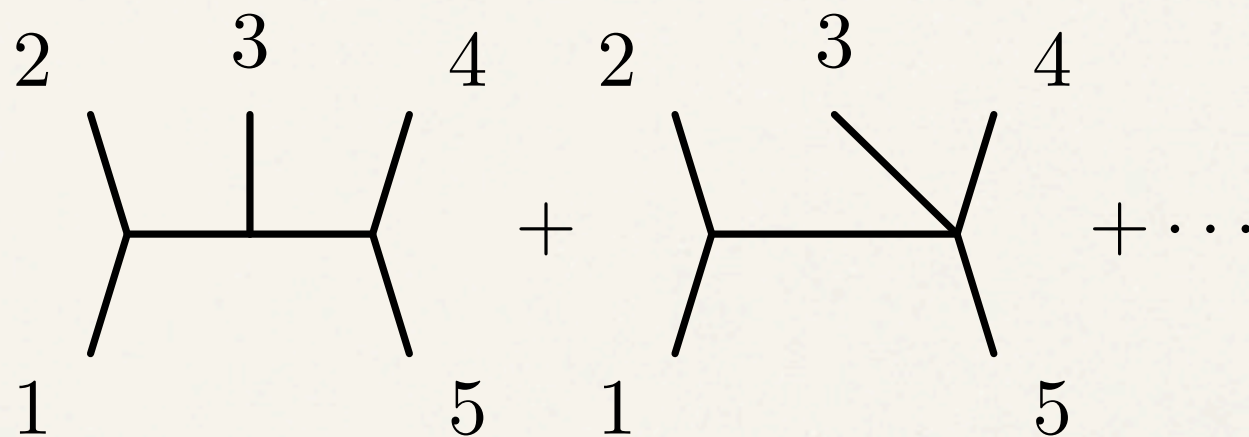
- ❖ Lots of diagrams, complicated



Structure in amplitudes

- ✧ But the colour-dressed amplitude itself has interesting structure

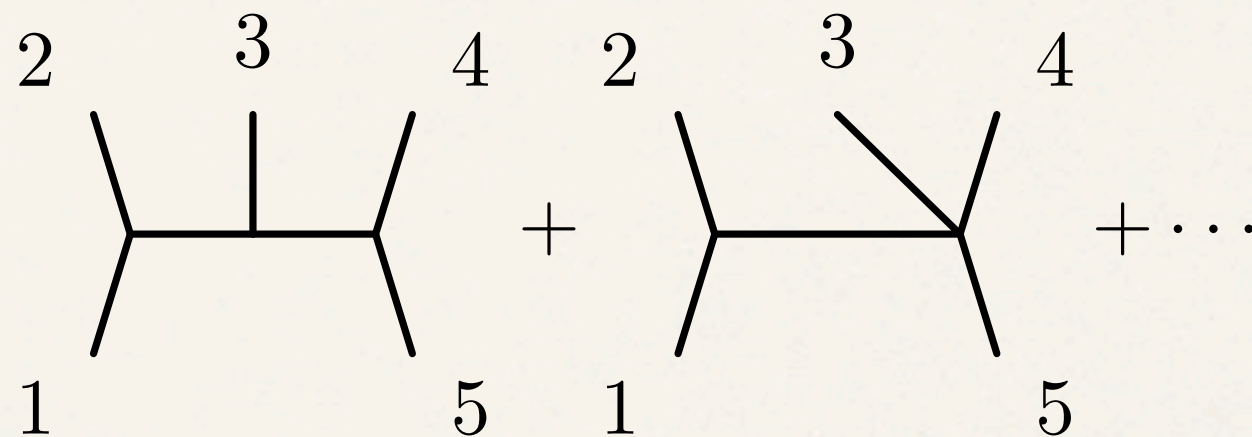
$$\mathcal{A}(1, 2, \dots, n) = \sum_{\text{diagrams}} \frac{c_i n_i}{D_i}$$



Structure in amplitudes

- ✧ But the colour-dressed amplitude itself has interesting structure

$$\mathcal{A}(1, 2, \dots, n) = \sum_{\text{diagrams}} \frac{c_i n_i}{D_i}$$

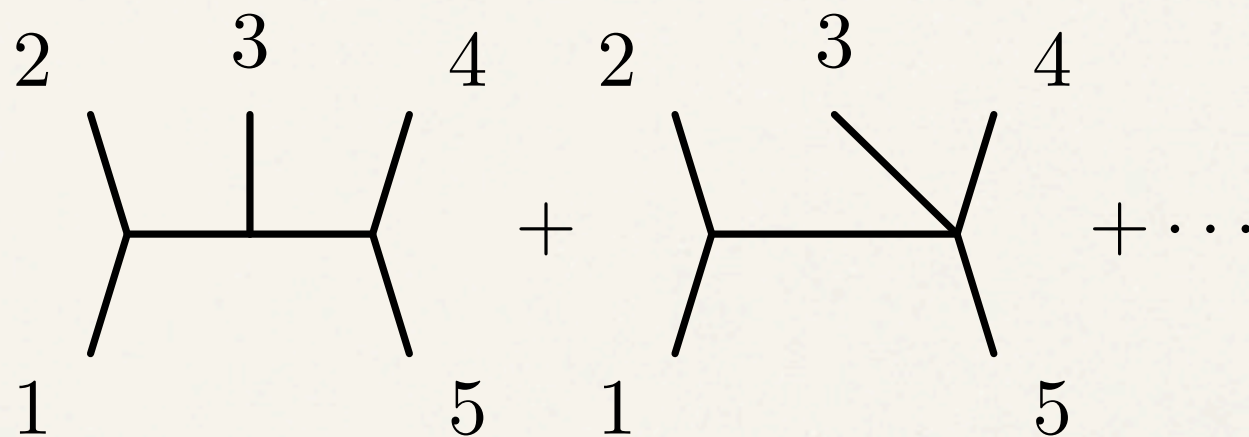


$$c = f^{a_1 a_2 b_1} f^{b_1 a_3 b_2} f^{b_2 a_4 a_5}$$

Structure in amplitudes

- ✧ But the colour-dressed amplitude itself has interesting structure

$$\mathcal{A}(1, 2, \dots, n) = \sum_{\text{diagrams}} \frac{c_i n_i}{D_i}$$



$$c = \begin{cases} f^{a_1 a_2 b_1} f^{b_1 a_3 b_2} f^{b_2 a_4 a_5} \\ f^{a_1 a_2 b_1} f^{b_1 a_4 b_2} f^{b_2 a_5 a_3} \\ f^{a_1 a_2 b_1} f^{b_1 a_5 b_2} f^{b_2 a_3 a_4} \end{cases}$$

Structure in amplitudes

- But the colour-dressed amplitude itself has interesting structure

$$\mathcal{A}(1, 2, \dots, n) = \sum_{\text{diagrams}} \frac{c_i n_i}{D_i}$$

The diagram shows the expansion of a four-point amplitude (represented by a shaded circle with four external lines labeled 1, 2, 3, 4) into a sum of tree-level diagrams. The first row shows two diagrams: a box diagram with external lines 2, 3, 4, 1 and an internal line 5, and a triangle diagram with external lines 2, 3, 4, 1 and an internal line 5. The second row shows three diagrams: a box diagram with external lines 2, 3, 4, 1 and an internal line 5, a triangle diagram with external lines 2, 3, 4, 1 and an internal line 5, and a triangle diagram with external lines 2, 3, 4, 1 and an internal line 5. The expansion is followed by an ellipsis.

$$c = \begin{cases} f^{a_1 a_2 b_1} f^{b_1 a_3 b_2} f^{b_2 a_4 a_5} \\ f^{a_1 a_2 b_1} f^{b_1 a_4 b_2} f^{b_2 a_5 a_3} \\ f^{a_1 a_2 b_1} f^{b_1 a_5 b_2} f^{b_2 a_3 a_4} \end{cases}$$

Structure in amplitudes

- ❖ Can remove quartic vertex
 - ❖ Assign terms coming from quartic to three point vertices according to colour factor
 - ❖ Not unique

The diagram illustrates the replacement of a quartic vertex with a sum of terms involving three-point vertices. On the left, a quartic vertex is represented by four external lines labeled 1, 2, 3, and 4, with lines 1 and 2 on the left and 3 and 4 on the right. This is set equal to a sum of terms. The first term in the sum is a diagram with a horizontal internal line. The left vertex of this internal line is connected to lines 1 and 2, and the right vertex is connected to lines 3 and 4. Above the horizontal line is the label $(p_1 + p_2)^2$. To the right of this term is an ellipsis $+\dots$.

$$\begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} = \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ (p_1 + p_2)^2 \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} + \dots$$

Structure in amplitudes

- ✧ Amplitude a sum over cubic diagrams Bern, Carrasco, Johansson (BCJ)

$$\mathcal{A}_{YM} = \sum_{i \in \text{cubic}} \frac{n_i c_i}{D_i}$$

$$c_i + c_j + c_k = 0 \quad \text{Jacobi identity, group theory}$$

$$\begin{array}{c}
 \begin{array}{c} 2 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array}
 \begin{array}{c} 3 \\ \diagup \\ \text{---} \\ \diagdown \\ 4 \end{array}
 +
 \begin{array}{c} 2 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array}
 \begin{array}{c} 3 \\ \diagup \\ \text{---} \\ \diagdown \\ 4 \end{array}
 +
 \begin{array}{c} 2 \\ \diagdown \\ \text{---} \\ \diagup \\ 1 \end{array}
 \begin{array}{c} 3 \\ \diagup \\ \text{---} \\ \diagdown \\ 4 \end{array}
 = 0
 \end{array}$$

Structure in amplitudes

- Amplitude a sum over cubic diagrams Bern, Carrasco, Johansson (BCJ)

$$\mathcal{A}_{YM} = \sum_{i \in \text{cubic}} \frac{n_i c_i}{D_i}$$

$$c_i + c_j + c_k = 0 \quad \text{Jacobi identity, group theory}$$

$$n_i + n_j + n_k = 0$$

$$\begin{array}{c} 2 \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ 1 \qquad 4 \end{array} + \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ 1 \qquad 4 \end{array} + \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ 1 \qquad 4 \end{array} = 0$$

Structure in amplitudes

- Amplitude a sum over cubic diagrams Bern, Carrasco, Johansson (BCJ)

$$\mathcal{A}_{YM} = \sum_{i \in \text{cubic}} \frac{n_i c_i}{D_i}$$

$$c_i + c_j + c_k = 0 \quad \text{Jacobi identity, group theory}$$

$$n_i + n_j + n_k = 0 \quad \text{Is there a kinematic group?}$$

$$\begin{array}{c} 2 \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ 1 \end{array}
 \begin{array}{c} 3 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ 4 \end{array}
 +
 \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ 1 \end{array}
 \begin{array}{c} 3 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ 4 \end{array}
 +
 \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ 1 \end{array}
 \begin{array}{c} 3 \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ 4 \end{array}
 = 0$$

Structure in amplitudes

- Amplitude a sum over cubic diagrams Bern, Carrasco, Johansson (BCJ)

$$\mathcal{A}_{YM} = \sum_{i \in \text{cubic}} \frac{n_i c_i}{D_i}$$

Colour-kinematic duality $\left\{ \begin{array}{l} c_i + c_j + c_k = 0 \\ n_i + n_j + n_k = 0 \end{array} \right.$ Jacobi identity, group theory
Is there a kinematic group?

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} = 0$$

Structure in amplitudes

- ✧ Amplitude a sum over cubic diagrams Bern, Carrasco, Johansson (BCJ)

$$\mathcal{A}_{YM} = \sum_{i \in \text{cubic}} \frac{n_i c_i}{D_i}$$

Colour-kinematic duality $\left\{ \begin{array}{ll} c_i + c_j + c_k = 0 & \text{Jacobi identity, group theory} \\ n_i + n_j + n_k = 0 & \text{Is there a kinematic group?} \end{array} \right.$

- ✧ Loops

$$\mathcal{A}_{YM} = \int \frac{d^{LD} \ell}{(2\pi)^{LD}} \sum_{i \in \text{cubic}} \frac{1}{S_i} \frac{n_i c_i}{D_i}$$

Some consequences

1. BCJ relations for colour-ordered amplitudes

$$s_{35}A_5(1, 2, 4, 3, 5) - (s_{13} + s_{23})A_5(1, 2, 3, 4, 5) - s_{13}A_5(1, 3, 2, 4, 5) = 0$$

* Basis of independent colour-ordered n -point amplitudes $(n - 3)!$

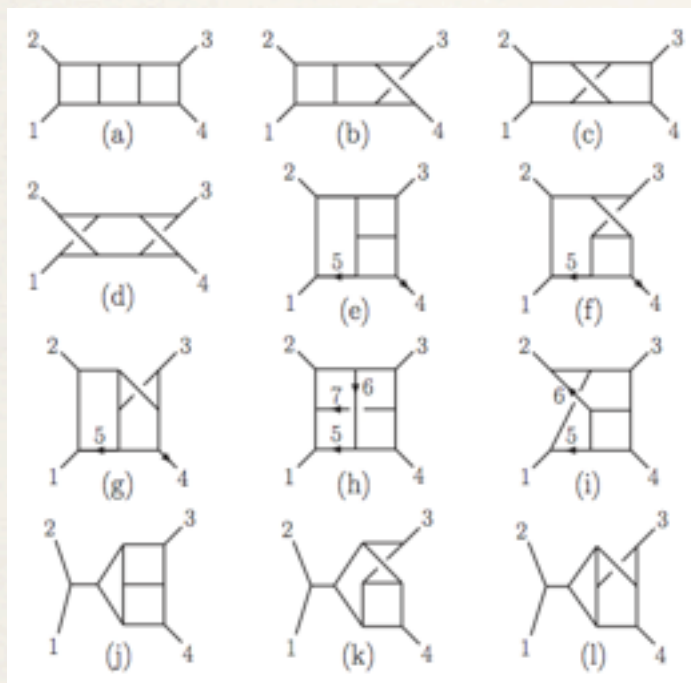
2. New expression for gravity amplitude (the double-copy formula)

$$A_{YM} = \sum_{i \in \text{cubic}} \frac{n_i c_i}{D_i} \qquad \mathcal{M} = \sum_{i \in \text{cubic}} \frac{n_i \tilde{n}_i}{D_i}$$

An example: 3 loops

- Complete 3 loop maximal SYM/SUGRA amplitude

BCJ



(a)–(d)	s^2
(e)–(g)	$(s(-\tau_{35} + \tau_{45} + t) - t(\tau_{25} + \tau_{45}) + u(\tau_{25} + \tau_{35}) - s^2)/3$
(h)	$(s(2\tau_{15} - \tau_{16} + 2\tau_{26} - \tau_{27} + 2\tau_{35} + \tau_{36} + \tau_{37} - u) + t(\tau_{16} + \tau_{26} - \tau_{37} + 2\tau_{36} - 2\tau_{15} - 2\tau_{27} - 2\tau_{35} - 3\tau_{17}) + s^2)/3$
(i)	$(s(-\tau_{25} - \tau_{26} - \tau_{35} + \tau_{36} + \tau_{45} + 2t) + t(\tau_{26} + \tau_{35} + 2\tau_{36} + 2\tau_{45} + 3\tau_{46}) + u\tau_{25} + s^2)/3$
(j)–(l)	$s(t - u)/3$

$$\tau_{ij} = 2l_i \cdot p_j$$

- Jacobi relations relate planar and non-planar diagrams
 - Simplicity beyond the planar theory; simplicity in gravity

But...

- ❖ This is great!
- ❖ How do we compute BCJ numerators?
- ❖ We don't know
- ❖ Make a guess, solve for undetermined coefficients, repeat

Current status

- ❖ Three main parts aspects of the BCJ story

1. Existence of the numerators
2. BCJ relations for amplitudes
3. Double copy formula

Current status

- ❖ Three main parts aspects of the BCJ story

1. Existence of the numerators

Proven (tree only)

2. BCJ relations for amplitudes

3. Double copy formula

Current status

- ❖ Three main parts aspects of the BCJ story

1. Existence of the numerators

Proven (tree only)

2. BCJ relations for amplitudes

Proven

3. Double copy formula

Current status

- ❖ Three main parts aspects of the BCJ story

- | | |
|---------------------------------|--------------------|
| 1. Existence of the numerators | Proven (tree only) |
| 2. BCJ relations for amplitudes | Proven |
| 3. Double copy formula | Proven (tree only) |

Bjerrum-Bohr, Damgaard, Vanhove, Stieberger,
Feng, Huang, Jia, Cachazo, Mafra, Schlotterer,
Monteiro, DOC, Bern, Carrasco, Johansson,
Dennen, Huang, Kiermaier

Manifest duality in loops?

- ❖ So what do we know about loops?

	1 loop	2 loop	3 loop	4 loop	L loop
4 pt	✓	✓	✓	✓	?
5 pt	✓	✓	✓*	?	?
6 pt	?	?	?	?	?
n pt	?	?	?	?	?

- ❖ Maximal supersymmetry only

Manifest duality in loops?

- ❖ So what do we know about loops?

	1 loop	2 loop	3 loop	4 loop	L loop
4 pt	✓	✓	✓	✓	?
5 pt	✓	✓	✓*	?	?
6 pt	?	?	?	?	?
n pt	?	?	?	?	?

Bern,
Carrasco,
Dixon,
Johansson,
Roiban

- ❖ Maximal supersymmetry only

Outline

1. Manifest colour-kinematic duality in self-dual Yang-Mills
2. Algebra- and superalgebra-based scalar theories as bases for gauge amplitudes

Manifest duality

SD Yang-Mills and Gravity

- ✧ Colour-kinematic duality is manifest in the self-dual sector

R. Monteiro & DOC

$$\partial^2 \Phi + ig[\partial_w \Phi, \partial_u \Phi] = 0$$

YM: Leznov & Mukhtarov, Parkes,
Bardeen, Chalmers & Siegel,
Cangemi

$$\partial^2 \phi + \kappa\{\partial_w \phi, \partial_u \phi\} = 0$$

GR: Plebanski equation of motion

w and u are spacetime coords, and

$$\{f, g\} = \partial_w f \partial_u g - \partial_u f \partial_w g$$

Kinematic algebra

- ✧ Brace generates diffeomorphism algebra

$$\{e^{-ip_1 \cdot x}, e^{-ip_2 \cdot x}\} = -X(p_1, p_2)e^{-i(p_1+p_2) \cdot x}, \quad X(p_1, p_2) = p_{1w}p_{2u} - p_{2w}p_{1u}$$

- ✧ Generators of infinitesimal diffeomorphisms in u, w plane

$$L_k = e^{-ik \cdot x} (-k_w \partial_u + k_u \partial_w)$$

- ✧ Lie algebra $[L_{p_1}, L_{p_2}] = iX(p_1, p_2)L_{p_1+p_2} = iF_{p_1 p_2}^{\quad k} L_k$

$$F_{p_1 p_2 p_3} = (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3) X(p_1, p_2)$$

- ✧ Structure constants of area-preserving diffeomorphisms

Kinematic algebra

- ✧ Gravity equation of motion

$$k^2 \phi(k) = \frac{1}{2} \kappa \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} F_{p_1 p_2}{}^k X(p_1, p_2) \phi(p_1) \phi(p_2)$$

- ✧ Gauge case

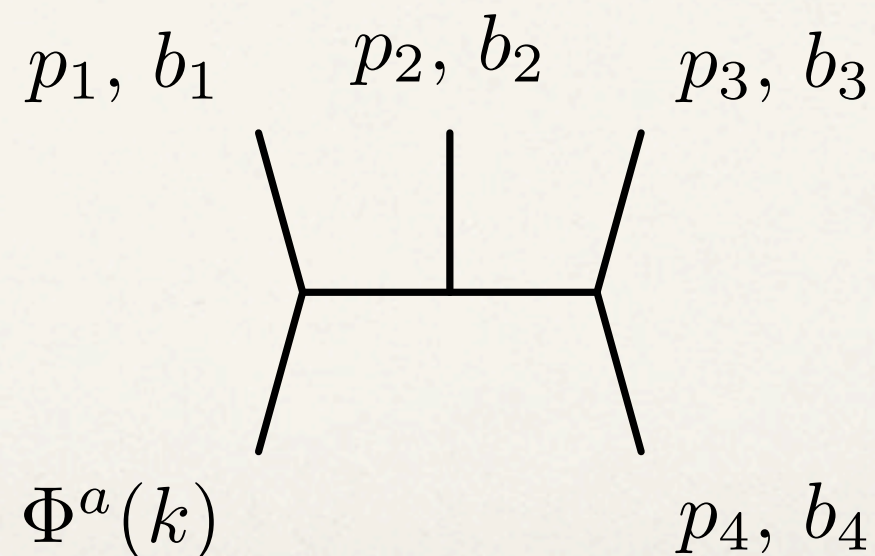
$$k^2 \Phi^a(k) = \frac{1}{2} g \int \frac{d^4 p_1}{(2\pi)^4} \frac{d^4 p_2}{(2\pi)^4} F_{p_1 p_2}{}^k f^{b_1 b_2 a} \Phi^{b_1}(p_1) \Phi^{b_2}(p_2)$$

- ✧ Let's see how the duality work in detail!

Kinematic algebra

- ❖ Solve equations of motion perturbatively

$$\Phi^a(k) = \cdots + \int \frac{dp_1 dp_2 dp_3 dp_4}{k^2 s_{34} s_{234}} F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \times \\ \times j^{b_1}(p_1) j^{b_2}(p_2) j^{b_3}(p_3) j^{b_4}(p_4) f^a{}_{b_1}{}^{c_1} f_{c_1 b_2}{}^{c_3} f_{c_3 b_3 b_4} + \cdots$$



Kinematic algebra

- ❖ Solve equations of motion perturbatively
- Manifest duality

$$\Phi^a(k) = \cdots + \int \frac{dp_1 dp_2 dp_3 dp_4}{k^2 s_{34} s_{234}} F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \times \\ \times j^{b_1}(p_1) j^{b_2}(p_2) j^{b_3}(p_3) j^{b_4}(p_4) f^a{}_{b_1}{}^{c_1} f_{c_1 b_2}{}^{c_3} f_{c_3 b_3 b_4} + \cdots$$

Kinematic algebra

- ❖ Solve equations of motion perturbatively

$$\Phi^a(k) = \cdots + \int \frac{dp_1 dp_2 dp_3 dp_4}{k^2 s_{34} s_{234}} F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \times \\ \times j^{b_1}(p_1) j^{b_2}(p_2) j^{b_3}(p_3) j^{b_4}(p_4) f^a{}_{b_1}{}^{c_1} f_{c_1 b_2}{}^{c_3} f_{c_3 b_3 b_4} + \cdots$$

- ❖ Square numerator?

$$F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} = X(-k, p_1) X(p_1 - k, p_2) X(p_3, p_4) \\ \delta^{(4)}(p_1 + p_2 + p_3 + p_4 - k)$$

Kinematic algebra

- ❖ Solve equations of motion perturbatively

$$\Phi^a(k) = \cdots + \int \frac{dp_1 dp_2 dp_3 dp_4}{k^2 s_{34} s_{234}} F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \times \\ \times j^{b_1}(p_1) j^{b_2}(p_2) j^{b_3}(p_3) j^{b_4}(p_4) f^a{}_{b_1}{}^{c_1} f_{c_1 b_2}{}^{c_3} f_{c_3 b_3 b_4} + \cdots$$

- ❖ Square numerator?

$$F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \rightarrow X(-k, p_1) X(p_1 - k, p_2) X(p_3, p_4)$$

Kinematic algebra

- ❖ Solve equations of motion perturbatively

$$\Phi^a(k) = \cdots + \int \frac{dp_1 dp_2 dp_3 dp_4}{k^2 s_{34} s_{234}} F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \times \\ \times j^{b_1}(p_1) j^{b_2}(p_2) j^{b_3}(p_3) j^{b_4}(p_4) f^a{}_{b_1}{}^{c_1} f_{c_1 b_2}{}^{c_3} f_{c_3 b_3 b_4} + \cdots$$

- ❖ Square numerator?

$$F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \rightarrow X(-k, p_1) X(p_1 - k, p_2) X(p_3, p_4)$$

$$\phi(k) = \cdots + \int \frac{dp_1 dp_2 dp_3 dp_4}{k^2 s_{34} s_{234}} F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \\ j(p_1) j(p_2) j(p_3) j(p_4) X(-k, p_1) X(p_1 - k, p_2) X(p_3, p_4) + \cdots$$

Kinematic algebra

- ❖ Solve equations of motion perturbatively

$$\Phi^a(k) = \cdots + \int \frac{dp_1 dp_2 dp_3 dp_4}{k^2 s_{34} s_{234}} F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \times \\ \times j^{b_1}(p_1) j^{b_2}(p_2) j^{b_3}(p_3) j^{b_4}(p_4) f^a{}_{b_1}{}^{c_1} f_{c_1 b_2}{}^{c_3} f_{c_3 b_3 b_4} + \cdots$$

- ❖ Square numerator?

$$F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \rightarrow X(-k, p_1) X(p_1 - k, p_2) X(p_3, p_4)$$

$$\phi(k) = \cdots + \int \frac{dp_1 dp_2 dp_3 dp_4}{k^2 s_{34} s_{234}} F^k_{p_1}{}^{q_1} F_{q_1 p_2}{}^{q_3} F_{q_3 p_3 p_4} \quad \text{Double copy} \\ j(p_1) j(p_2) j(p_3) j(p_4) X(-k, p_1) X(p_1 - k, p_2) X(p_3, p_4) + \cdots$$

Kinematic algebra

- ❖ Side remark, can express solution of gauge theory EOM as

$$\begin{aligned}\Phi^{(n)}(k) = & (ig)^n \int dp_1 dp_2 \dots dp_{n+1} \delta(p_1 + p_2 + \dots + p_{n+1} - k) \\ & \times j(p_1)j(p_2) \dots j(p_{n+1}) \frac{1}{Q(p_1, p_2)Q(p_2, p_3) \dots Q(p_n, p_{n+1})}\end{aligned}$$

where

$$\begin{aligned}Q(p) &= \frac{p_{\bar{u}}}{p_u} \\ Q(p_1, p_2) &= Q(p_1) - Q(p_2)\end{aligned}$$

$$\begin{aligned}X(1, 2) &= [12] \\ Q(1, 2) &= \langle 12 \rangle\end{aligned}$$

Kinematic algebra

- ❖ Side remark, can express solution of gauge theory EOM as

$$\Phi^{(n)}(k) = (ig)^n \int dp_1 dp_2 \dots dp_{n+1} \delta(p_1 + p_2 + \dots + p_{n+1} - k) \\ \times j(p_1) j(p_2) \dots j(p_{n+1}) \frac{1}{Q(p_1, p_2) Q(p_2, p_3) \dots Q(p_n, p_{n+1})}$$

where

$$Q(p) = \frac{p_{\bar{u}}}{p_u} \\ Q(p_1, p_2) = Q(p_1) - Q(p_2)$$

Appears elsewhere in mathematics

$$X(1, 2) = [12] \\ Q(1, 2) = \langle 12 \rangle$$

Kinematic algebra

- ✧ Consider space $\mathbb{C}^{n+1}$ coords Q_i , $i = 1, 2, \dots, n + 1$
- ✧ Now omit lines where Q 's coincide $\{\mathbb{C}^{n+1} | Q_i \neq Q_j\}$ and define

$$\theta_{ij} = \frac{dQ_i - dQ_j}{Q_i - Q_j} = \frac{dQ_i - dQ_j}{Q(i, j)}$$

- ✧ Basis for the cohomology group H^{n-1} is Arnold

$$\theta_{12}\theta_{23} \cdots \theta_{n,n+1} = \frac{(-1)^{n+1} \sum (-1)^i dQ_1 \cdots \widehat{dQ_i} \cdots dQ_n}{Q(1, 2)Q(2, 3) \cdots Q(n, n + 1)}$$

- ✧ Space of these objects is the same as space $\text{Lie}(n + 1)$ of multilinear Lie polynomials in $n + 1$ formal variables

Kapranov

Kinematic algebra

- ✧ Three point function from gravity equation

$$\partial^2 \phi + \kappa \{ \partial_w \phi, \partial_u \phi \} = 0 \quad \Rightarrow \quad \mathcal{M}_3 = X(p_1, p_2)^2 \delta(p_1 + p_2 + p_3)$$

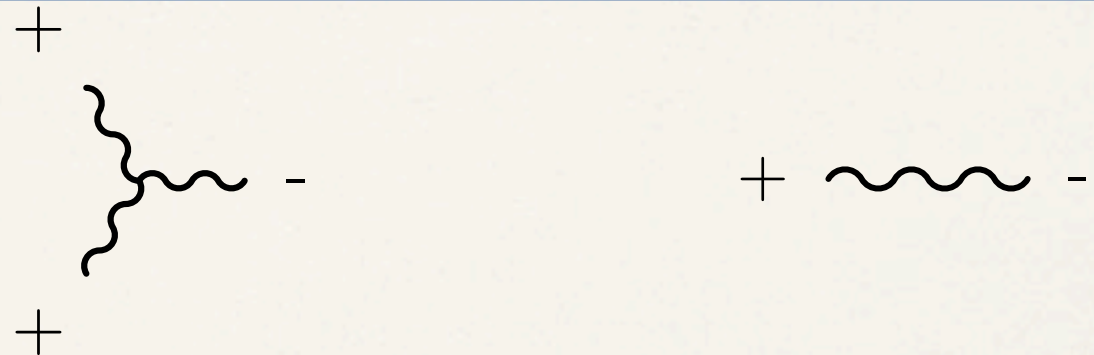
- ✧ “Reverse KLT”

$$\mathcal{A}_3 = X(p_1, p_2) \delta(p_1 + p_2 + p_3) f^{abc} = F_{p_1 p_2 p_3} f^{abc}$$

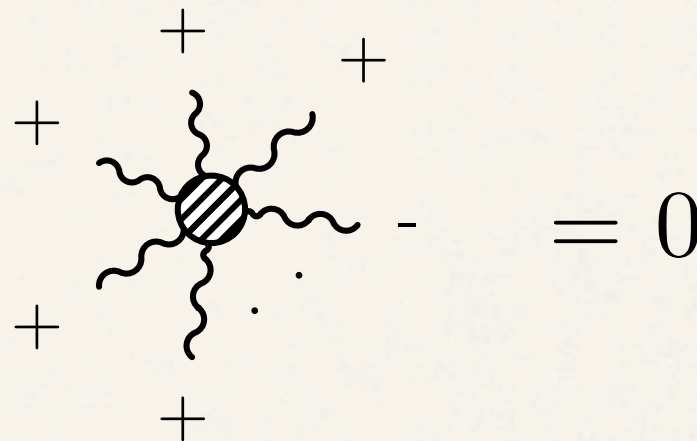
- ✧ So same algebra appears in YM
- ✧ Physics of colour-kinematics is gravitational

Self-dual amplitudes

- ✧ Feynman rules in SD theory

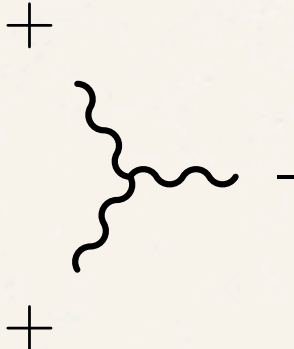


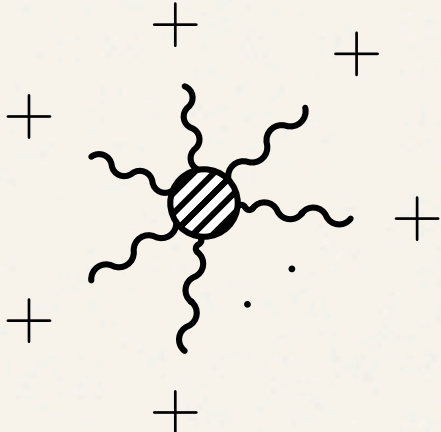
- ✧ Tree amplitudes



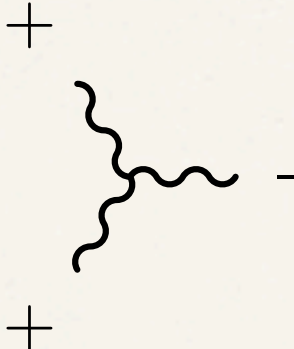
- ✧ MHV amplitudes inherit structure of kinematic algebra from SD
 - ✧ Choose an appropriate gauge

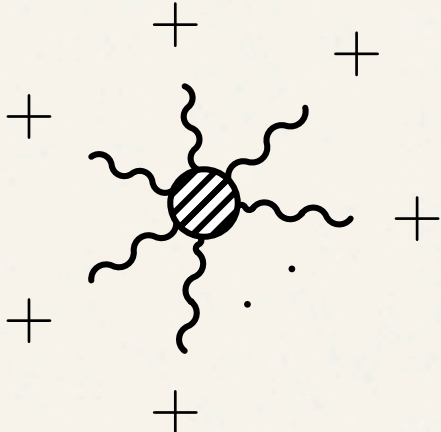
Self-dual amplitudes

- ✧ Only vertex in SD theory is 

- ✧ One loop amplitudes  $\neq 0$

Self-dual amplitudes

- ❖ Only vertex in SD theory is 

- ❖ One loop amplitudes  $\neq 0$

- ❖ Same as usual YM all-plus amplitudes

All-plus amplitudes

- ✧ All one-loop all-plus amplitudes can be computed in self-dual theory

$$\mathcal{A}_{++++} = \text{[Diagram 1]} + \text{[Diagram 2]} + \dots$$

The first diagram is a hexagon with an internal diagonal line. The top-left and top-right edges are labeled F . The bottom-left and bottom-right edges are labeled F . The internal diagonal line is labeled F . The second diagram is a more complex structure with multiple internal lines, also labeled with F .

	1 loop	2 loop	3 loop	4 loop	L loop
4 pt	✓	✓	✓	✓	?
5 pt	✓	✓	?	?	?
6 pt	?	?	?	?	?
n pt	?	?	?	?	?

All-plus amplitudes

- ✧ All one-loop all-plus amplitudes can be computed in self-dual theory

$$\mathcal{A}_{++++} = \text{[Diagram 1]} + \text{[Diagram 2]} + \dots$$

The first diagram is a hexagon with two external lines on the left and two on the right. The top-left internal line is labeled F , the top-right is F , the bottom-left is F , and the bottom-right is F . The second diagram is a more complex structure with a central vertex connected to four other vertices, forming a loop. The top-left internal line is labeled F , the top-right is F , the bottom-left is F , and the bottom-right is F .

	1 loop	2 loop	3 loop	4 loop	L loop
4 pt	✓	✓	✓	✓	?
5 pt	✓	✓	?	?	?
6 pt	✓ / ?	?	?	?	?
n pt	✓ / ?	?	?	?	?

A scalar basis using algebras

A linear space

- ✧ YM amplitudes satisfy KK, BCJ relations
- ✧ These are linear equations
 - ✧ There is a linear operator acting on $(n - 1)!$ space of amplitudes

$$\sum_{\beta} \tilde{S}[\alpha_{2,n} | \beta_{2,n}] A(1, \beta_{2,n}) = 0$$

Damgaard, Bjerrum-Bohr,
Søndergaard, Vanhove

- ✧ Kernel of $\tilde{S}$ has dimensions $(n - 3)!$

Scalar theories

- ❖ YM is not only theory with KK, BCJ relations
- ❖ Consider cubic scalar theory with two different structure constants at each vertex

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \text{---} = g f^{abc} f'^{a'b'c'}$$

Scalar theories

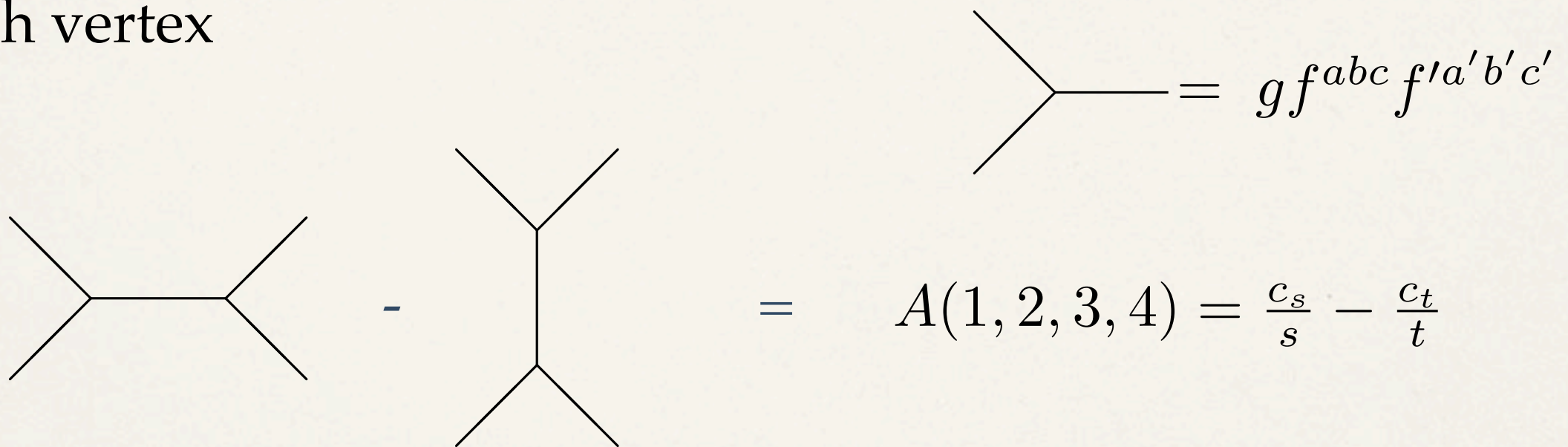
- ✧ YM is not only theory with KK, BCJ relations
- ✧ Consider cubic scalar theory with two different structure constants at each vertex

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \text{---} \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} - \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \\ \diagdown \end{array} = g f^{abc} f'^{a'b'c'}$$

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} - \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = A(1, 2, 3, 4) = \frac{n_s}{s} - \frac{n_t}{t}$$

Scalar theories

- ✧ YM is not only theory with KK, BCJ relations
- ✧ Consider cubic scalar theory with two different structure constants at each vertex

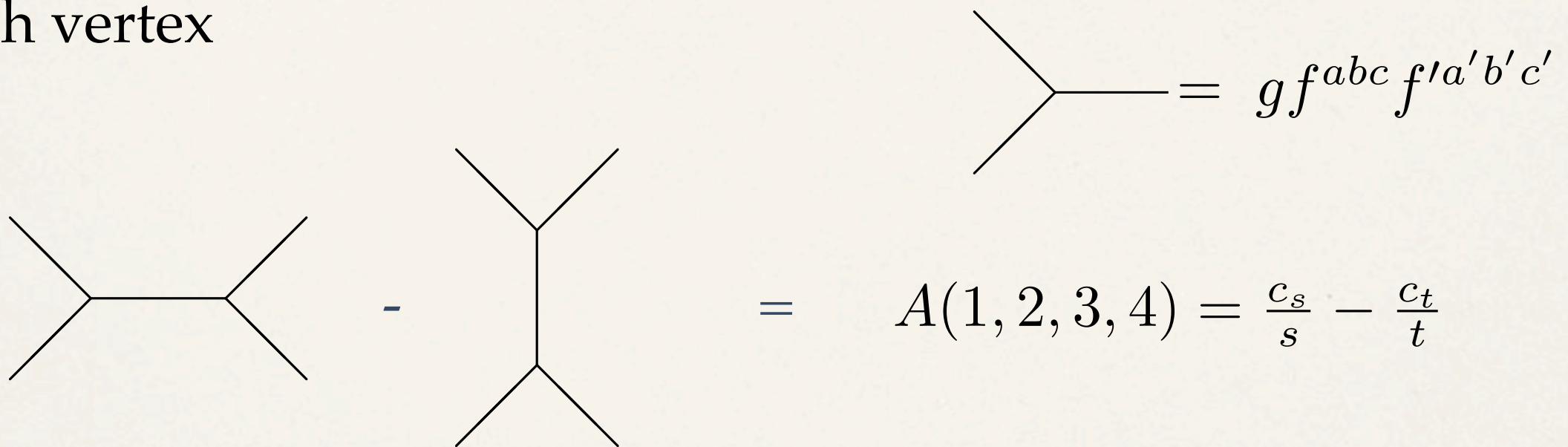


$$\begin{array}{c} \diagup \\ \diagdown \end{array} \text{---} \begin{array}{c} \diagup \\ \diagdown \end{array} - \begin{array}{c} \diagup \\ \diagdown \end{array} \text{---} \begin{array}{c} \diagup \\ \diagdown \end{array} = g f^{abc} f'^{a'b'c'}$$

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \text{---} \begin{array}{c} \diagup \\ \diagdown \end{array} - \begin{array}{c} \diagup \\ \diagdown \end{array} \text{---} \begin{array}{c} \diagup \\ \diagdown \end{array} = A(1, 2, 3, 4) = \frac{c_s}{s} - \frac{c_t}{t}$$

Scalar theories

- ✧ YM is not only theory with KK, BCJ relations
- ✧ Consider cubic scalar theory with two different structure constants at each vertex



$$\begin{aligned}
 & \text{Diagram 1} - \text{Diagram 2} = A(1, 2, 3, 4) = \frac{c_s}{s} - \frac{c_t}{t} \\
 & \text{Diagram 3} = g f^{abc} f'^{a'b'c'}
 \end{aligned}$$

- ✧ KK true
- ✧ BCJ true (explicit proof by Du, Feng and Fu)

A basis

- ❖ YM, scalar theories all in kernel of $\tilde{S}$
- ❖ Pick $(n-3)!$ independent cubic scalar theories; YM a linear combination

J th colour-ordered amplitude
of I th scalar theory

$$A_{(J)} = \sum_{I=1}^{(n-3)!} \alpha_I \theta_{IJ}$$

- ❖ When θ is non-singular, unique expansion coefficients α exist

$$\mathcal{A} = \sum_{I=1}^{(n-3)!} \alpha_I \Theta_I \leftarrow \text{Colour dressed}$$

Manifest colour-kinematics duality

- ✧ Scalar basis has cubic numerators with manifest Jacobi

$$\Theta_I = \sum_k \frac{\tilde{c}_{I,k} c_k}{D_k}$$

- ✧ So YM numerators represented as

$$n_k^{YM} = \sum_{I=1}^{(n-3)!} \alpha_I \tilde{c}_{I,k}$$

- ✧ Jacobi relations follow by linearity

What about loops?

- * $\mathcal{N} = 4$ one loop diagrams: no bubbles, no triangles
 - * Scalar theories using superalgebras can reproduce this
- * Killing form $a \text{ --- } \bigcirc \text{ --- } b = (-1)^{|c|} f_{ac}{}^d f_{bd}{}^c = K_{ab}$
- * K vanishes for superalgebras $\mathfrak{psu}(n|n), \dots$
- * Scalar theories based on such algebras: no bubbles
 - * Similar ideas in (partially) quenched chiral perturbation theory

Jacobi for Triangle

- ✧ Using bosonic external states, Jacobi removes triangle

The diagram illustrates the Jacobi identity for a triangle diagram with bosonic external states. It is presented as an equation: $\text{Triangle}(3, 2, 1) = \text{Bubble}(3, 2, 1) - \text{Triangle}(3, 1, 2)$.
- The first term on the left is a triangle diagram with external legs labeled 3 (top), 2 (bottom), and 1 (right).
- The middle term is a bubble diagram with external legs labeled 3 (top), 2 (bottom), and 1 (right).
- The third term is another triangle diagram with external legs labeled 3 (top), 2 (bottom), and 1 (right), representing a different orientation of the triangle.

- ✧ So superalgebra-based scalar theories also have no triangles or bubbles

Four point $\mathcal{N} = 4$ amplitude

- ✧ Integrand is

$$\mathcal{I} = st\mathcal{A}_{\text{tree}} \frac{1}{l^2(l+p_1)^2(l+p_1+p_2)^2(l-p_4)^2}$$

- ✧ Scalar theory

$$\theta = (-1)^{|d|} f^{d1a} f^{a2b} f^{b3c} f^{c4d} \frac{1}{l^2(l+p_1)^2(l+p_1+p_2)^2(l-p_4)^2}$$

- ✧ Eg $\mathfrak{psu}(2|2)$

$$(-1)^{|d|} f^{d1a} f^{a2b} f^{b3c} f^{c4d} = 2$$

(completely symmetric in 1,2,3,4)

- ✧ So $\mathcal{I} = \alpha\theta$, $\alpha = \frac{st\mathcal{A}_{\text{tree}}}{2}$

Conclusions

- ❖ Self-dual sector: manifest colour-kinematics
 - ❖ Kinematic algebra is an area-preserving diffeomorphism algebra
 - ❖ All one-loop all-plus numerators can be chosen to manifest colour-kinematic duality
 - ❖ MHV sector: inherits kinematic algebra from self-dual
- ❖ Scalar theories promising route to understand duality at loop level