The black hole stability problem

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Outline

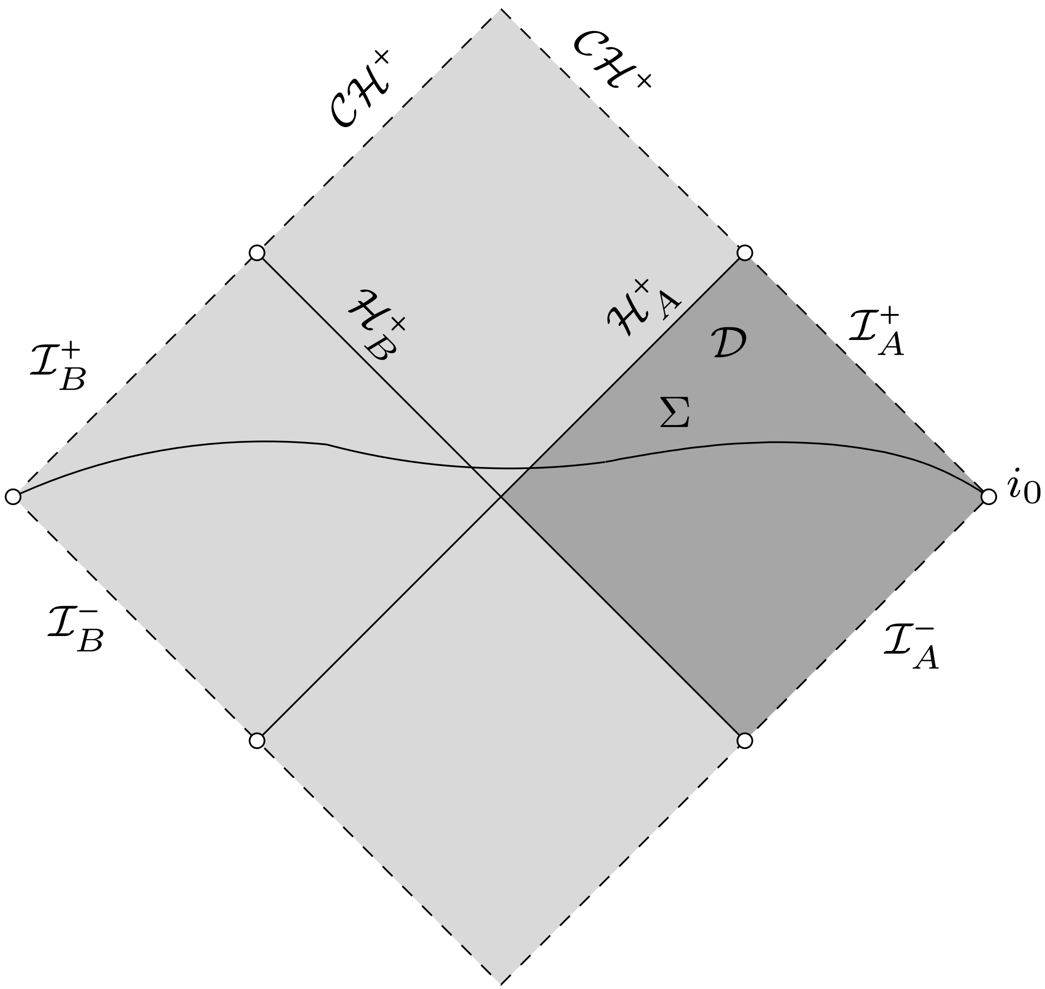
1. The classical black hole stability problem: a status report

2. Extremal black holes

3. The asymptotically AdS case

Goal: the nonlinear stability problem of Kerr

Perturbations of a Kerr metric should dynamically approach the Kerr family **in the exterior-to-the-black-hole region**.



More precise formulation

Conjecture (Stability of Kerr)**.** *Let* $(\Sigma, \bar{g}, K)$ *be a vacuum initial data set sufficiently close to the initial data on a Cauchy hypersurface* Σ *in the Kerr solution* $(\mathcal{M}, g_{M_i,a_i})$ *for some parameters* $0 \leq |a_i| < M_i$*. Then the maximal vacuum Cauchy development* $(\mathcal{M}, g)$ *possesses a complete null infinity* $\mathcal{I}^+$ *such that the metric restricted to* $J^-(\mathcal{I}^+)$ *approaches a Kerr solution* $(\mathcal{M}, g_{M_f,a_f})$ *in a uniform way with quantitative decay rates, where* M_f*,* a_f *are near* M_i*,* a_i *respectively.*

Note: $a_i = 0$ **will not imply** that $a_f = 0$! Thus, one cannot study separately the stability of Schwarzschild without understanding Kerr for $|a| \ll M$.

Cf. stability of Minkowski space

Theorem (Christodoulou–Klainerman, 1993). *Consider an asymptotically flat vacuum initial data set* $(\Sigma, \bar{g}, K)$ *which moreover is globally "small". Let* $(\mathcal{M}, g)$ *be the arising maximal vacuum Cauchy development spacetime. Then* $(\mathcal{M}, g)$ *is geodesically complete and approaches the Minkowski metric (with quantitative decay rates) along all causal geodesics.*

In fact, the theorem proves much more: The "Penrose diagramme" of Minkowski space is stable, one can define a notion of null infinity (with $\mathcal{M} = J^{-}(\mathcal{I}^{+})$), Bondi mass, news function, etc., giving the non-linear laws of gravitational radiation at infinity, Christodoulou memory, etc.

***Why* is Minkowski space stable?**

The heuristic idea of the proof is simple: Minkowski space is stable because perturbations radiate and decay *sufficiently fast*.

It is hard to prove, because, the rate of decay is *just fast enough*.

The analogue of stability of Minkowski space is **much easier** for spatial dimension $n \geq 4$, and can now be given in just a few pages (see recent paper of Choquet-Bruhat–Chrusciel–Loizelet).

More recent developments: Lindblad–Rodnianski, Bieri.

There is no shortcut to orbital stability. Without proving sufficiently fast convergence to Minkowski space, one cannot prove any stability statement whatsoever!

Turning back to the black hole case, it follows that if one can understand properly quantitative dispersion for a suitable formulation of the "linearised problem", one expects that this can be used to prove non-linear stability of Kerr.

The main difficulty in the nonlinear stability problem of Kerr is that the linear problem has not been understood.

“Poor man’s” linear theory: Boundedness and decay for $\Box_g \psi = 0$ on Schwarzschild and Kerr:

Early contributors include: Carter, Chandrasekhar, Hartle–Wilkins, Kay–Wald, Moncrief, Press–Teukolsky, Price, Regge–Wheeler, Vishveshwara, Whiting, . . .

Lots of recent work in the last 10 years surveyed in:

[See M. D.–Rodnianski *Lectures on black holes and linear waves*, arXiv:0811.0354, *The black hole stability problem for linear scalar perturbations*, arXiv:1010.5137]

Current state of the art for $\Box_g \psi = 0$

1. **Boundedness** of ψ on a general class of C^1 stationary axisymmetric metrics g near Schwarzschild (M.D.–Rodnianski)

2. **"Integrated local energy decay"** for ψ on exactly Kerr (or approaching Kerr):

   1. $|a| \ll M$ (M.D.–Rodnianski, Tataru–Tohaneanu, Andersson–Blue), and

   2. $|a| < M$ (M.D.–Rodnianski)

3. Pointwise-in-time decay from 1. and 2. Two methods available:

   1. pure physical-space-based energy method using *new* version of vector field method [M.D.–Rodnianski]
      robust to quasilinear applications, Einstein equations!

   2. resolvent or fund. solution estimates, coupled with usual vector field technique [Tataru, Metcalfe–Tataru–Tohaneanu])

In particular:

Theorem (M.D.–Rodnianski, 2010). *Solutions to* $\Box_g \psi = 0$ *on Kerr exterior spacetimes* $\boxed{\textit{in the full subextremal range } |a| < M}$ *decay quantitatively at a polynomial rate. The quantitative decay estimate is sharp in the class of admissible data.*

For the proof of the above theorem, see: *The black hole stability problem for linear scalar perturbations*, arXiv:1010.5137

The proof applies also to spacetimes asymptoting to a Kerr solution in a manner compatible with the type of decay proven (cf. stability of Minkowski space). *Were the non-linear stability of Kerr a "scalar problem", then in principle it could be addressed using the above theorem.* See work of Luk.

The main difficulties

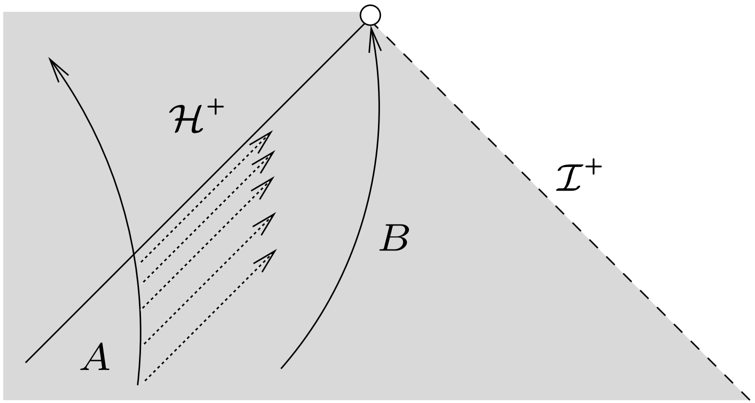
1. Red-shift effect
2. Superradiance
3. Trapping

(and

4. The coupling of 1–3)

The red-shift

The **redshift** is classically understood in the geometric optics approximation in terms of signals sent by two observers A and B.



First discussed in the Schwarzschild setting by Oppenheimer–Snyder, 1939.

In fact, properly thought of, only depends on positivity of surface gravity.

Extremal case $a = M$: The red-shift factor at the horizon vanishes.

Superradiance

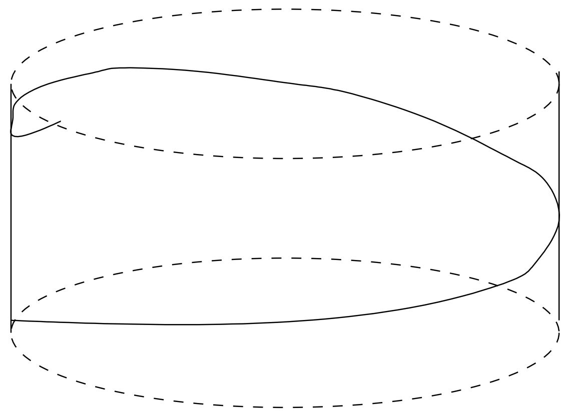
In Schwarzschild ($a = 0$), the Killing vector field ∂_t is timelike in the exterior, becoming null on the horizon. Thus there is a **conserved** *(by Noether)* **non-negative definite** *(by the timelike condition)* energy. The only subtlety is that this energy degenerates at the horizon.

In stationary perturbations of Schwarzschild, ∂_t in general becomes **spacelike** near the horizon. This happens already for Kerr with $0 \neq |a| \ll M$. The corresponding energy is conserved but does not have a sign. For particle motion, this leads to the so-called Penrose process. For waves, this leads to the phenomenon of *superradiance* (Zeldovich).

In particular, using the conservation law associated to ∂_t one cannot prove *a priori* boundedness, even away from the horizon.

Trapping

On Schwarzschild, the "photon sphere" $r = 3M$ has the property that it contains null geodesics. These null geodesics thus neither escape to $\mathcal{I}^+$ nor to the horizon $\mathcal{H}^+$.



In Kerr, the behaviour persists, but it is more complicated!

One can concentrate energy for arbitrarily large times near trapped null geodesics. One has to capture this to prove dispersive results.

In particular, pointwise-in-time decay estimates for energy must lose derivatives (Ralston).

Redshift II

The redshift effect can be captured by a robust generalised energy identity associated to a suitable vector field multiplier N and vector field commutators.

That is to say, one can construct a suitable vector field N (which is timelike and transverse to the horizon) such that the divergence $\mathbf{K}^N[\psi]$ of its energy current $\mathbf{J}^N[\psi]$ has good positivity properties near the horizon. These good properties survive under commutation, i.e. considering $\mathbf{J}^N[N \cdots N\psi]$. (M.D.–Rodnianski)

This construction is completely general and only depends on the positivity of the so-called surface gravity.

(C.f. earlier understanding of waves on the horizon on Schwarzschild in pioneering work of Wald, Kay–Wald)

Superradiance II: small perturbations of Schwarzschild

In the presence of 2 Killing fields ∂_t and ∂_φ which span the null generator of the horizon (like in Kerr), ψ can be heuristically decomposed $\psi = \psi_\flat + \psi_\sharp$ into its superradiant part $\psi_\flat$ and non-superradiant part $\psi_\sharp$.

Restricted to the non-superradiant part $\psi_\sharp$ the conserved energy is indeed nonnegative, $\Longrightarrow \psi_\sharp$ is bounded.

For the superradiant part $\psi_\sharp$, the only mechanism for boundedness is dispersion. **For small perturbations of Schwarzschild, however, one can show that the superradiant part $\psi_\flat$ is not trapped!** $\Longrightarrow \psi_\flat$ in fact disperses. This insight allows for a very general boundedness theorem for $\psi = \psi_\flat + \psi_\sharp$ (M.D.–RODNIANSKI) which includes as a special case the result for Kerr with $|a| \ll M$.

In particular, the boundedness property does not depend on the details of geodesic flow of the underlying metric!

Trapping II

History: In the Schwarzschild case pioneering work of LABA–SOFFER, adapted by BLUE–SOFFER.

Followed by M.D.–RODNIANSKI, BLUE–STERBENZ, MARZUOLA–METCALFE–TATARU–TOHANEANU, ALINHAC, etc.

Idea: construct vector field multiplier currents $\mathbf{J}^X[\phi]$ whose divergence $\mathbf{K}^X[\phi]$ enjoys nonnegativity properties with degeneration precisely at $r = 3M$.

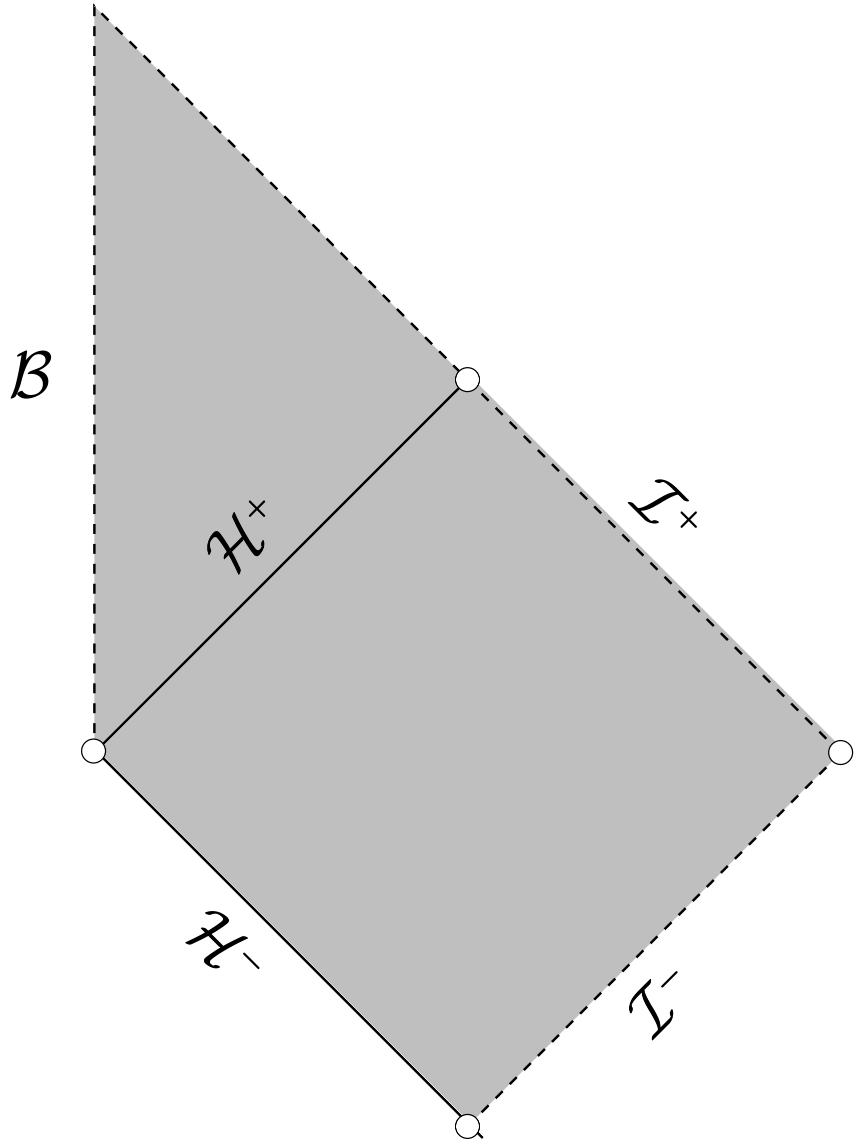
In the Kerr case, the set where there is degeneration must be understood in phase space and the above methods are insufficient (ALINHAC). Two original approaches (M.D.–RODNIANSKI, TATARU–TOHANEANU) in the case $|a| \ll M$. Interesting new approach due to ANDERSSON–BLUE, also for $|a| \ll M$. Promising recent work of WUNSCH–ZWORSKI.

The coupling of redshift, superradiance and trapping

For understanding decay in the general $|a| < M$ case, one must return to the observation of the general boundedness result: **Remarkably, it is still the case that superradiant frequencies are not trapped.** One can construct Morawetz multiplier currents $\mathbf{J}^X[\phi]$ localised in frequency *both* to capture trapping and to distinguish between superradiant/non-superradiant modes. (M.D.–Rodnianski)

The above insight is in fact also related to **unique-continuation** properties at the heart of the "uniqueness" of the Kerr family in the class of stationary solutions to the Einstein vacuum equations. See recent work of Ionescu–Klainerman, Alexakis–Ionescu–Klainerman.

Extremal black holes



Extremal black holes are characterized by the property that the "local redshift factor" vanishes on the horizon.

Simplest examples: extremal Reissner–Nordström $Q = M$ and extremal Kerr $|a| = M$.

Theorem (Aretakis 2010). *For solutions to $\Box_g \psi = 0$ on extremal Reissner–Nordstrom, one has analogous boundedness results as in the non-extremal case, and analogous decay results* **away from the horixon**.

Theorem (Aretakis 2010). *For generic initial data above, the higher order non-degenerate derivatives of ψ transversal to the horizon blow up polynomially in time. In this sense, extremal Reissner–Nordström is (mildly) linearly unstable.*

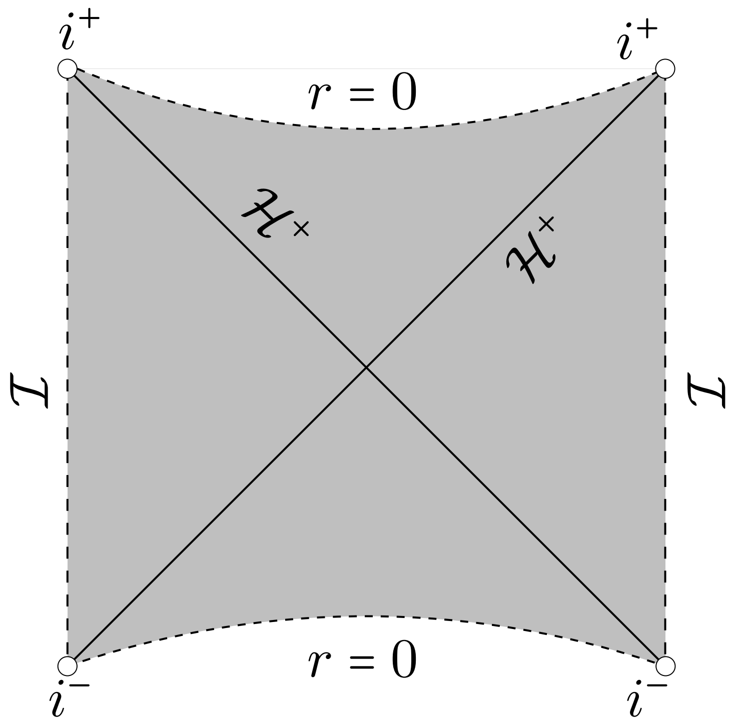
Recently, ARETAKIS has extended his first theorem to *axisymmetric* solutions on extremal Kerr. (Recall that such solutions are not subject to superradiance.)

As far as *non-axisymmetric* solutions, there is an additional difficulty: The fundamental insight allowing for the resolution of the problem in the subextremal case, namely that superradiant frequencies are not trapped, **breaks down exactly at extremality**.

This is related to a limit of quasinormal modes approaching the real axis as $a \to M$, studied by ANDERSSON–GLAMPEDAKIS.

The significance of this for the quantitative study of the wave equation is yet to be understood.

The asymptotically AdS case



The Klein–Gordon equation on Kerr–AdS

Theorem (Holzegel 2009, see also Vasy). *The Klein–Gordon equation is well-posed on asymptotically AdS spacetimes, provided the KG mass is above the Breitenlohner–Freedman bound.*

Theorem (Holzegel 2010). *On Kerr–AdS (in fact, on suitable perturbations of Kerr–AdS which keep only the Hawking–Reall Killing field), solutions of the Klein–Gordon equation as above are uniformly bounded in the exterior.*

Theorem (Holzegel–Smulevici 2011). *(1) On Kerr-AdS spacetimes, general solutions ψ decay* **logarithmically** *in time.*

(2) On spherical Schwarzschild–AdS, **individual spherical harmonics** *decay* **exponentially***.*

(3) Schwarzschild–AdS is asymptotically stable as a solution of the **coupled** *Einstein-Klein Gordon system* **under spherical symmetry***.*

Part (3) of Holzegel–Smulevici argument requires a trapped surface to be present in the data. It did not apply to pure AdS.

Indeed, for pure AdS, since there are an infinite number of stationary solutions of the wave equation, there is no dispersive mechanism. Naively plugging in the results of linearisation back into the equation, this suggests that initially-arbitrarily small solutions grow without bound. Moreover, since there is a threshold after which singularities form, this suggests that initially arbitrarily small solutions in fact form singularities in finite time. On the basis of the above, it was natural to conjecture:

Conjecture (M.D.–Holzegel 2006)**.** *Pure AdS is dynamically unstable.*

Instability results are harder to prove than stability, but, following Holzegel–Smulevici, this has been studied numerically by Bizon–Rostworowski in the context of the above mentioned spherically symmetric model.

Returning to Kerr-AdS, heuristics on quasinormal modes indicate that the Holzegel–Smulevici logarithmic decay result is sharp.

From the point of view of the previous considerations, logarithmic decay is no better than no decay at all.

Again, this slow decay suggests that these black holes are unstable in the nonlinear theory. Don't be fooled by the fact that by part (3) of the Holzegel–Smulevici result, this instability can obviously not be seen in the spherically symmetric model.

This suggests in fact:

Conjecture (Holzegel–Smulevici). *All asymptotically AdS spacetimes (with spherical infinity) are dynamically unstable.*

General small perturbations of AdS may generate an infinite cascade of small black holes–**or worse**.