Daegu lecture series (2018): Category, iconography, and topology

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Abstract. The naturally monoidal, strictly 2-pivotal, weakly 3-pivotal, rotationally commutative strictly 3-tortile 3-category with one object that is freely generated by a weakly self-invertible non-identity 1-morphism is equivalent to the 3-category of isotopy classes of surfaces that are properly embedded in 3-dim'l space. This result is known among the experts, but written versions are scarce. In exploring the result, one quickly comes to the notion that diagrammatic notation can facilitate topological descriptions.

1. Introduction

We will continue and expand upon the things that we touched upon last summer. Perhaps the students, with a little urging, can develop a very good set of notes that can be adapted as a separate chapter with Professor Przytycki's notes from last summer or as an independent opus. My hope is that together we can develop a new system of fonts (using TiKs, for example) that will allow us to express diagrammatic calculations in an iconographic fashion.

Fonts and different scripts for differing languages have become a new fascination since I hope to understand and to read Hangul and Japanese by the end of this nearly year-long tour. While the structure of Hangul characters is meant to mimic the way the mouth and throat are shaped when one pronounces a given sound, Kanji, Hiragana, and Katakana fonts are derived from Chinese characters. Novices in Asian languages often manufacture meaning to the associated Kanji to abstract its true meaning. In the case of Hiragana, many on-line tutorials encourage developing mnemonics that associate a character's sound to a comic depiction. To me, the character め (me) looks like a mother holding a baby, and the Vietnamese word for mother is *mẹ* with a pronunciation (meh) that is close to the Japanese sound of the character.[1] That is how I learned the name of that Hiragana character.

There is no sense to Roman characters. One might expect the characters b,d,p,q to share the long e sound since they are related by the action of the dihedral group; q doesn't, and what about c,g,t,v?

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[1]It cannot be identical since Vietnamese includes inflection for vowels as an integral part of the language.

I noticed recently that the iconography that android operating systems are using — a stylized envelope for mail, a paper airplane for the verb "send message," an early 20th century camera for taking photos or video calls, and the hand-piece of a telephone from my youth as the tool to make telephone calls — might be creating metaphors that young people may not understand. I think my children still say, "dial the number" even if they never experienced a telephone with a dial. Who among those who were born in the 21st century will recognize these shapes as graphical depictions of the machines that once performed these functions? Do young people know that there is such a thing as a telephone cord? Will all the functional icons of the future be small rectangular solids?

In math, some symbols are developed so that they create metaphors. The symbol $=$ was defined to suggest that the line segments which comprise the font are identical. The symbol % suggests a fraction as does the symbol $\div$; both suggest a comparison by putting one quantity over or beside the other. Parallel lines are denoted $\|$ and perpendicular lines are denoted by $\perp$. Our understanding of mathematical concepts is informed by choosing names that reflect the metaphors that inspired the math (gradient flow, curl, divergence, etc). Moreover good symbolism develops math intuition more quickly.

The first talk will begin with an overview of (small) categories, the associated iconography, and the higher dimensional categories that result. Straightforward categoricial axioms have topological descriptions. Let us state a theorem.

Theorem 1.1. *The naturally monoidal, strictly* 2*-pivotal, weakly* 3*-pivotal, rotationally commutative, strictly* 3*-tortile* 3*-category with one object that is freely generated by a weakly self-invertible non-identity* 1*-morphism is equivalent to the* 3*-category of isotopy classes of surfaces that are properly embedded in* $\mathbb{R}^2 \times [0,1]$.

The higher categories involved are globular in nature: n-morphisms fill a globe that has $(n-1)$-morphisms defining the hemispherical faces. Thus the program Globular [**?**] that is available at globular.science will facilitate the exposition. In particular, we will link to a variety of globular work sheets to present and exemplify the results.

The main result of today's lecture is certainly known to experts, and it could have been articulated twenty or more years ago. It is the context of the $n = 2$, $k = 1$, entry in the Baez Dolan [**?**] cobordism hypothesis as stated in Table 1 of [**?**]. As such, it is folklore; the experts with whom I corresponded did not know of a written proof. The result stated here has its categorical degree shifted up from cobordism hypothesis's statement. We will decategorify towards the end of the paper to make the statement here coincide with the statement from the cobordism hypothesis.

I am not trained as a higher dimensional category theorist. So to that audience, this presentation will appear awkward. I want to thank John Baez, Masahico Saito, and Jamie Vicary for advice.

Throughout, there are two principles to follow:

(1) Different things are not equal. At best, they are naturally isomorphic.
(2) Critical events do not occur simultaneously.

Let us proceed.

2. Categories and n-categories

2.1. Preliminaries. First, all n-categories will be really small. The collection of objects is a set, and the category of k-morphisms between any pair of $(k-1)$-morphisms will also form a set, for $1 \leq k \leq 4$. A 0-morphism is also called *an object.* A 1-morphism between a pair of objects is called a *(single) arrow.* In general, a k-morphism will be also called a *double, triple, or quadruple* arrow, for the obvious values of k. An overly simplistic definition of a *really small n-category* is that it is a category in which the set of morphisms between $(n-1)$-morphisms is a category. Composition should be unital and associative. Natural families of isomorphisms are used to disambiguate compositions that are defined across categorical levels. These notions will be made precise as we climb the categorical ladder up through 4-morphisms.

A really small category[2] is defined as follows. Given objects a and b, the collection of arrows $b \xleftarrow{f} a$ from the *source* $a = s(f)$ to the *target* $b = t(f)$ is a set. If $c \xleftarrow{g} b$ and $b \xleftarrow{f} a$ are arrows so that $s(g) = t(f)$, then their composition is an arrow $c \xleftarrow{g \circ f} a$ with source $s(g \circ f) = a$ and target $t(g \circ f) = b$. Compositions of arrows is associative:

$$\left[d \xleftarrow{h} c \xleftarrow{g \circ f} a\right] = \left[d \xleftarrow{h \circ g} b \xleftarrow{f} a\right].$$

For any object a there is an arrow $a \!-\!\!-\! a$ that behaves as an identity under compositions:

$$\left(b \!-\!\!-\! b \xleftarrow{f} a\right) = \left(b \xleftarrow{f} a\right) = \left(b \xleftarrow{f} a \!-\!\!-\! a\right).$$

In order to put an inductively defined category structure upon the set of n-morphism (or multiple arrows) between a pair of $(n-1)$-morphisms, we write

$$\begin{array}{ccc} b & \xleftarrow{g} & a \\ | & \Uparrow_F & | \\ b & \xleftarrow{f} & a \end{array}$$

[2] The standard terminology is small and locally small. Since the higher morphisms are also categories, here we want every collection of morphisms to form a set.

for an n-morphism F whose source is the $(n-1)$-morphism $f = s(F)$ and whose target is the $(n-1)$-morphism $g = t(F)$. In this case, note that $s(f) = s(g) = a$ while $t(f) = t(g) = b$.

Since the sources and targets for f and g agree, it is customary to draw the n-morphism F as on the left side of Fig. 1. The (vertical) composition of 2-morphisms is illustrated on the right of the same figure. Often authors define a horizontal composition of 2-morphisms that resembles the central drawing in Fig. 2. In this case, the horizontal composition is ambiguously interpreted via either the left or right drawing. The ambiguity is resolved by interpreting these as being equal. But that condition is too strong for our purposes. So, in fact, we suppose that there is a natural 3-morphism connecting the two interpretations. We will return to the point later.

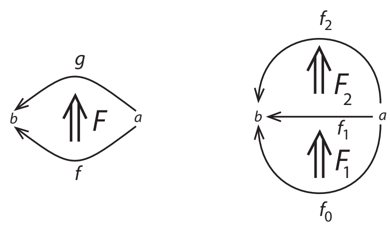


FIGURE 1. Composition of 2-morphims

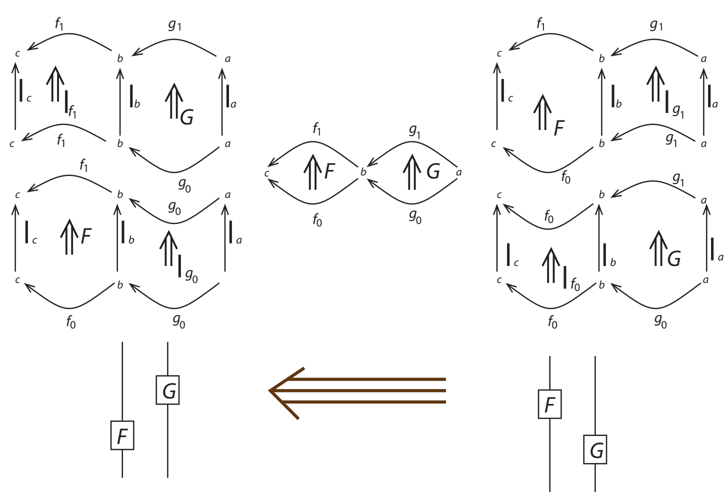


FIGURE 2. Horizontal composition is not well-defined

Digression. In general, any commutative diagram that involves 1-morphisms can be reinterpreted as a 2-morphism, and often these 2-morphisms are understood to be isomorphisms — all the higher relations among them are deemed to be true.

For example, define tautological 2-morphisms

$$\left[\begin{array}{c} c \xleftarrow{g\circ f} a \\ \curlywedge \\ c \xleftarrow{g} b \xleftarrow{f} a \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} c \xleftarrow{g} b \xleftarrow{f} a \\ \mathsf{Y} \\ c \xleftarrow{g\circ f} a \end{array}\right];$$

these will be denoted as $\curlywedge_{g,f}$ (*the composer*) and $\mathsf{Y}^{g,f}$ (*the factorizer*). For each 1-arrow $b \xleftarrow{f} a$ there is an *identity 2-morphism*:

$$\left[\begin{array}{c} b \xleftarrow{f} a \\ \Uparrow [\mathsf{I}_f] \\ b \xleftarrow{f} a \end{array}\right]$$

that is denoted as I_f. It does not affect any composition with another 2-morphism, and so it also is tautological. The associative law reads as

$$\begin{pmatrix} \curlywedge \\ \curlywedge\,\mathsf{I} \end{pmatrix} = \begin{pmatrix} \curlywedge \\ \mathsf{I}\,\curlywedge \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \mathsf{Y}\,\mathsf{I} \\ \mathsf{Y} \end{pmatrix} = \begin{pmatrix} \mathsf{I}\,\mathsf{Y} \\ \mathsf{Y} \end{pmatrix}$$

where double arrows are stacked vertically to indicate their composition. Including information about sources and targets, and writing in a linear format (with $\circ_2$ denoting the composition of double arrows), these read as

$$(\curlywedge_{h\circ g,f}) \circ_2 (\curlywedge_{h,g} \otimes \mathsf{I}_f) = (\curlywedge_{h,g\circ f}) \circ_2 (\mathsf{I}_h \otimes \curlywedge_{g,f}),$$

or

$$\left(\mathsf{Y}^{h,g} \otimes \mathsf{I}^f\right) \circ_2 \left(\mathsf{Y}^{h\circ g,f}\right) = \left(\mathsf{I}^h \otimes \mathsf{Y}^{g,f}\right) \circ_2 \left(\mathsf{Y}^{h,g\circ f}\right).$$

The tensor symbol $\otimes$ in this context simply indicates horizontal juxtaposition, but here and below, horizontal juxtaposition can only occur when one of the double arrows involved is an identity. Both Y and $\curlywedge$ should be interpreted as being 2-invertible, and each is an inverse of the other. Explicitly, we assert that

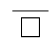
$$\left(\mathsf{Y}^{g,f}\right) \circ_2 (\curlywedge_{g,f}) = (\mathsf{I}_g \otimes \mathsf{I}_f),$$

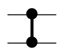
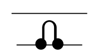
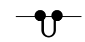
and

$$(\curlywedge_{g,f}) \circ_2 \left(\mathsf{Y}^{g,f}\right) = \left(\mathsf{I}_{(g\circ f)}\right).$$

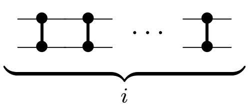
In the second equality, it is vital to index the branches in the central portion of the composition by the 1-morphisms f and g. A given composition $(g \circ f)$ has many possible factorization, and the analogous expression $\left[\begin{smallmatrix} \curlywedge \\ \mathsf{Y} \end{smallmatrix}\right]$, without adornment, would be interpreted as putting in all possible factorizations in an abstract tensor context. The composer $\curlywedge$ and the factorizer Y can also be interpreted as giving a method to keep track of rescaling the arrows in the composition.

2.2. Primary example. Suppose there is a unique object x in a category, and there is a non-identity morphism $x \xleftarrow{-\bullet-} x$. Then define $(-\bullet-)^k$ inductively as the k-fold composition of $-\bullet-$ with itself: $(-\bullet-)^k = (-\bullet-)^{k-1} \circ -\bullet-$. Of course, $(-\bullet-)^0 = \text{---} = x \text{---} x$. Now observe that the set of powers of $-\bullet-$ corresponds to the non-negative integers $\mathbb{N} = \{0, 1, 2, \ldots\}$, and indeed the correspondence is given by expressing a non-negative integer in unary notation. Moreover, the set $\mathbb{N}$ under the operation of addition coincides with the monoid structure on the powers of $-\bullet-$ under composition. This is the *free monoid on a single generator*.

The identity double arrow on --- is  . The identity 2-morphism on

$-\bullet-$ is  . We define double arrows  and  . Ordinarily, it will not be necessary to specify sources and targets of double arrows by bullets. Instead, caps, and cups will be indexed according to the number of bullets to the left or right. The following paragraph explains.

We begin by defining $\mathsf{I}_1 = \mathsf{I}$, and more generally inductively define $\mathsf{I}_i = (\mathsf{I}_{i-1}) \otimes \mathsf{I}$ to be the horizontal juxtaposition; thus I_i is the identity on $(-\bullet-)^i$. Explicitly,

$\mathsf{I}_i =$  .

Let $\mathsf{U}_{i,j} = \mathsf{U}(i,j) = \mathsf{I}_i \otimes \mathsf{U} \otimes \mathsf{I}_j$ and $\cap_{i,j} = \cap(i,j) = \mathsf{I}_i \otimes \cap \otimes \mathsf{I}_j$ indicate the horizontal juxtaposition of either cup or cap with the identity on $(-\bullet-)^i$ on its left and the identity on $(-\bullet-)^j$ on its right. These are double arrows between non-negative integers with $s(\mathsf{U}(i,j)) = t(\cap(i,j)) = i + j$ while $t(\mathsf{U}(i,j)) = s(\cap(i,j)) = i + 2 + j$. We can also write these in terms of the factorizer Y or the composer $\curlywedge$. Thus the sub- or super-scripts on $\curlywedge_{i,j}$ or $\mathsf{Y}^{i,j}$ are a pair of positive integers. We can rewrite $\cap_{i,j} = (\curlywedge_{i,j}) \circ_2 (\mathsf{I}_i \cap \mathsf{I}_j)$, and $\mathsf{U}^{i,j} = (\mathsf{I}_i \,\mathsf{U}\, \mathsf{I}_j) \circ_2 \left(\mathsf{Y}^{i,j}\right)$. The tensor $\otimes$ notation is dropped in favor of juxtaposition.

In a later context, we will consider U (resp. $\cap$) to be the time evolution of the birth (resp. death) of a pair of identical particles. But here, these are simply double arrows that transform between 0 and 2. More generally $(i+j) \Leftarrow (i+2+j) : \cap_{i,j}$ while $(i+2+j) \Leftarrow (i+j) : \mathsf{U}^{i,j}$.

We say that an arrow $-\bullet-$ is *weakly self-invertible* provided there are double arrows U and $\cap$ as above. For further context, $-\bullet-$ is *strongly self-invertible* if these double arrows are 2-isomorphisms. To assert strong self-invertibility is to assert also that certain triple arrows $\dot{\smile}$, $\underset{\cdot}{\frown}$, $\underset{\cdot}{\cup}$, and $\dot{\cap}$ are invertible, as well, with $(\dot{\smile})^{-1} = \underset{\cdot}{\frown}$, and $\left(\underset{\cdot}{\cup}\right)^{-1} = \dot{\cap}$. These conditions are *too strong* for the puposes here.

$(\cap_{i,j}) \circ_2 \left(\cup^{i,j}\right) = (\mathsf{I}_i \cap \mathsf{I}_j) \circ_2 (\mathsf{I}_i \cup \mathsf{I}_j)$
$(\cup_{i,j}) \circ_2 \left(\cap^{i,j}\right) = (\mathsf{I}_i \cup \mathsf{I}_j) \circ_2 (\mathsf{I}_i \cap \mathsf{I}_j)$
$(\cap_{i,1+j}) \circ_2 \left(\cup^{i+1,j}\right) = (\mathsf{I}_i \cap \mathsf{I}_{1+j}) \circ_2 (\mathsf{I}_{i+1} \cup \mathsf{I}_j)$
$(\cap_{i+1,j}) \circ_2 \left(\cup^{i,1+j}\right) = (\mathsf{I}_{i+1} \cap \mathsf{I}_j) \circ_2 (\mathsf{I}_i \cup \mathsf{I}_{1+j})$
$\left(\cup^{i+1,1+j}\right) \circ_2 (\cup_{i,j}) = (\mathsf{I}_{i+1} \cup \mathsf{I}_{1+j}) \circ_2 (\mathsf{I}_i \cup \mathsf{I}_j)$
$\left(\cap^{i,j}\right) \circ_2 \left(\cap^{i+1,1+j}\right) = (\mathsf{I}_i \cap \mathsf{I}_j) \circ_2 (\mathsf{I}_{i+1} \cap \mathsf{I}_{1+j})$
$(\cap_{i,j+2+k}) \circ_2 \left(\cup^{i+2+j,k}\right) = (\mathsf{I}_i \cap \mathsf{I}_{j+2+k}) \circ_2 (\mathsf{I}_{i+2+j} \cup \mathsf{I}_k)$
$\left(\cup^{i+j,k}\right) \circ_2 (\cap_{i,j+k}) = (\mathsf{I}_{i+j} \cup \mathsf{I}_k) \circ_2 (\mathsf{I}_i \cap \mathsf{I}_{j+k})$
$(\cap_{i+2+j,k}) \circ_2 (\cup_{i,j+2+k}) = (\mathsf{I}_{i+2+j} \cap \mathsf{I}_k) \circ_2 (\mathsf{I}_i \cup \mathsf{I}_{j+2+k})$
$\left(\cup^{i,j+k}\right) \circ_2 (\cap_{i+j,k}) = (\mathsf{I}_i \cup \mathsf{I}_{j+k}) \circ_2 (\mathsf{I}_{i+j} \cap \mathsf{I}_k)$
$(\cap_{i+j,k}) \circ_2 (\cap_{i,j+2+k}) = (\mathsf{I}_{i+j} \cap \mathsf{I}_k) \circ_2 (\mathsf{I}_i \cap \mathsf{I}_{j+2+k})$
$(\cap_{i,j+k}) \circ_2 (\cap_{i+2+j,k}) = (\mathsf{I}_i \cap \mathsf{I}_{j+k}) \circ_2 (\mathsf{I}_{i+2+j} \cap \mathsf{I}_k)$
$\left(\cup^{i+2+j,k}\right) \circ_2 \left(\cup^{i,j+k}\right) = (\mathsf{I}_{i+2+j} \cup \mathsf{I}_k) \circ_2 (\mathsf{I}_i \cup \mathsf{I}_{j+k})$
$\left(\cup^{i,j+2+k}\right) \circ_2 \left(\cup^{i+j,k}\right) = (\mathsf{I}_i \cup \mathsf{I}_{j+2+k}) \circ_2 (\mathsf{I}_{i+j} \cup \mathsf{I}_k)$

TABLE 1. A table of possible vertical compositions

Vertical juxtaposition of diagrams will correspond to composition of double arrows. As is usual in topological situations, after these are composed, they too will be rescaled. It is wise to step back and compute the various compositions that can be made among $\cap_{i,j}$ and $\cup_{k,\ell}$. These compositions are compiled in Table 1.

The more visual representation of these double arrows are depicted among the stills in the movie illustrations found in Fig. 3 in which the vertical composition is achieved. The figure serves two purposes. Its primary purpose is to indicate a number of generating triple arrows. Since many of these involve the items Table 1, the double arrows which involve nested $\cup$s or $\cap$s are encircled at the top of the illustration and these are included. Before these triple arrows are discussed (and represented via movies), the definition of the tensor product of a pair of non-identity double arrows is needed. Meanwhile, the reader is encouraged to find the representative 2-morphisms among the stills of movies that are indicated in this illustration.

Notice also that if two items in the table are separated by a single line, then they are related by a triple arrow in Fig 3.

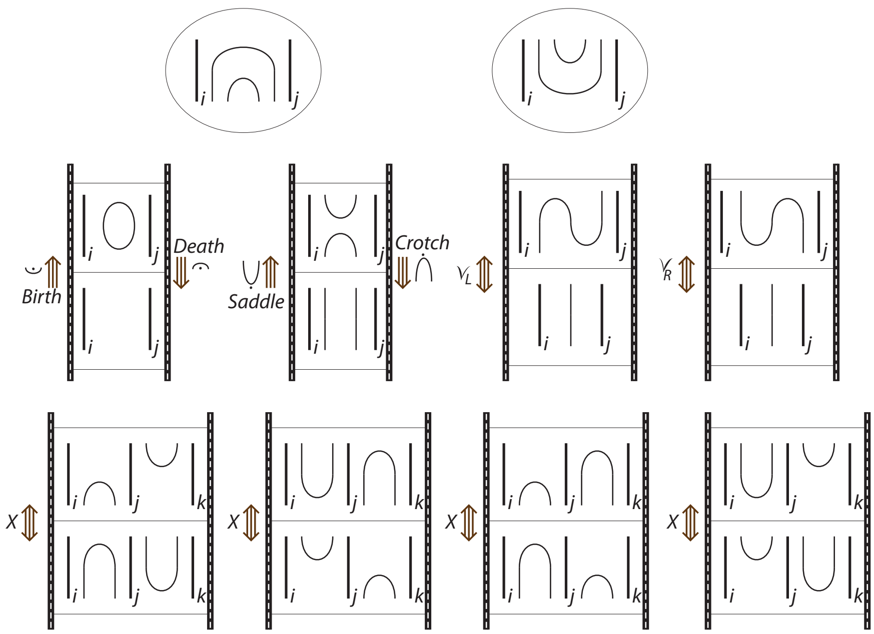


FIGURE 3. Some compositions of double arrows and triple arrows that relate them

2.3. Tensor products of double arrows and the exchanger. Suppose that a pair of arbitrary double arrows $\boxed{F}^{\,i}_{\,k}$ and $\boxed{G}^{\,j}_{\,\ell}$ are given. Here the double arrow F has source $s(F) = (\!-\!\bullet\!-\!)^k$ and target $t(F) = (\!-\!\bullet\!-\!)^i$ while those for G are $s(G) = (\!-\!\bullet\!-\!)^\ell$ and target $t(G) = (\!-\!\bullet\!-\!)^j$. Then define

$$\boxed{F}^{\,i}_{\,k} \otimes \boxed{G}^{\,j}_{\,\ell} = \left(\boxed{F}^{\,i}_{\,k} \otimes \Big|_{j} \right) \circ_2 \left(\Big|_{i} \otimes \boxed{G}^{\,j}_{\,\ell} \right) = \begin{pmatrix} \boxed{F}^{\,i} & \Big|^{\,j} \\ \Big|_{\,k} & \boxed{G}_{\,\ell} \end{pmatrix}.$$

Recall that the tensor product of any 2-morphism with the identity 2-morphism $|_k$ is obtained by juxtaposing the identity horizontally. From the diagram, we see that $S(F \otimes G) = k + \ell = s(F) + s(G)$ while $t(F \otimes G) = i + j = t(F) + t(G)$.

In globular categories in general, most authors assert that the 4-square relation that is indicated in Fig. 4 holds. That is the diagram on the right should be well-defined. However, since the horizontal composition of double

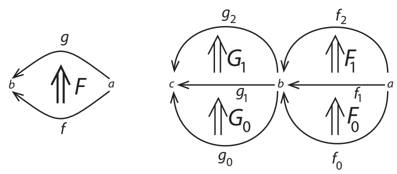
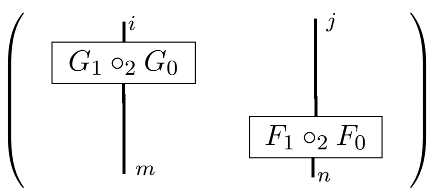
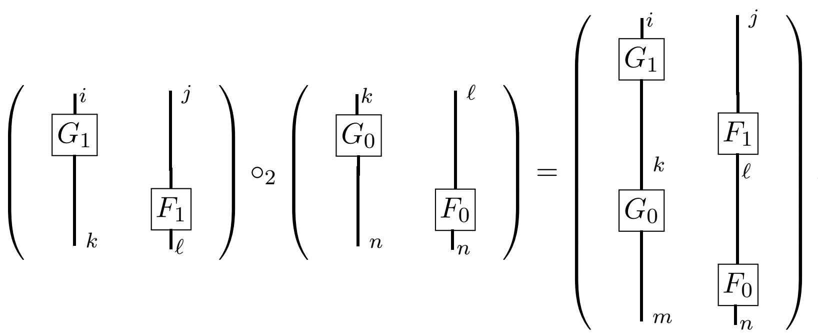


FIGURE 4. The 4-square relation

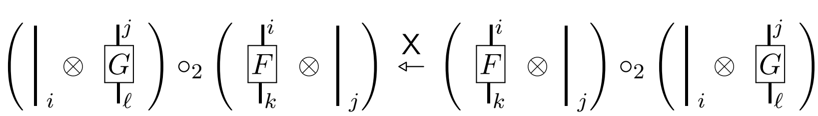
arrows is ambiguous, we will have to look more closely. One possible reading is of the composition is



Another is this:



From the literalist point of view that we take here, these are not equal, but instead they are related via a triple arrow (3-morphism) that is called the *interchanger* in the context of GLOBULAR or the *tensorator* in [**?**]. Here we will define a natural family of 3-isomorphisms



any of which is called an *exchanger.* The notation is cumbersome when the source and target of an exchanger is specified. But clearly, an exchanger depends upon both two morphisms F and G and consequently, also depends upon their sources and targets.

In Fig. 5, the exchanger is indicated towards the upper left of the illustration as a triple arrow that points leftward. On the upper right of the

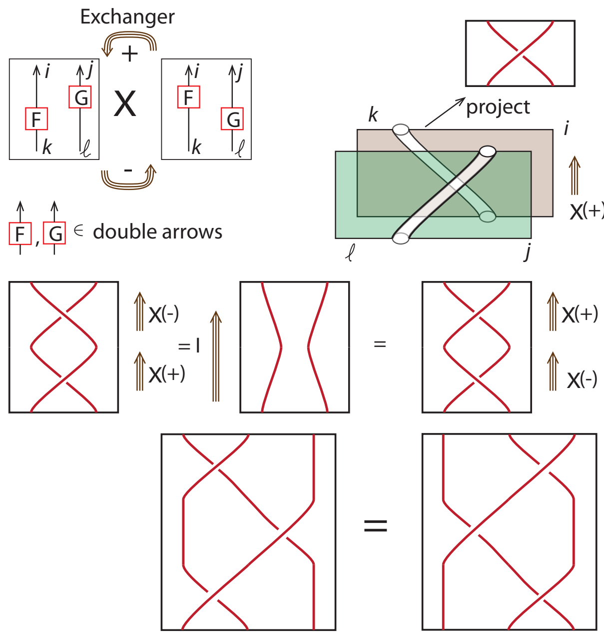


FIGURE 5. The invertible exchanger and naturality condition

illustration is a schematic diagram that indicates the exchanger as a kinematic process. In our case, the translucent sheets that are labeled i, j, k and ℓ indicate parallel disks that may be folded along the tubes. So the white tubes indicate a collection of stacked $\mathsf{U} \times [0,1]$s and $\mathsf{\cap} \times [0,1]$s that occur within the tubes. The exchanger $\mathsf{X}(+)$ is schematized as the crossing throughout the illustration. Traditionally when the arc with positive slope crosses in front of the arc with negative slope, the crossing is considered positive.

We assume that the exchanger is invertible so that the equalities $\mathsf{X}(-) \circ_3 \mathsf{X}(+) = \mathsf{I}_G \otimes \mathsf{I}_F = \mathsf{X}(+) \circ_3 \mathsf{X}(-)$ hold for any pair of double arrows F and G. These identities are indicated below the definition of the exchanger within Fig. 5. The final condition for the exchanger is that it is natural with respect to any other triple arrow. This implies, in particular, that the Yang-Baxter type relation that is indicated at the bottom of the illustration holds. The naturality condition also implies that the exchanger commutes with the other triple arrows that we will define in a moment.

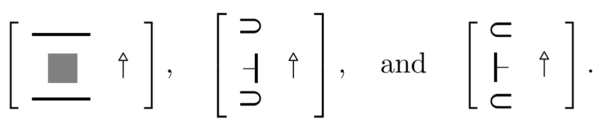
In case a 2-category has an ambiguous horizontal composition, but the ambiguity is resolved by asserting that there is a natural family of 3-morphisms X which satisfy the conditions asserted in Fig. 7, the 2-category is said to be *naturally monoidal*.

2.4. Triple arrows. In the abstract, triple arrows or 3-morphisms are simply arrows of the form:

$$\left(\begin{array}{c} {}^{j} \\ \boxed{F} \\ {}_{i} \end{array} \right) \overset{\mathcal{R}}{\Leftarrow} \left(\begin{array}{c} {}^{j} \\ \boxed{G} \\ {}_{i} \end{array} \right),$$

so that $s(\mathcal{R}) = G$ and $t(\mathcal{R}) = F$. On the other hand, under the categorical isomorphisms that we will establish, we will imagine these as being geometrically meaningful. Thus the kinematic description of the interchanger X that is indicated in Fig. 5 in which we imagine the strings of the identity as evolving into disks, is only a metaphor. Obviously, the purpose here is to evoke that metaphor, but be aware that even though the triple arrows that are named birth, death, saddle, crotch, and left and right cusp, they are only triple arrows with explicit sources and targets.

Since the collection of triple arrows is also meant to be a category, we need to define the identity upon the identity double arrow I, and the two non-trivial double arrows U, and $\mathsf{\cap}$. We lay these out as follows:



When necessary, *folds* which are the identities upon $\mathsf{U}_{i,j}$ and $\cap_{i,j}$ can be adorned with double indices (i, j) to indicate the location (i sheets to to left [behind] and j sheets to the right [in front]) of them.

Let us examine Fig. 3 carefully. We have the following generating triple arrows:
Birth $[\dot{\smile}]$:

$$(\cap) \circ_2 (\mathsf{U}) \Leftarrow \left(\overline{\underline{\square}} \right),$$

Death $[\dot{\frown}]$:

$$\left(\overline{\underline{\square}} \right) \Leftarrow (\cap) \circ_2 (\mathsf{U}),$$

Saddle $[\dot{\cup}]$:

$$(\mathsf{U}) \circ_2 (\cap) \Leftarrow (\mathsf{I} \otimes \mathsf{I}),$$

Crotch[3] $[\dot{\cap}]$:

$$(\mathsf{I} \otimes \mathsf{I}) \Leftarrow (\mathsf{U}) \circ_2 (\cap),$$

and 3-isomorphisms
Left cusp $[\curlyvee_L]$:

$$(\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \mathsf{U}) \leftrightarrows (\mathsf{I}),$$

Right cusp $[\curlyvee_R]$:

$$(\mathsf{I} \otimes \cap) \circ_2 (\mathsf{U} \otimes \mathsf{I}) \leftrightarrows (\mathsf{I}).$$

[3]This mildly naughty term is meant to be used in the same way that a seamstress or tailor would use the word: as if it were the junction of the legs in a pair of pants.

Let us explicate some of the notation. The names *birth, death, saddle, crotch, left cusp, and right cusp* are the names of the represented triple arrows, and one should also pronounce the associated icons in the same way. On the other hand, in the cases of the cusps, the leftward pointing triple arrows are called *left cusp down:* $\curlyvee_L$ and *right cusp down*: $\curlyvee_R$. We define *left cusp up*: $\curlywedge^L = \curlyvee_L^{-1}$ and *right cusp up*: $\curlywedge^R = \curlyvee_R^{-1}$. To say that $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ are inverses is to assert that the compositions

$$\mathsf{I} \xleftarrow{\curlywedge^L} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \xleftarrow{\curlyvee_L} \mathsf{I},$$

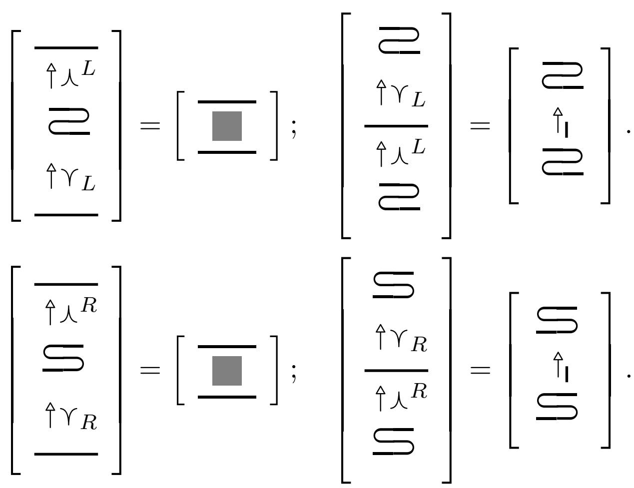
$$(\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \xleftarrow{\curlyvee_L} \mathsf{I} \xleftarrow{\curlywedge^L} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup),$$

$$\mathsf{I} \xleftarrow{\curlywedge^R} (\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \xleftarrow{\curlyvee_R} \mathsf{I},$$

and

$$(\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \xleftarrow{\curlyvee_R} \mathsf{I} \xleftarrow{\curlywedge^R} (\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I})$$

are the identity 3-morphisms on their (coincident) sources and targets. These relations are easier to imagine when written vertically:



Here the triple arrow $\uparrow_{\mathsf{I}}$ is meant to indicate the identity triple arrow on the zig-zagged compositions $(\cap\ \mathsf{I}) \circ_2 (\mathsf{I}\ \cup)$ and $(\mathsf{I}\ \cap) \circ_2 (\cup\ \mathsf{I})$.

In fact, the invertibility assertion is quite a bit stronger. That these are identities is to assert the existence of quadruple arrows to fill-in the obvious commutative diagrams. The compositions of these quadruple arrows and all the higher arrows derived therefrom are all identities. The directions left and right refer to the direction of optimal point at the bottom of the figure. Of course the names of the triple arrows and their iconography are chosen to suggest some geometric phenomena.

We say that a 2-category that has a weakly self-invertible arrow $-\!\bullet\!-$ is *strictly* 2*-pivotal* if there are 2-isomorphism, $\curlyvee^D$ and $\curlywedge^D$, for $D = L, R$ in the sense that is defined above.

Please be aware. Even though each of the compositions $((\frown) \circ_3 (\smile))$, $((\smile) \circ_3 (\frown))$, $\big((\cup) \circ_3 (\dot{\cap})\big)$, and $\big((\dot{\cap}) \circ_3 (\cup)\big)$ have coincident sources and targets, we are not asserting here that they are identities. Their existence is suggested by the assertion that $-\!\bullet\!-$ is weakly self-invertible, but we do not insist that these are isomorphisms or that the typographically similar triple arrows are inverses of each other.

Before we articulate the existence of *all* of the quadruple arrows that will be asserted to be identities we turn to the question of dimensionality with respect to our (multi) arrows.

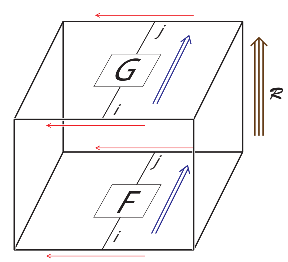


FIGURE 6. Higher morphisms as aspects of dimensionality

2.5. Dimensionality. In the n-category that we are in the process of describing, each level of morphism lives in one more dimension than the previous one. Let us explain. Our object x which has remained tacit for some time is a 0-dimensional object. The arrow (or 1-morphism) $x \xleftarrow{-\bullet-} x$ defines a 1-dimensional space, as does any sequence

$$(-\!\bullet\!-) \circ (-\!\bullet\!-) \circ \ldots \circ (-\!\bullet\!-) \,.$$

The compilation of $(-\!\bullet\!-)^k$ as the integer k represented in unary notation suggests a metaphor in which tally rocks are arranged along a horizontal line. The double arrows (2-morphism) $\mathsf{I}_k, \cap(i,j)$, and $\cup(i,j)$ suggest creating a second dimension:

$$\left(\begin{array}{c} |_j \\ \boxed{F} \\ |_i \end{array} \quad \Uparrow \right).$$

Furthermore, we can lay a pair of 2-morphisms that are connected by a 3-morphism along the floor and ceiling of a 3-dimensional space, and thereby we envision that the 3-morphism occurs as a geometric process in a cubical space as indicated in Fig. 6. Similarly, 4-morphisms or quadruple arrows can be schematized as hypercubes, or in case we are imagining the 3-morphism as a surface inside the box, the 4-morphism is a time parameterized deformation or isotopy of the surface that takes place inside the box.

Let us return to the generating 3-morphisms $\dot{\smile}$, $\dot{\frown}$, $\underset{\cdot}{\cup}$, $\dot{\cap}$, Υ_L, Υ_R, $\curlywedge^L$, $\curlywedge^R$ and the various forms of the exchanger X that are depicted in Fig. 3. These can either be depicted as occurring in the interior of boxes, or schematically as two frames of a movie as illustrated in Fig. 3.

2.6. Quadruple arrows are identities. Rather than trying to illustrate quadruple arrows as occurring along the boundary of a 4-dimensional cube, we will borrow from the technique exemplfied in [**?**], and arrange these as movie moves.

We are now at the topmost level of the categorical structure that is being constructed. So every quadruple arrow described here is asserted to be an identity. These identities fall into several groups.

(1) Invertibility of the exchanger,
(2) Naturality of the exchanger,
(3) Invertibility of cusps,
(4) Adjointness between birth/crotch and death/saddle,
(5) commutations between cusps and saddles/crotches,
(6) Swallowtail identities.

(1) Please note that the exchanger can be factored into the exchanges between Us and $\mathsf{\cap}$s that are depicted at the bottom of Fig. 3. By our conventions, the upward pointing arrows indicate $\mathsf{X}(+)$ and the downward pointing arrows indicate $\mathsf{X}(-)$. Within Fig. 5, a schematic depiction of the identities

$$\mathsf{X}(+) \circ_3 \mathsf{X}(-) = \mathsf{I}_{G \otimes F} = \mathsf{X}(-) \circ_3 \mathsf{X}(+)$$

is given.

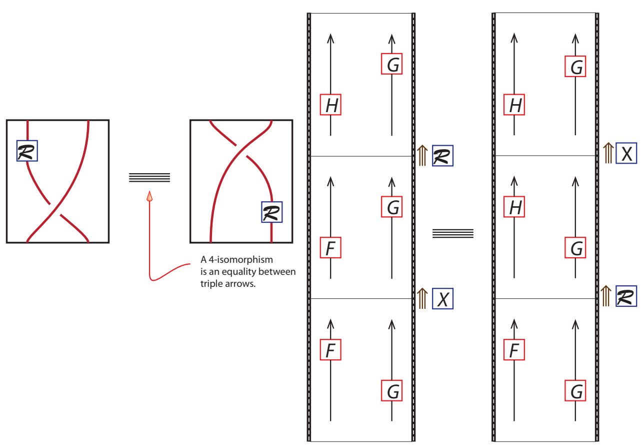


FIGURE 7. The naturality of the exchanger X

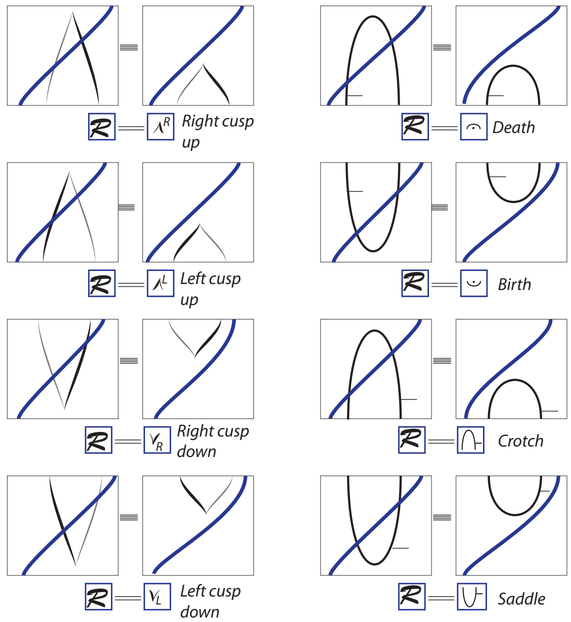


FIGURE 8. The naturality of the exchanger X in many specific instances

(2) Within Fig. 7, we depict a schematic representation of the naturality of the exchanger with respect to any triple arrow $\mathcal{R}$. In Fig. 8, we demonstrate the naturality condition in case the triple arrow is a birth $\dot{\smile}$, a death $\underset{\cdot}{\frown}$, a saddle $\underset{\cdot}{\cup}$, a crotch $\dot{\cap}$, or any of the cusps $\curlyvee_L$, $\curlyvee_R$, $\curlywedge^L$ or $\curlywedge^R$ in terms of some iconagraphy. The iconography can be reinterpreted as a composition of triple arrows. Half of these identities can be written in the form:

$$\left(\mathcal{R}\otimes |_F\right)\circ_3\left(|\otimes \mathsf{X}(\pm)\right)\circ_3\left(\mathsf{X}(\pm)\otimes|\right)=\left(|_F\otimes\mathcal{R}\right),$$

the others are of the form:

$$\left(|\otimes \mathsf{X}(\pm)\right)\circ_3\left(\mathsf{X}(\pm)\otimes|\right)\circ_3\left(|_F\otimes\mathcal{R}\right)=\left(\mathcal{R}\otimes |_F\right);$$

here F is the double arrow whose identity crosses in front of or behind the remaining triple arrows.

(3) The invertibility of the various cusps comes in two different families. In Fig. 9, the invertibility of $(\curlywedge_L)\circ_3(\curlyvee_L)$ and that of $(\curlywedge_R)\circ_3(\curlyvee_R)$ are depicted in movie form and in an iconographic manner below the movies. In Fig. 10, the invertibilities of $(\curlyvee_L)\circ_3(\curlywedge_L)$ and $(\curlyvee_R)\circ_3(\curlywedge_R)$ are given in both movie move and iconographic depictions.

(4) The adjoint relations read as follows:

$$\left(\dashv\otimes\dot{\cap}\right)\circ_3\left(\dot{\smile}\otimes\dashv\right)=\dashv=\left(\dashv\otimes\underset{\cdot}{\frown}\right)\circ\left(\underset{\cdot}{\cup}\otimes\dashv\right),$$

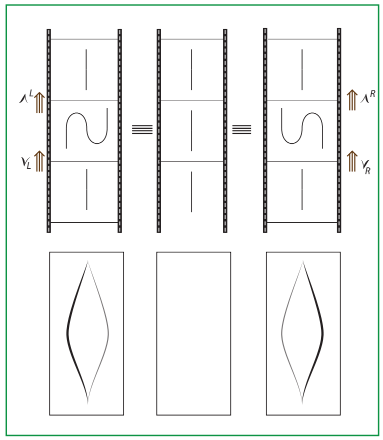


FIGURE 9. The "lips" invertibility $\curlyvee_L^{-1} = \curlywedge^L$ and $\curlyvee_R^{-1} = \curlywedge^R$

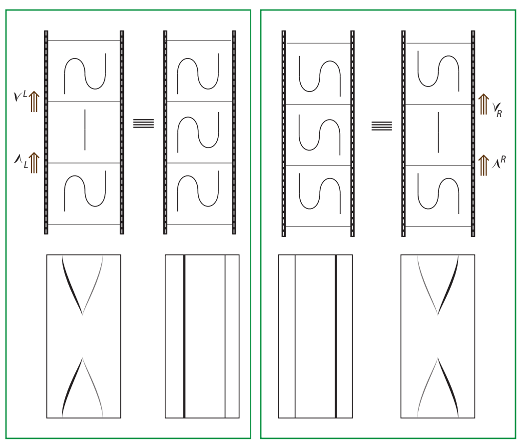


FIGURE 10. The beak-to-beak invertibilities for $\curlyvee_L^{-1} = \curlywedge^L$ and $\curlyvee_R^{-1} = \curlywedge^R$

and

$$\left(\frown \otimes \vdash \right) \circ_3 \left(\vdash \otimes \underset{\cdot}{\cup} \right) = \vdash = \left(\vdash \otimes \dot{\cap} \right) \circ \left(\smile \otimes \vdash \right).$$

Please examine the right hand drawings in Fig. 11. These indicate that in the icongraphic representation, the adjoint relations satisfy co-oriented zig-zag identities. Also note that the source and target of the triple arrows do not determine the arrow. For example both $\smile$ and $\underset{\cdot}{\cup}$ represent triple arrows with source $(\cap)$ and target $(\cap) \circ_2 (\cup) \circ_2 (\cap)$ within the movie move that is illustrated in the top.

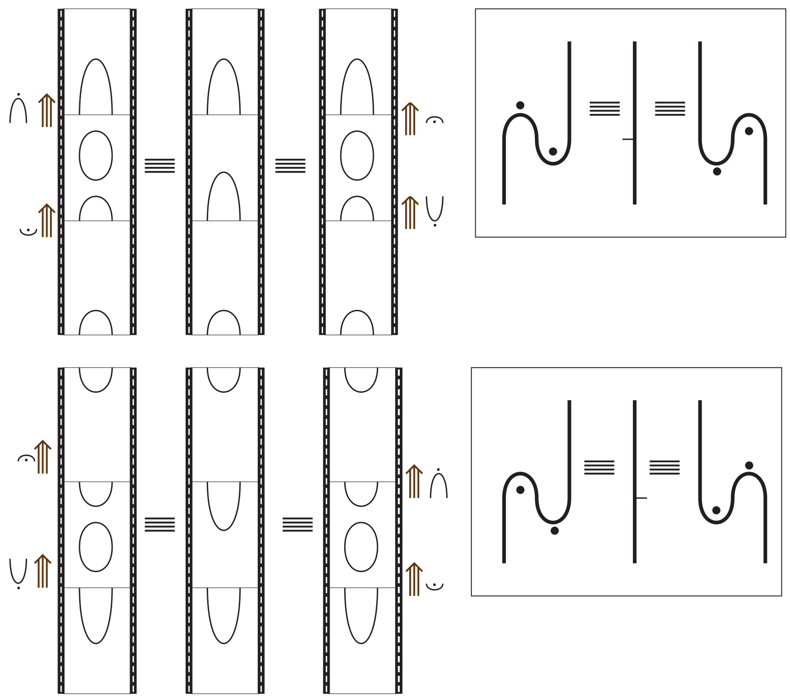


FIGURE 11. The adjoint relations

We say that a 3-category which has a weakly self-invertible non-identity arrow is *weakly* 3-*pivotal* in case the non-invertible 3-morphisms $\underset{\cdot}{\smile}$, $\overset{\cdot}{\frown}$, $\underset{\cdot}{\cup}$, and $\dot{\cap}$ satisfy the adjoint relations (4) that are indicated immediately above.

---

(5) There are four commutation relations between cusps and saddles or crotches. These are expressed in the following four equations. Here, in order to demonstrate the symmetries among these relations we denote $\curlywedge^R = \curlywedge^{\bullet}$, $\curlywedge^L = {}^{\bullet}\curlywedge$, $\curlyvee_R = \curlyvee_{\bullet}$, $\curlyvee_L = {}_{\bullet}\curlywedge$, and $\circ_3 = \circ$.

(i)

$$\left(\vdash \otimes \dot{\cap}\right) \circ \left(\curlyvee_{\bullet} \otimes \vdash\right) = \left({}^{\bullet}\curlywedge \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right)$$

(ii)

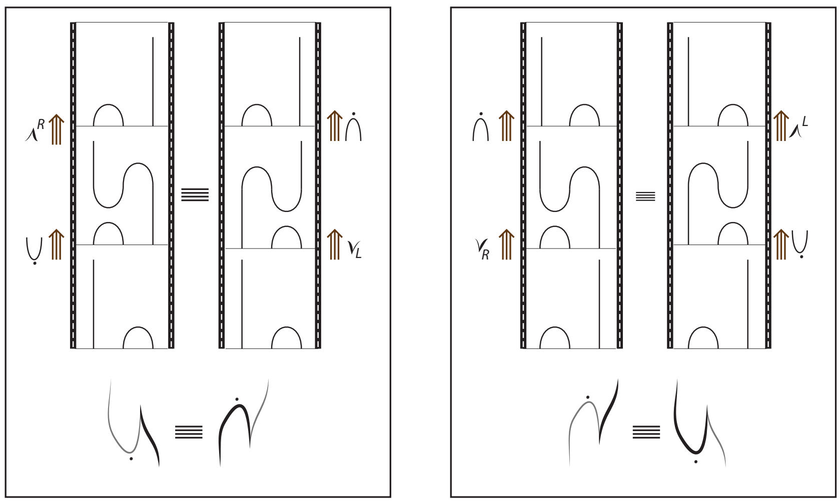
$$\left(\curlywedge^{\bullet} \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right) = \left(\vdash \otimes \dot{\cap}\right) \circ \left({}_{\bullet}\curlyvee \otimes \vdash\right)$$

(iii)

$$\left(\dashv \otimes \curlywedge^{\bullet}\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right) = \left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes {}_{\bullet}\curlyvee\right)$$

(iv)

$$\left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes \Upsilon_{\bullet}\right) = \left(\dashv \otimes {}^{\bullet}\curlywedge\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right)$$



$(\curlywedge^R) \circ_3 (\underset{\cdot}{\cup}) = (\dot{\cap}) \circ_3 (\Upsilon_L)$ $\quad$ $(\dot{\cap}) \circ_3 (\Upsilon_R) = (\curlywedge^L) \circ_3 (\underset{\cdot}{\cup})$

$(\dashv \otimes \curlywedge^R) \circ_3 (\underset{\cdot}{\cup} \otimes \dashv) = (\dot{\cap} \otimes \dashv) \circ_3 (\dashv \otimes \Upsilon_L)$ $\quad$ $(\dot{\cap} \otimes \dashv) \circ_3 (\dashv \otimes \Upsilon_R) = (\vdash \otimes \curlywedge^L) \circ_3 (\underset{\cdot}{\cup} \otimes \dashv)$

FIGURE 12. Commutations $\dot{\cap}$, $\Upsilon \leftrightharpoons \underset{\cdot}{\cup}$, $\curlywedge$ — part 1

Obviously, we are asserting these relations because of topological considerations. But let us pause to observe that when they are written in these forms, equation (i) and (iv) are upside down versions of each other, as are (ii) and (iii). Furthermore, (ii) can be obtains from (i) by interchanging the left and right sides while also moving the indicators of the Υ and the $\curlywedge$ to the other side of the font. Equations (iii) and (iv) are similarly related. In an effort to obtain a concise version, we write

$$\boxed{\cap \circ \Upsilon = \curlywedge \circ \cup}$$

to encapsulate all four relations. The movie move versions of these relations are found in Figs. 12 and 13.

Even though these relations $\cap \circ \Upsilon = \curlywedge \circ \cup$ are deemed to be invertible, there are sequences of triple arrows that might seem to be identities, but are not since $\underset{\cdot}{\cup}$ and $\dot{\cap}$ are non-invertible triple arrows. For example, compose the film strip on the left sides of movie moves in Fig. 13. The composition is $\left(\curlywedge^R\right) \circ_3 \left(\underset{\cdot}{\cup}\right) \circ \left(\dot{\cap}\right) \circ \left(\Upsilon^R\right)$, but the inner pair of triple arrows $\left(\underset{\cdot}{\cup}\right) \circ \left(\dot{\cap}\right)$ do not cancel.

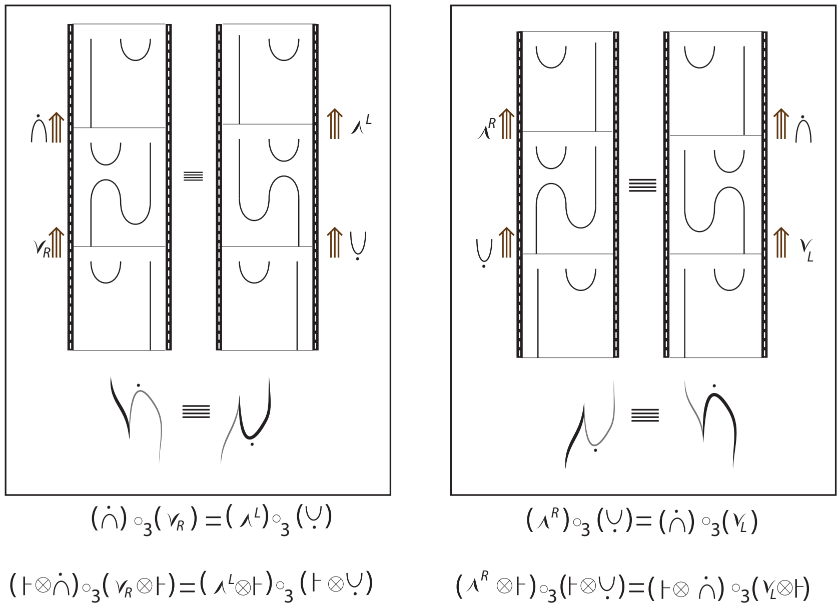


FIGURE 13. Commutations $\curlyvee, \dot{\cap} \leftrightarrows \curlywedge, \underset{\cdot}{\cup}$ — part 2

(5) The swallowtail identities do not lend themselves to a clever algebraic description. The identities take the form

$$\left(\curlywedge^L \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(+) \right) \circ \left(\curlyvee^L \otimes \dashv \right) = \dashv ,$$

$$\dashv = \left(\curlywedge^R \otimes \dashv \right) \circ_3 \left(\vdash \otimes \mathsf{X}(-) \right) \circ \left(\curlyvee^L \otimes \dashv \right),$$

$$\left(\vdash \otimes \curlywedge^L \right) \circ_3 \left(\mathsf{X}(+) \otimes \dashv \right) \circ \left(\dashv \otimes \curlyvee^L \right) = \vdash ,$$

and

$$\vdash = \left(\vdash \otimes \curlywedge^L \right) \circ_3 \left(\mathsf{X}(-) \otimes \dashv \right) \circ \left(\dashv \otimes \curlyvee^L \right).$$

In the next section we will supply a collection of adjectives that describe the 3-category that this example encompasses.

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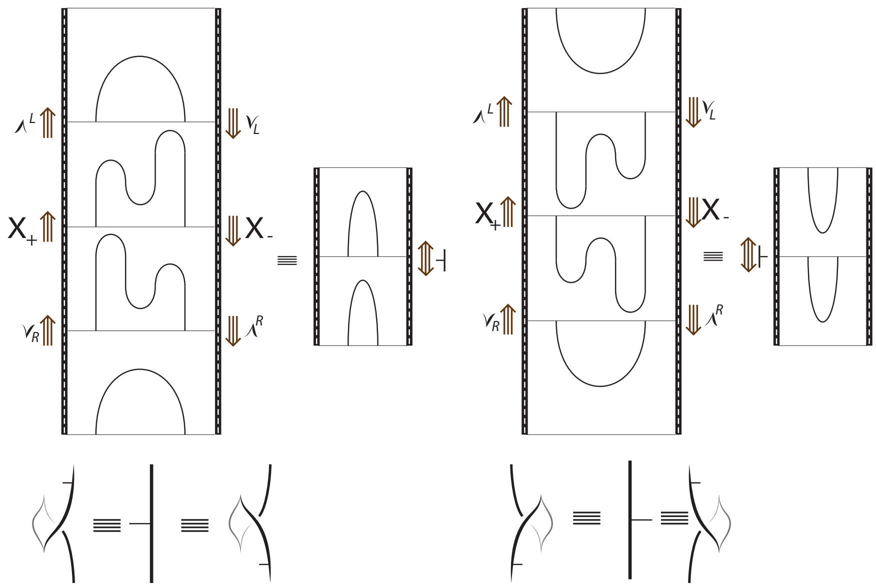

Figure 14. Swallowtail identities