Options

Shashi Jain, IISc

SUMMER RESEARCH PROGRAM

DYNAMICS OF COMPLEX SYSTEMS

2019

- What are options ?
- Can options be risk-free ?

Options

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Equity

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Nifty50 11,861.10 -67.65 -0.57%

Normal Market has Closed. May 29, 2019

Next Trading Date : May 30 , 2019

(All prices in ₹)

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View : Stock Options

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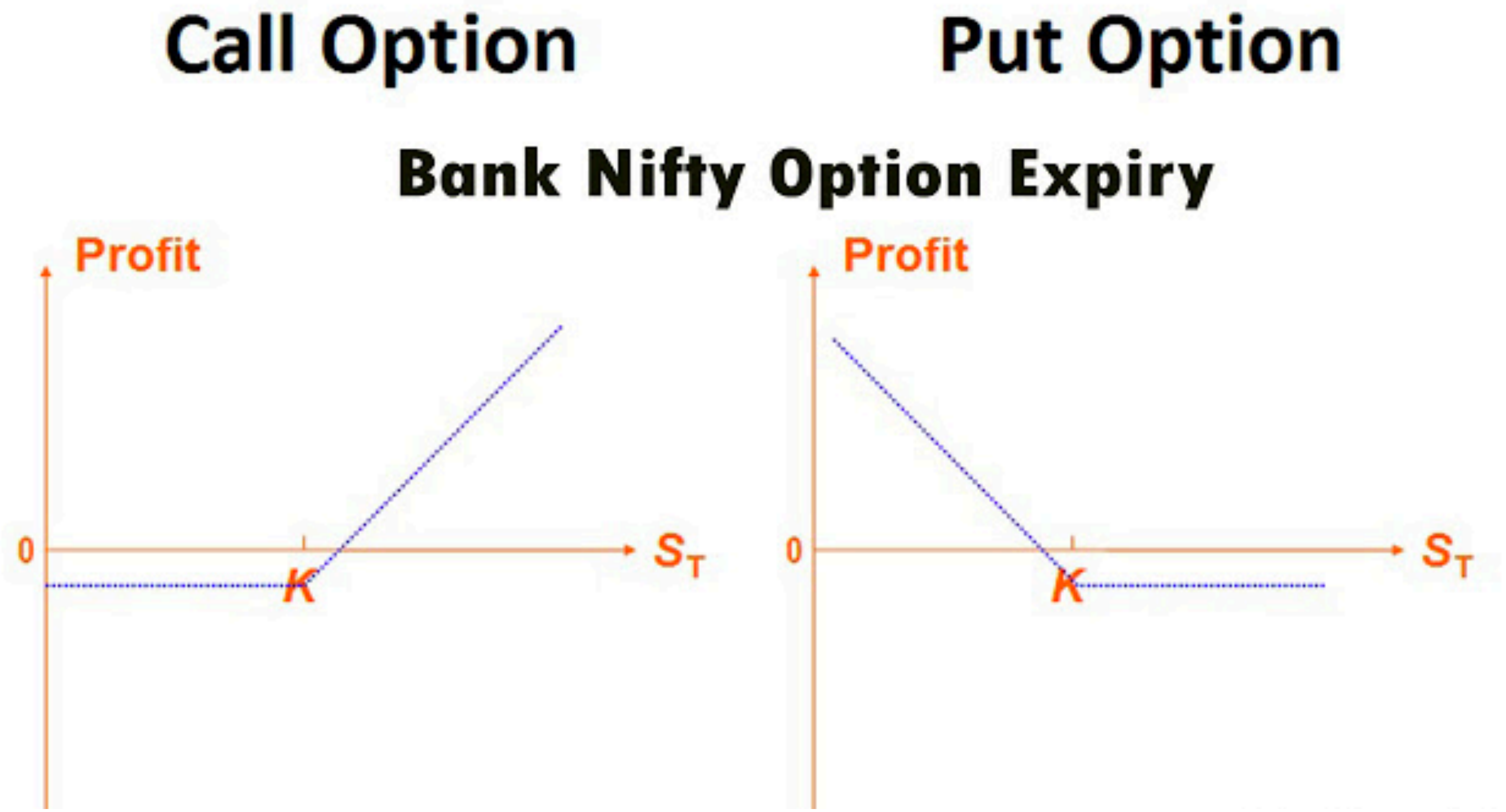
as on May 29, 2019 15:30:30 IST

Instrument Type	Underlying	Expiry Date	Option Type	Strike Price	Prev Close	Open Price	High Price	Low Price	Last Price	Volume (Contracts)	Turnover * (lacs)	Premium Turnover(lacs)	Underlying Value
Stock Options	SBIN	30MAY2019	CE	360.00	4.45	3.15	3.30	0.30	0.40	13,102	1,41,886.80	385.20	348.20
Stock Options	RELIANCE	30MAY2019	CE	1,340.00	8.05	7.10	9.90	1.65	2.20	18,328	1,23,293.37	495.77	1315.60
Stock Options	RELIANCE	30MAY2019	PE	1,300.00	4.75	4.45	8.30	2.55	3.15	14,629	95,422.77	334.27	1315.60
Stock Options	SBIN	30MAY2019	CE	355.00	7.65	5.25	5.75	0.65	0.65	8,543	91,446.83	463.88	348.20
Stock Options	RELIANCE	30MAY2019	CE	1,360.00	3.45	4.35	4.35	0.80	1.00	11,326	77,135.72	118.92	1315.60

Option definition

- Option gives you (the holder) right but not the obligation to buy(call) or sell(put) the underlying at a pre-agreed price (strike).
  - You can buy or sell standard options on an underlying in an exchange.
  - The price would depend upon
    - the maturity,
    - type of the option,
    - Current price of the underlying
    - The strike or the pre-agreed price
    - **volatility of the underlying**

At Maturity



Call Option
Put Option
Bank Nifty Option Expiry
Profit
0
K
S_T
Profit
0
K
S_T

Risk Free

- If you know for certain a cashflow in future, you discount it using the risk free rate
- Is it possible to discount future cashflows from entering into option position using a risk free rate ?

Brownian Motion

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What we cover

- Building the intuition and definition of Brownian motion
- Properties of Brownian motion
- Concept of Quadratic variation.

Symmetric Random Walks

$$X_j = \begin{cases} 1 & \text{if } \omega_j = H, \\ -1 & \text{if } \omega_j = T, \end{cases}$$

$$M_k = \sum_{j=1}^{k} X_j, \quad k = 1, 2, \ldots.$$

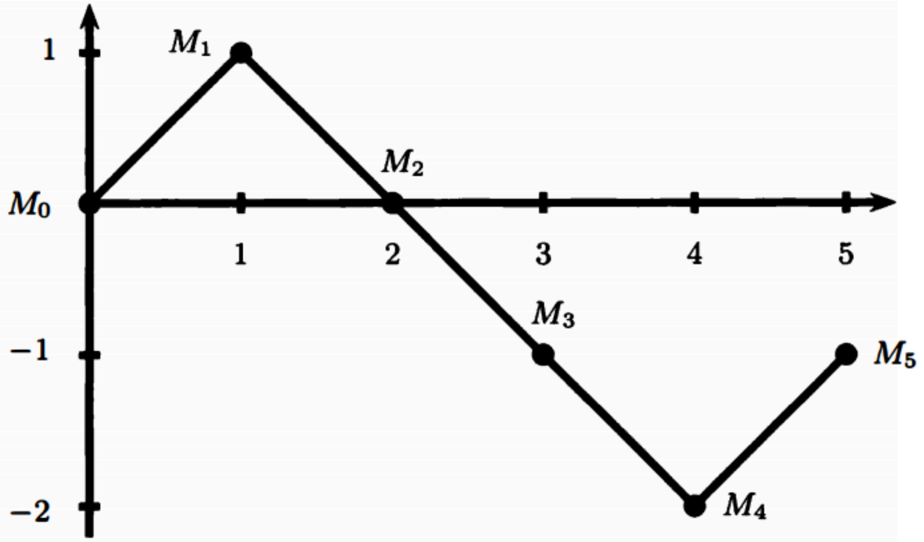


Fig. 3.2.1. Five steps of a random walk.

Properties of SRW

- The increments are independent

A random walk has *independent increments*. This means that if we choose nonnegative integers $0 = k_0 < k_1 < \cdots < k_m$, the random variables

$$M_{k_1} = (M_{k_1} - M_{k_0}),\ (M_{k_2} - M_{k_1}), \ldots, (M_{k_m} - M_{k_{m-1}})$$

$$M_{k_{i+1}} - M_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j,$$

- What is the expected value of each increment ?
- What is the variance of the increments ?

Martingale Property

- When k < l

$$\begin{aligned}\mathbb{E}\left[M_\ell \middle| \mathcal{F}_k\right] &= \mathbb{E}\left[(M_\ell - M_k) + M_k \middle| \mathcal{F}_k\right] \\ &= \mathbb{E}\left[M_\ell - M_k \middle| \mathcal{F}_k\right] + \mathbb{E}\left[M_k \middle| \mathcal{F}_k\right] \\ &= \mathbb{E}\left[M_\ell - M_k \middle| \mathcal{F}_k\right] + M_k \\ &= \mathbb{E}\left[M_\ell - M_k\right] + M_k \;=\; M_k.\end{aligned}$$

Quadratic Variation of SRW

- The quadratic variation is defined along a path
- QV up to time k along a path is computed by taken all the one step increments of M along the path and squaring these increments

$$[M,M]_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2 = k.$$

- What is the difference between the variance and QV ?

Scaled Symmetric Random Walk

- We want to speed up the coin toss and scale down the step up and down values of symmetric random walk

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt},$$

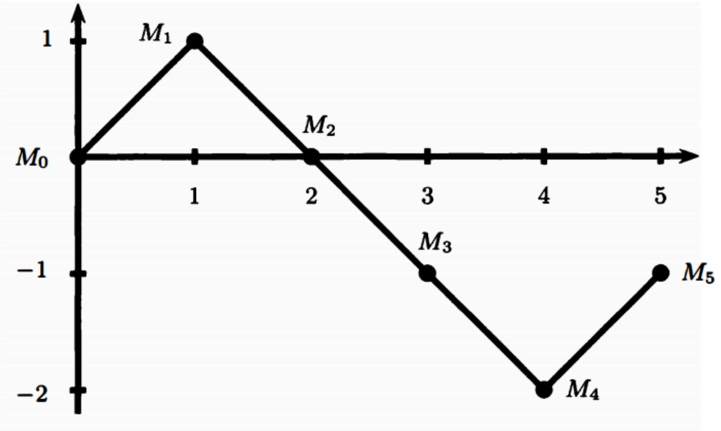


Fig. 3.2.1. Five steps of a random walk.

3.2 Scaled Random Walks

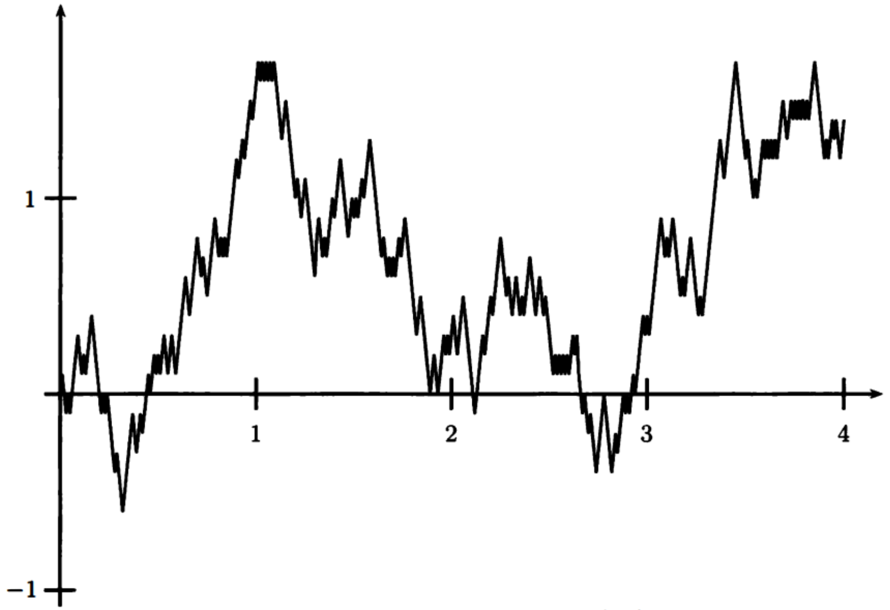


Fig. 3.2.2. A sample path of $W^{(100)}$.

Properties of SSRW

- The increments are independent

$$(W^{(n)}(t_1) - W^{(n)}(t_0)),\ (W^{(n)}(t_2) - W^{(n)}(t_1)), \ldots, (W^{(n)}(t_m) - W^{(n)}(t_{m-1}))$$

- The expectation of increments is:

$$\mathbb{E}\big(W^{(n)}(t) - W^{(n)}(s)\big) = 0,$$

- The variance of increments is

$$\mathrm{Var}\big(W^{(n)}(t) - W^{(n)}(s)\big) = t - s.$$

- SSRW is a martingale

$$\mathbb{E}\big[W^{(n)}(t) \big| \mathcal{F}(s)\big] = W^{(n)}(s)$$

Quadratic variation of SSRW

- The QV for a W(100) up till time 1.37 is

$$
\begin{aligned}
[W^{(100)}, W^{(100)}](1.37) &= \sum_{j=1}^{137}\left[W^{(100)}\left(\frac{j}{100}\right) - W^{(100)}\left(\frac{j-1}{100}\right)\right]^2 \\
&= \sum_{j=1}^{137}\left[\frac{1}{10}X_j\right]^2 = \sum_{j=1}^{137}\frac{1}{100} = 1.37.
\end{aligned}
$$

For any t

$$
\begin{aligned}
[W^{(n)}, W^{(n)}](t) &= \sum_{j=1}^{nt}\left[W^{(n)}\left(\frac{j}{n}\right) - W^{(n)}\left(\frac{j-1}{n}\right)\right]^2 \\
&= \sum_{j=1}^{nt}\left[\frac{1}{\sqrt{n}}X_j\right]^2 = \sum_{j=1}^{nt}\frac{1}{n} = t.
\end{aligned}
$$

Brownian motion

- Limiting distribution of a SSRW is a Brownian Motion
- The figure shows the outcome of one series of coin tosses
- You can also fix time t and look at outcome of several series of coin tosses
- If t = 0.25, the possible values of $W^{(100)}(0.25)$?
- What is the probability of $W^{(100)}(0.25)=0.1$?

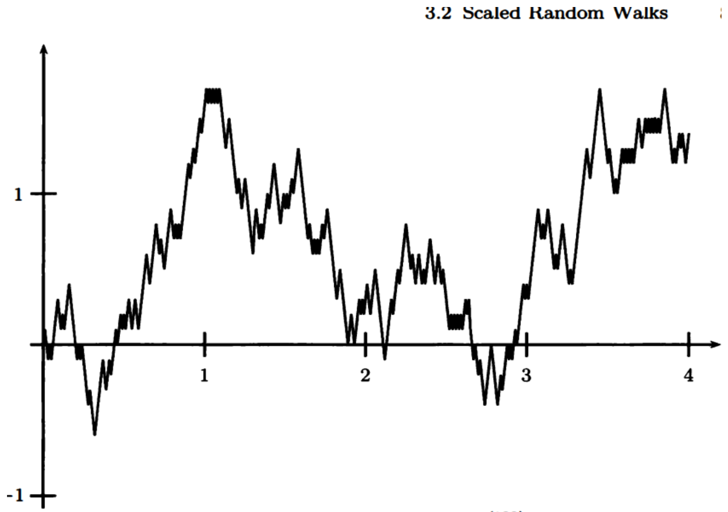
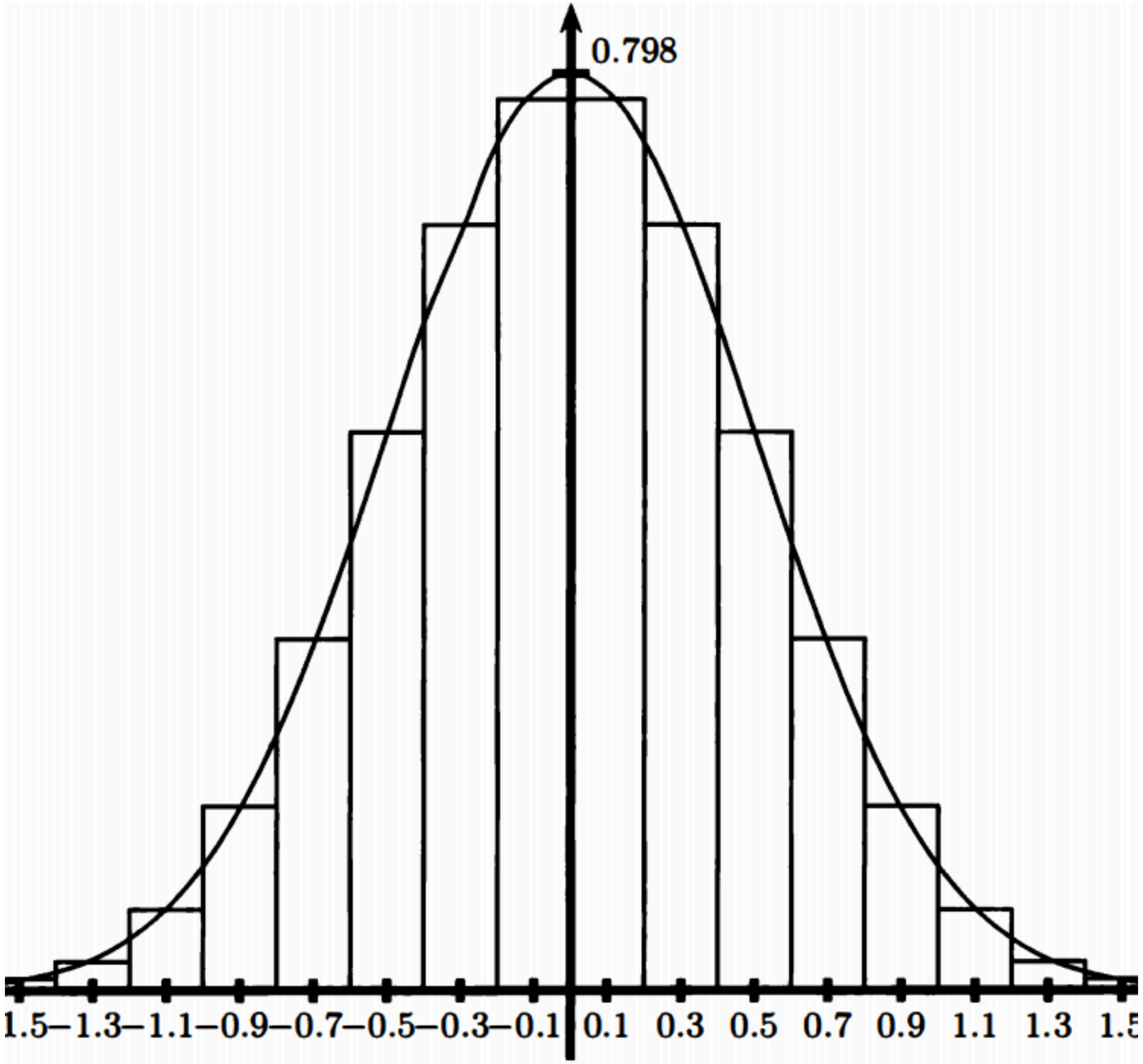


Fig. 3.2.2. A sample path of $W^{(100)}$.

Distribution of SSRW at t

- Figure shows the histogram bar centered at 0.1
- The width of the bar is 0.2, then what should be the height of the bar such that the probability of $W^{(100)}(0.25)=0.1555$?
- What is the variance of $W^{(100)}(0.25)$?



ig. 3.2.3. Distribution of $W^{(100)}(0.25)$ and normal curve $y = \frac{2}{\sqrt{2\pi}}e^{-2}$

Limiting distribution of SSRW

- For a fixed t, as n goes to infinity the distribution of scaled random walk converges to normal distribution, with mean ?, and variance ?
- PROOF
  - In order to identify a distribution one can identify their moment generating functions.
  - For normal density it would be

$$\begin{aligned}
\varphi(u) &= \int_{-\infty}^{\infty} e^{ux} f(x)\,dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left\{ux - \frac{x^2}{2t}\right\} dx \\
&= e^{\frac{1}{2}u^2 t} \cdot \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-ut)^2}{2t}\right\} dx \\
&= e^{\frac{1}{2}u^2 t}
\end{aligned}$$

Proof continued

- Moment generating function for $W^{(n)}(t)$ will be

$$\begin{aligned}\varphi_n(u) = \mathbb{E}e^{uW^{(n)}(t)} &= \mathbb{E}\exp\left\{\frac{u}{\sqrt{n}}M_{nt}\right\} \\ = \mathbb{E}\exp\left\{\frac{u}{\sqrt{n}}\sum_{j=1}^{nt}X_j\right\} &= \mathbb{E}\prod_{j=1}^{nt}\exp\left\{\frac{u}{\sqrt{n}}X_j\right\}.\end{aligned}$$

- The expectation can be taken inside, because of independence of X

$$\prod_{j=1}^{nt}\mathbb{E}\exp\left\{\frac{u}{\sqrt{n}}X_j\right\} = \prod_{j=1}^{nt}\left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right) = \left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right)^{nt}.$$

- We need to show that as n goes to infinity the above converges to the moment generating function of normal distribution

Proof continued

- We need show that

$$\log \varphi_n(u) = nt \log\left(\frac{1}{2}e^{\frac{u}{\sqrt{n}}} + \frac{1}{2}e^{-\frac{u}{\sqrt{n}}}\right)$$

Converges to $\log \varphi(u) = \frac{1}{2}u^2 t.$

- Take x = 1/sqrt(n)

$$\lim_{n\to\infty} \log \varphi_n(u) = t \lim_{x \downarrow 0} \frac{\log\left(\frac{1}{2}e^{ux} + \frac{1}{2}e^{-ux}\right)}{x^2}.$$

Apply L’ Hopital’s rule

- Numerator

$$\frac{\partial}{\partial x}\log\left(\frac{1}{2}e^{ux}+\frac{1}{2}e^{-ux}\right)=\frac{\frac{u}{2}e^{ux}-\frac{u}{2}e^{-ux}}{\frac{1}{2}e^{ux}+\frac{1}{2}e^{-ux}},$$

- Denominator is 2x
- Therefore,

$$\lim_{n\to\infty}\log\varphi_n(u)=t\lim_{x\downarrow 0}\frac{\frac{u}{2}e^{ux}-\frac{u}{2}e^{-ux}}{2x\left(\frac{1}{2}e^{ux}+\frac{1}{2}e^{-ux}\right)}=\frac{t}{2}\lim_{x\downarrow 0}\frac{\frac{u}{2}e^{ux}-\frac{u}{2}e^{-ux}}{x},$$

Again apply L' Hopital's rule

- Numerator

$$\frac{\partial}{\partial x}\left(\frac{u}{2}e^{ux}-\frac{u}{2}e^{-ux}\right)=\frac{u^2}{2}e^{ux}+\frac{u^2}{2}e^{-ux},$$

- Denominator is 1. Hence

$$\lim_{n\to\infty}\log\varphi_n(u)=\frac{t}{2}\lim_{x\downarrow 0}\left(\frac{u^2}{2}e^{ux}+\frac{u^2}{2}e^{-ux}\right)=\frac{1}{2}u^2t,$$

Definition of Brownian Motion

- Brownian motion is obtained as the limiting case of scaled random walk

Definition 3.3.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. For each* $\omega \in \Omega$, *suppose there is a continuous function* $W(t)$ *of* $t \geq 0$ *that satisfies* $W(0) = 0$ *and that depends on* ω. *Then* $W(t)$, $t \geq 0$, *is a* Brownian motion *if for all* $0 = t_0 < t_1 < \cdots < t_m$ *the increments*

$$W(t_1) = W(t_1) - W(t_0),\ W(t_2) - W(t_1),\ \ldots,\ W(t_m) - W(t_{m-1}) \quad (3.3.1)$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}\big[W(t_{i+1}) - W(t_i)\big] = 0, \quad (3.3.2)$$

$$\operatorname{Var}\big[W(t_{i+1}) - W(t_i)\big] = t_{i+1} - t_i. \quad (3.3.3)$$

Stochastic Calculus

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Quadratic Variation

- Before we look at Quadratic Variation let us consider first order variation (FOV).
- FOV gives the amount of up down movement of a function between 0 and T, with down moves adding to rather than subtracting.

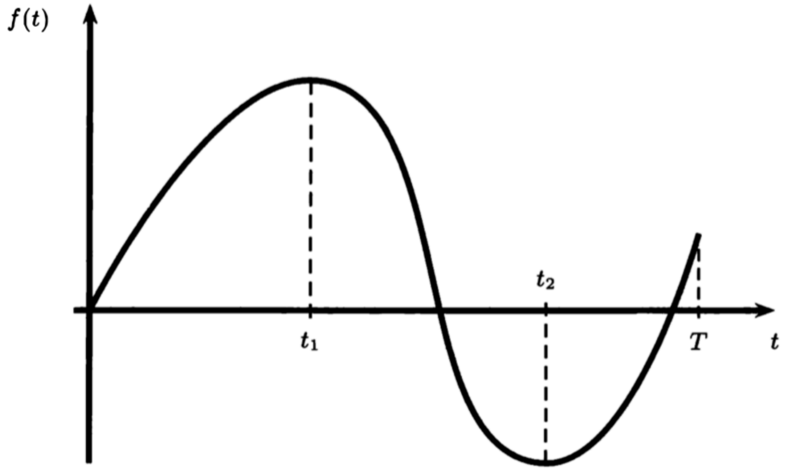

Fig. 3.4.1. Computing the first-order variation.

$$
\begin{aligned}
\mathrm{FV}_T(f) &= [f(t_1) - f(0)] - [f(t_2) - f(t_1)] + [f(T) - f(t_2)] \\
&= \int_0^{t_1} f'(t)\,dt + \int_{t_1}^{t_2} (-f'(t))\,dt + \int_{t_2}^{T} f'(t)\,dt \\
&= \int_0^{T} |f'(t)|\,dt.
\end{aligned}
$$

FOV

- In general you first partition [0, T] as

$$0 = t_0 < t_1 < \cdots < t_n = T. \qquad \Pi = \{t_0, t_1, \ldots, t_n\}$$

- The maximum step size of the partition is denoted as

$$\|\Pi\| = \max_{j=0,\ldots,n-1}(t_{j+1} - t_j)$$

- Then

$$\mathrm{FV}_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|.$$

FOV continued

- We use Mean Value Theorem

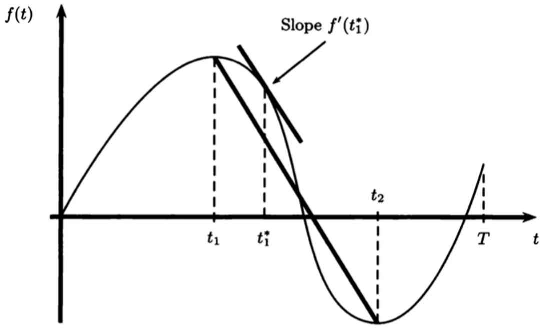

Fig. 3.4.2. Mean Value Theorem.

$$\frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} = f'(t_j^*).$$

- Therefore, we can write

$$\mathrm{FV}_T(f) = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} |f'(t_j^*)|(t_{j+1} - t_j) = \int_0^T |f'(t)|\, dt,$$

Quadratic Variation

- According to definition of Quadratic variation

Definition 3.4.1. *Let $f(t)$ be a function defined for $0 \leq t \leq T$. The* quadratic variation *of f up to time T is*

$$[f, f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2, \tag{3.4.5}$$

where $\Pi = \{t_0, t_1, \ldots, t_n\}$ and $0 = t_0 < t_1 < \cdots < t_n = T$.

Quadratic variation of f (continuous and differentiable)

- Using MVT we have

$$\sum_{j=0}^{n-1}[f(t_{j+1})-f(t_j)]^2 = \sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j)^2 \leq \|\Pi\|\cdot\sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j),$$

- Which translates to

$$\begin{aligned}
[f,f](T) &\leq \lim_{\|\Pi\|\to 0}\left[\|\Pi\|\cdot\sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j)\right] \\
&= \lim_{\|\Pi\|\to 0}\|\Pi\|\cdot\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1}|f'(t_j^*)|^2(t_{j+1}-t_j) \\
&= \lim_{\|\Pi\|\to 0}\|\Pi\|\cdot\int_0^T|f'(t)|^2\,dt \;=\; 0.
\end{aligned}$$

Quadratic variation of W (Brownian Motion)

- Define sampled Quadratic variation for a partition of [0,T] as

$$Q_\Pi = \sum_{j=0}^{n-1} \big(W(t_{j+1}) - W(t_j)\big)^2.$$

- Prove that the above **random variable** converges to T as $\|\Pi\| \to 0.$
- This can be shown if the expected value of Q is T and its variance converges to 0.

Definition of Brownian Motion

- Definition of Brownian Motion:

Definition 3.3.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. For each* $\omega \in \Omega$, *suppose there is a continuous function* $W(t)$ *of* $t \geq 0$ *that satisfies* $W(0) = 0$ *and that depends on* ω. *Then* $W(t)$, $t \geq 0$, *is a* Brownian motion *if for all* $0 = t_0 < t_1 < \cdots < t_m$ *the increments*

$$W(t_1) = W(t_1) - W(t_0),\ W(t_2) - W(t_1),\ \ldots,\ W(t_m) - W(t_{m-1}) \tag{3.3.1}$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}\big[W(t_{i+1}) - W(t_i)\big] = 0, \tag{3.3.2}$$

$$\mathrm{Var}\big[W(t_{i+1}) - W(t_i)\big] = t_{i+1} - t_i. \tag{3.3.3}$$

In order to prove QV of W

- We first use:

$$\mathbb{E}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] = \mathrm{Var}\left[W(t_{j+1}) - W(t_j)\right] = t_{j+1} - t_j$$

- Therefore,

$$\mathbb{E}Q_\Pi = \sum_{j=0}^{n-1} \mathbb{E}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] = \sum_{j=0}^{n-1}(t_{j+1} - t_j) = T,$$

- Next we determine that variance of Q

Variance of Q

$$
\begin{aligned}
&\mathrm{Var}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] \\
&= \mathbb{E}\left[\left(\left(W(t_{j+1}) - W(t_j)\right)^2 - (t_{j+1} - t_j)\right)^2\right] \\
&= \mathbb{E}\left[\left(W(t_{j+1}) - W(t_j)\right)^4\right] - 2(t_{j+1} - t_j)\mathbb{E}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] \\
&\qquad + (t_{j+1} - t_j)^2.
\end{aligned}
$$

- The 4th moment of normal random variable with zero mean is three times its variance squared (home work exercise)

$$
\mathbb{E}\left[\left(W(t_{j+1}) - W(t_j)\right)^4\right] = 3(t_{j+1} - t_j)^2,
$$

- Therefore

$$
\begin{aligned}
\mathrm{Var}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] &= 3(t_{j+1} - t_j)^2 - 2(t_{j+1} - t_j)^2 + (t_{j+1} - t_j)^2 \\
&= 2(t_{j+1} - t_j)^2, \qquad (3.4.7)
\end{aligned}
$$

Continued ..

$$\begin{aligned}\mathrm{Var}(Q_\Pi) &= \sum_{j=0}^{n-1} \mathrm{Var}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] = \sum_{j=0}^{n-1} 2(t_{j+1} - t_j)^2 \\ &\le \sum_{j=0}^{n-1} 2\|\Pi\|(t_{j+1} - t_j) = 2\|\Pi\| T.\end{aligned}$$

- Therefore

$$\lim_{\|\Pi\|\to 0} \mathrm{Var}(Q_\Pi) = 0,$$

$$\lim_{\|\Pi\|\to 0} Q_\Pi = \mathbb{E} Q_\Pi = T.$$

Implication

- We have

$$\mathbb{E}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] = t_{j+1} - t_j$$

$$\mathrm{Var}\left[\left(W(t_{j+1}) - W(t_j)\right)^2\right] = 2(t_{j+1} - t_j)^2.$$

- When t_{j+1}-t_j is small then $(t_{j+1}$-$t_j)^2$ is very small

$$\left(W(t_{j+1}) - W(t_j)\right)^2 \approx t_{j+1} - t_j.$$

- Or $E[\Delta W \Delta W] \approx \Delta t$

$$dW(t)\,dW(t) = dt$$

Ito's Integral

- For a normal integral

$$\int_0^T \Delta(t)\, dg(t) = \int_0^T \Delta(t) g'(t)\, dt,$$

- However, W(t) is not differentiable, so

$$\int_0^T \Delta(t)\, dW(t).$$

- You need to use Ito's integral

Constructing the Ito's integral

- Figure shows a simple path of Δ(t)
- Then define

$$I(t) = \Delta(t_0)[W(t) - W(t_0)] \ = \ \Delta(0)W(t), \quad 0 \le t \le t_1,$$
$$I(t) = \Delta(0)W(t_1) + \Delta(t_1)[W(t) - W(t_1)], \quad t_1 \le t \le t_2,$$
$$I(t) = \Delta(0)W(t_1) + \Delta(t_1)[W(t_2) - W(t_1)] + \Delta(t_2)[W(t) - W(t_2)], \quad t_2 \le t \le t_3,$$

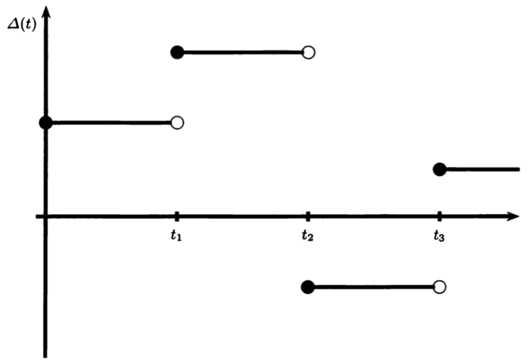

Fig. 4.2.1. A path of a simple process.

- Ito's integral of the process is

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)].$$

$$I(t) = \int_0^t \Delta(u)\, dW(u).$$

Properties of Ito’s integral

1. Ito’s isometry

$$\mathbb{E}\, I^2(t) = \mathbb{E} \int_0^t \Delta^2(u)\, du.$$

- Proof

2. Quadratic variation of I

$$[I, I](t) = \int_0^t \Delta^2(u)\, du.$$

Ito Doeblin Formula

- If W was differentiable, we could write

$$df(W(t)) = f'(W(t))\,W'(t)\,dt = f'(W(t))\,dW(t).$$

- However, as W has non zero quadratic variation, the correct formula is

$$df(W(t)) = f'(W(t))\,dW(t) + \frac{1}{2}f''(W(t))\,dt.$$

Theorem 4.4.1 (Itô-Doeblin formula for Brownian motion). *Let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous, and let $W(t)$ be a Brownian motion. Then, for every $T \geq 0$,*

$$f\big(T,W(T)\big) = f\big(0,W(0)\big) + \int_0^T f_t\big(t,W(t)\big)\,dt + \int_0^T f_x\big(t,W(t)\big)\,dW(t) + \frac{1}{2}\int_0^T f_{xx}\big(t,W(t)\big)\,dt. \quad (4.4.3)$$

Proof

- Let x_j, x_{j+1} be numbers, then by Taylor expansion:

$$f(x_{j+1}) - f(x_j) = f'(x_j)(x_{j+1} - x_j) + \frac{1}{2} f''(x_j)(x_{j+1} - x_j)^2.$$

- We want to find f(W(T))-f(W(0))
- Divide [0,T] into intervals. We look at one such interval $[t_j, t_{j+1}]$

$$\begin{aligned} f(W(T)) - f(W(0)) &= \sum_{j=0}^{n-1} \left[f\big(W(t_{j+1})\big) - f\big(W(t_j)\big) \right] \\ &= \sum_{j=0}^{n-1} f'(W(t_j))\big[W(t_{j+1}) - W(t_j)\big] \\ &\quad + \frac{1}{2} \sum_{j=0}^{n-1} f''(W(t_j))\big[W(t_{j+1}) - W(t_j)\big]^2. \end{aligned}$$

Properties of Ito's integral

Theorem 4.3.1. *Let T be a positive constant and let $\Delta(t)$, $0 \le t \le T$, be an adapted stochastic process that satisfies (4.3.1). Then $I(t) = \int_0^t \Delta(u)\,dW(u)$ defined by (4.3.3) has the following properties.*

(i) **(Continuity)** *As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.*

(ii) **(Adaptivity)** *For each t, $I(t)$ is $\mathcal{F}(t)$-measurable.*

(iii) **(Linearity)** *If $I(t) = \int_0^t \Delta(u)\,dW(u)$ and $J(t) = \int_0^t \Gamma(u)\,dW(u)$, then $I(t) \pm J(t) = \int_0^t \big(\Delta(u) \pm \Gamma(u)\big)\,dW(u)$; furthermore, for every constant c, $cI(t) = \int_0^t c\Delta(u)\,dW(u)$.*

(iv) **(Martingale)** *$I(t)$ is a martingale.*

(v) **(Itô isometry)** $\mathbb{E}I^2(t) = \mathbb{E}\int_0^t \Delta^2(u)\,du$.

(vi) **(Quadratic variation)** $[I, I](t) = \int_0^t \Delta^2(u)\,du$.

Black Scholes Equation

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Evolution of Portfolio value

- Consider you have a portfolio which at any time t has a value denoted by X(t).
- This portfolio invests
    a. In a money market(bank) account that pays a constant rate of interest r
    b. In a stock modeled by the Geometric Brownian Motion (GBM)

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t).$$

The portfolio $X(t)$

- At each t, suppose the investor holds $\Delta(t)$ shares of the above stock.
- $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $W(t)$.
- The remainder of the portfolio value, $X(t)- \Delta(t)S(t)$ is invested in the money market.

Change in the portfolio value

- The change is the investor's portfolio at each time t is due to
  1. Capital gains Δ(t)dS(t) on the stock position
  2. Interest on the bank account r(X(t)- Δ(t)S(t))dt
- Therefore

$$
\begin{aligned}
dX(t) &= \Delta(t)\, dS(t) + r\big(X(t) - \Delta(t)S(t)\big)\, dt \\
&= \Delta(t)\big(\alpha S(t)\, dt + \sigma S(t)\, dW(t)\big) + r\big(X(t) - \Delta(t)S(t)\big)\, dt \\
&= rX(t)\, dt + \Delta(t)(\alpha - r)S(t)\, dt + \Delta(t)\sigma S(t)\, dW(t).
\end{aligned}
$$

Discounted portfolio value

- If we are interested in the present value of the future change in the portfolio

$$
\begin{aligned}
&d\big(e^{-rt}X(t)\big)\\
&= df\big(t, X(t)\big)\\
&= f_t\big(t, X(t)\big)\,dt + f_x\big(t, X(t)\big)\,dX(t) + \frac{1}{2}f_{xx}\big(t, X(t)\big)\,dX(t)\,dX(t)\\
&= -re^{-rt}X(t)\,dt + e^{-rt}\,dX(t)\\
&= \Delta(t)(\alpha - r)e^{-rt}S(t)\,dt + \Delta(t)\sigma e^{-rt}S(t)\,dW(t)
\end{aligned}
$$

European Option

- A European option gives the option holder the right but not obligation to buy (call) or sell (put) an asset at a prespecified price (called the strike).
- A call option pays $(S(T)-K)^+$ at time T.
- The value of this option at any time (t) would depend upon
  - The time to expiry, i.e. (T-t)
  - Value of the stock, i.e. S(t)
  - Agreed strike price (K)
  - It also depends upon the model parameters (r, sigma) which are assumed to be constant
- Therefore we can use c(t,x) to denote the value of a call option, where x at time t is S(t).

Change in option price

- Our goal is to determine a function c(t,x), so that if we plug in the stock price at any time we get the corresponding option price.

$$
\begin{aligned}
&dc\big(t, S(t)\big) \\
&= c_t\big(t, S(t)\big)\,dt + c_x\big(t, S(t)\big)\,dS(t) + \frac{1}{2}c_{xx}\big(t, S(t)\big)\,dS(t)\,dS(t) \\
&= c_t\big(t, S(t)\big)\,dt + c_x\big(t, S(t)\big)\big(\alpha S(t)\,dt + \sigma S(t)\,dW(t)\big) \\
&\qquad + \frac{1}{2}c_{xx}\big(t, S(t)\big)\sigma^2 S^2(t)\,dt \\
&= \left[c_t\big(t, S(t)\big) + \alpha S(t)c_x\big(t, S(t)\big) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}\big(t, S(t)\big)\right] dt \\
&\qquad + \sigma S(t)c_x\big(t, S(t)\big)\,dW(t).
\end{aligned}
$$

Change in the discounted option price

- Using the Ito's expansion for f(t,x) = $e^{-rt}x$, where x is c(t,S(t)).

$$
\begin{aligned}
d\big(e^{-rt}c(t,S(t))\big) \\
&= df\big(t, c(t,S(t))\big) \\
&= f_t\big(t,c(t,S(t))\big)\,dt + f_x\big(t,c(t,S(t))\big)\,dc(t,S(t)) \\
&\qquad + \frac{1}{2} f_{xx}\big(t,c(t,S(t))\big)\,dc(t,S(t))\,dc(t,S(t)) \\
&= -re^{-rt}c(t,S(t))\,dt + e^{-rt}\,dc(t,S(t)) \\
&= e^{-rt}\Big[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) \\
&\qquad + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\Big]\,dt + e^{-rt}\sigma S(t)c_x(t,S(t))dW(t).
\end{aligned}
$$

Hedging an option

- When you sell an option you start with an initial capital X(0) and invest it in stock and money market account
- The aim of the portfolio is to ideally at each time perfectly replicate the option value, c(t,S(t)).
- This will happen if the present value of the future portolio is always equal to the present value of the option, i.e. for all t
  - $e^{-rt}X(t) = e^{-rt}c\big(t, S(t)\big)$
- The above equality implies

$$d\big(e^{-rt}X(t)\big) = d\big(e^{-rt}c(t, S(t))\big) \text{ for all } t \in [0, T)$$

Hedging continued

- Substituting the LHS and RHS, we have

$$\begin{aligned}&\Delta(t)(\alpha - r)S(t)\,dt + \Delta(t)\sigma S(t)\,dW(t)\\&= \left[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right]dt\\&\qquad +\sigma S(t)c_x(t,S(t))\,dW(t).\end{aligned} \tag{4.5.10}$$

- First looking at dW(t) term:

$$\Delta(t) = c_x(t,S(t)) \text{ for all } t \in [0,T).$$

- This is the *delta-hedging* rule. At each time prior to the expiration, the number of shares held in the hedge portfolio is equal to the partial derivative with respect to the stock price of the option value at that time.
- $c_x(t,S(t))$ is called the delta of the option.

Equating the dt term

- We next equate the dt terms to obtain

$$
\begin{aligned}
&(\alpha - r)S(t)c_x(t, S(t)) \\
&= -rc(t, S(t)) + c_t(t, S(t)) + \alpha S(t)c_x(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t, S(t)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } t \in [0, T). \quad (4.5.12)
\end{aligned}
$$

- The alpha terms cancel out leaving

$$c_t(t, x) + rxc_x(t, x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t, x) = rc(t, x) \text{ for all } t \in [0, T),\ x \geq 0,$$

- c(t,x) is the solution of the above pde

Solution to the pde

- The pde is a backward parabolic type.
- For such an equation, in addition to the terminal condition one needs boundary conditions.
  - The terminal condition is ?
  - The boundary condition at x= 0 ?
  - The boundary condition at x goes to infinity ?
- The solution of the BS equation with these boundary and terminal conditions is

Black Scholes equation

$$c(t,x) = xN\big(d_+(T-t,x)\big) - Ke^{-r(T-t)}N\big(d_-(T-t,x)\big),\ 0 \le t < T,\ x > 0,$$

- Where

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right],$$

- And N is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}\,dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-\frac{z^2}{2}}\,dz.$$

- Sometimes the notation used is

$$\mathrm{BSM}(\tau,x;K,r,\sigma) = xN\big(d_+(\tau,x)\big) - Ke^{-r\tau}N\big(d_-(\tau,x)\big),$$

Forward contract

- A forward contract on a stock with delivery K obligates its holder to buy one stock at expiration T for K.
- At T the value of the contract is S(T)-K.
- Let f(t,x) denote the value of this contract at any time t in [0, T].
- We can show that
  - $f(t,x) = x - e^{(-r(T-t))}K$.
- This can be shown using the no-arbitrage argument.

Forward contract

- If you sell forward contract at time zero for
  - $f(0,S(0)) = S(0)-e^{-r(T)}K$
- You can use the proceed to buy one unit of stock (price S(0))
- You will have to borrow $e^{-rT}K$ from money market.
- At the expiration he has one unit of stock, i.e. S(T), owes the bank an amount K
- He needs to give the buyer of the forward contract a stock for an amount K.
- As he is able to replicate the payoff of the forward contract with this portfolio whose value at time t is
  - $S(t)-e^{-r(T-t)}K$, this should be the forward price at any time t.

Put Call Parity

- At any time

$$x - K = (x - K)^+ - (K - x)^+$$

- The payoff of a forward contract at T is equal to

$$f(T, S(T)) = c(T, S(T)) - p(T, S(T));$$

- It should also then agree at any time t

$$f(t, x) = c(t, x) - p(t, x), \quad x \geq 0,\ 0 \leq t \leq T.$$