Inter- and intra-host dynamics: Lecture 5 problems

1. Disease fitness in the simple SI model with host demography Consider the simple model of disease transmission from infected individuals ($I(t)$) to susceptible individuals ($S(t)$).

$$\frac{dS}{dt} = b - \delta S - \beta SI \quad (1)$$

$$\frac{dI}{dt} = \beta SI - \alpha I. \quad (2)$$

We can modify this model to have two "competing" strains of the disease:

$$\frac{dS}{dt} = b - \delta S - (\beta_1 I_1 + \beta_2 I_2)S \quad (3)$$

$$\frac{dI_1}{dt} = \beta_1 SI_1 - (\alpha_1 + \delta)I_1 \quad (4)$$

$$\frac{dI_2}{dt} = \beta_2 SI_2 - (\alpha_2 + \delta)I_2. \quad (5)$$

a. Define $R^{(i)} = \frac{\beta_i}{\alpha_i+\delta}$. In lecture, I made the statement that if two strains are present, the strain with the higher value of R^i will outcompete the other strain and eventually the weaker strain will be excluded. Prove the related statement that a strain with higher R^i can invade a resident strain with lower R^i. To do this, suppose that strain 1 is at steady state and then introduce a small number of strain 2. Linearize and show that strain 2 will grow only if it has a higher value of R^i.
b. (challenge) Prove the statement from class. Hint: Lyapunov function.

2. Viral competition in the standard viral dynamics model
The standard viral dynamics model, modified to allow for two competing strains within-host, looks like this:

$$\frac{dT}{dt} = \lambda - dT - (k_1 V_1 + k_2 V_2)T \quad (6)$$

$$\frac{dT_1^*}{dt} = k_1 V_1 T - \mu_1 T_1^* \quad (7)$$

$$\frac{dT_2^*}{dt} = k_2 V_2 T - \mu_2 T_2^* \quad (8)$$

$$\frac{dV_1}{dt} = p_1 T_1^* - c_1 V_1 \quad (9)$$

$$\frac{dV_2}{dt} = p_2 T_2^* - c_2 V_2 \quad (10)$$

Define the viral fitness $\rho_i = \frac{\lambda k_i p_i}{d\mu_i c_i}$. I made the statement that if two virus strains are present, the strain with the higher value of ρ will outcompete the other strain and exclude it from the population. This statement is almost equivalent to the statement in the previous problem 1a and the proof is similar.
b. Linearize this model around the uninfected steady state and calculate the leading eigenvalue (growth rate) for a single strain invading a new host, in this model. Show that the invading strain can invade if $r_i = \lambda k_i p_i - d\mu_i c_i > 0$.
c. Show that $r_i > 0$ if and only if $\rho_i > 1$.

d. Take $k_1 = k_2 = k$, $c_1 = c_2 = c$ and set parameters as below. Find example parameters p_i and μ_i so that $\rho_1 > \rho_2$ but $r_1 < r_2$.
$c = 10$/day, $k = 0.01$/day/viruses/μl, $\lambda = 10$cells/μl/day, $d = 1$/day.