Viral dynamics: Lecture 2 problems

The Forward Chapman-Kolmogorov equation for the birth-death process is:

$$\frac{dP_{n,n_0}}{dt} = d(n+1)P_{n+1,n_0} + b(n-1)P_{n-1,n_0} - (b+d)nP_{n,n_0}.$$

Here the per-capita birth rate is b and the death rate is d.

1. **Pure birth process**
Suppose that there are no deaths, and there is a single individual at $t = 0$.

a. Verify that

$$P_{n,1}(t) = e^{-nbt}\left(e^{bt} - 1\right)^{(n-1)}$$

valid for $n \geq 1$. (Challenge: derive this formula)

b. Calculate the mean population size $\bar{P}(t) = \sum_{n=0}^{\infty} nP_{n,1}(t)$.
(Hint: $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ and $\sum_{n=0}^{\infty} nq^n = \frac{q}{(1-q)^2}$)

2. **Pure death process**
Now suppose that

$$P_{n,n_0}(t=0) = \delta_{n,n_0} = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases}$$

and there are no births - only deaths.

a. Find $P_{n_0,n_0}(t)$ and also $P_{n,n_0}(t)$ valid for $t \geq 0$.

b. Find the probability of extinction by time t. In other words, find $P_{0,n_0}(t)$.

c. Find the mean time to extinction. Hint: consider $n_0 = 1$, $n_0 = 2$, etc and then work out the general case.

Challenge: calculate the mean time to extinction directly from part a. Calculate

$$\int_0^{\infty} tP_{0,n_0}(t)dt.$$

d. Suppose a population of size N goes extinct in exactly T years on average. How long will it take for a population of size $N+1$ to go extinct?