Lecture 6

An introduction to Macroscopic Fluctuation Theory.

Ref: B. Derrida Review in JStat Mech 2007

G. JONA-LASINIO in ArXiv 2014

P. Krapivsky and B. Meerson PRE 2012

Density Fluctuations in the open ASEP

Recall that Density Fluctuations in a gas at thermal equilibrium were obtained as

$$\Pr\{\rho(x)\} \sim \mathrm{e}^{-\beta V\, \mathcal{F}(\{\rho(x)\}}$$

where the Large-Deviation Functional is local and is given by

$$\mathcal{F}(\{\rho(x)\} = \int_0^1 \left(f(\rho(x), T) - f(\bar{\rho}, T)\right) d^3x$$

What do the Density Fluctuations in the ASEP look like?

The probability of observing an **atypical density profile in the steady state of the ASEP** was calculated starting from the exact microscopic solution of the exclusion process, with the help of the Matrix Ansatz (B. Derrida, J. Lebowitz E. Speer, 2002).

Large Deviations of the Density Profile in ASEP

The Large Deviation Functional for the symmetric case $q = 0$ is given by

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left(B(\rho(x), F(x)) + \log \frac{F'(x)}{\rho_2 - \rho_1} \right)$$

where $B(u, v) = (1-u) \log \frac{1-u}{1-v} + u \log \frac{u}{v}$ and $F(x)$ satisfies

$$F\left(F'^2 + (1-F)F''\right) = F'^2 \rho \quad \text{with} \quad F(0) = \rho_1 \text{ and } F(1) = \rho_2\,.$$

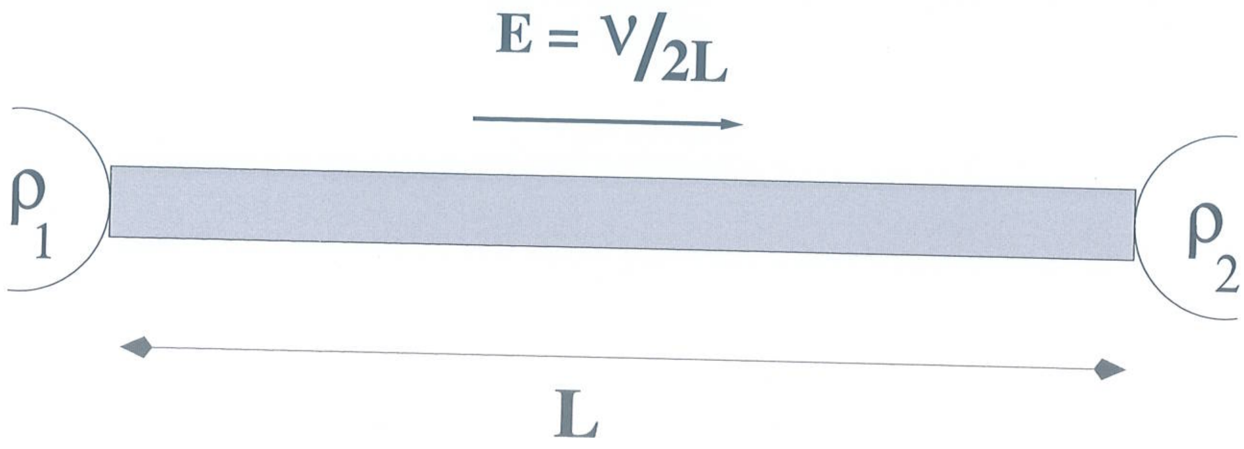
This functional is non-local as soon as $\rho_1 \neq \rho_2$.

This functional is NOT identical to the one given by local equilibrium.

Note that in the case of equilibrium, for $\rho_1 = \rho_2 = \bar{\rho}$, *we recover*

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left\{ (1-\rho(x)) \log \frac{1-\rho(x)}{1-\bar{\rho}} + \rho(x) \log \frac{\rho(x)}{\bar{\rho}} \right\}$$

The Hydrodynamic Limit: Diffusive case



Starting from the microscopic level, define local density $\rho(x,t)$ and current $j(x,t)$ with macroscopic space-time variables $x = i/L, t = s/L^2$ (diffusive scaling).
The typical evolution of the system is given by the hydrodynamic behaviour (Burgers-type equation):

$$\partial_t \rho = \nabla\left(D(\rho)\nabla\rho\right) - \nu\nabla\sigma(\rho) \quad \text{with} \quad D(\rho) = 1 \text{ and } \sigma(\rho) = 2\rho(1-\rho)$$

(Lebowitz, Spohn, Varadhan)

How can Fluctuations be taken into account?

Fluctuating Hydrodynamics

Consider Y_t the total number of particles transfered from the left reservoir to the right reservoir during time t.

- $\lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = D(\rho)\frac{\rho_1-\rho_2}{L} + \sigma(\rho)\frac{\nu}{L}$ for $(\rho_1 - \rho_2)$ small
- $\lim_{t\to\infty} \frac{\langle Y_t^2 \rangle}{t} = \dfrac{\sigma(\rho)}{L}$ for $\rho_1 = \rho_2 = \rho$ and $\nu = 0$.

Then, the equation of motion is obtained as:

$$\partial_t \rho = -\partial_x j \quad \text{with} \quad j = -D(\rho)\nabla\rho + \nu\sigma(\rho) + \sqrt{\sigma(\rho)}\xi(x,t)$$

where $\xi(x,t)$ is a Gaussian white noise with variance

$$\langle \xi(x',t')\xi(x,t)\rangle = \frac{1}{L}\delta(x-x')\delta(t-t')$$

For the symmetric exclusion process, the 'phenomenological' coefficients are given by

$$D(\rho) = 1 \quad \text{and} \quad \sigma(\rho) = 2\rho(1-\rho)$$

Large Deviations at the Hydrodynamic Level

What is the probability to observe an atypical current $j(x,t)$ and the corresponding density profile $\rho(x,t)$ during $0 \leq s \leq L^2\, T$?

$$\Pr\{j(x,t), \rho(x,t)\} \sim \mathrm{e}^{-L\,\mathcal{I}(j,\rho)}$$

Use fluctuating hydrodynamics to write the Large-Deviation Functional as a path-integral: the current and the density evolve $(\rho(x,t), j(x,t))$ according to a stochastic dynamics. The weight of a trajectory between 0 and t can written as:

$$\text{Weight}\left(\{\rho(x,t'), j(x,t')\}_{\substack{0\leq x\leq 1\\ 0\leq t'\leq t}}\right) =$$

$$\int \mathcal{D}\xi(x,t')\ \exp\left(-\frac{L}{2}\int_0^t dt' \int_0^1 dx\, \xi^2(x,t)\right)$$

$$\prod_{\substack{0\leq x\leq 1\\ 0\leq t'\leq t}} \delta\left(\frac{\partial\rho}{\partial t'} + \frac{\partial j}{\partial x}\right) \prod_{\substack{0\leq x\leq 1\\ 0\leq t'\leq t}} \delta\left(j + D(\rho)\frac{\partial\rho}{\partial x} - \nu\sigma(\rho) + \sqrt{\sigma(\rho)}\xi\right)$$

This formula is analogous to the one used to change variables in probability theory:
If X is a random variable distributed according to $P(X)$ and if $Y = F(X)$ (F being known function) then the distribution of Y is given by

$$\mathrm{Prob}(Y) = \int dX\, P(X)\, \delta(Y - F(X))$$

Now the probability of observing $\rho(x,t)$ and $j(x,t)$ at time t knowing that we started with $\rho_0(x), j_0(x)$ is given by the sum of the weights of all possible trajectories beginning with $\rho_0(x), j_0(x)$ and ending up at $\rho(x,t)$ and $j(x,t)$:

$$\text{Proba}\left(\rho(x,t), j(x,t) | \rho_0(x), j_0(x)\right)$$
$$= \int_{\substack{\rho_0 \to \rho_t \\ j_0 \to j_t}} \mathcal{D}\rho(x,t')\, \mathcal{D}j(x,t')\, \text{Weight}\left(\{\rho(x,t'), j(x,t')\}_{\substack{0 \le x \le 1 \\ 0 \le t' \le t}} \right)$$

Using the previous expression for the Trajectory Weight and performing the integral over the noise ξ, we obtain:

$$\text{Proba}\left(\rho(x,t), j(x,t) | \rho_0(x), j_0(x)\right) = \int_{\substack{\rho_0 \to \rho_t \\ j_0 \to j_t}} \mathcal{D}\rho \mathcal{D}j \prod_{\substack{0 \le x \le 1 \\ 0 \le t' \le t}} \delta\left(\frac{\partial \rho}{\partial t'} + \frac{\partial j}{\partial x}\right)$$

$$\exp\left(-\frac{L}{2}\int_0^t dt' \int_0^1 dx \frac{(j + D(\rho)\frac{\partial \rho}{\partial x} - \nu\sigma(\rho))^2}{\sigma(\rho)}\right)$$

We are interested in the large L limit: the integral will be dominated by the optimal value of the exponent (saddle-point).

The value at the saddle-point will provide us with the large deviation functional.

Macroscopic Fluctuation Theory

The large deviation functional can be written as the solution of an optimal path problem (G. Jona-Lasinio et al.)

$$\mathcal{I}(j,\rho) = \min_{\rho,j}\left\{\int_0^T dt \int_0^1 dx \frac{(j - \nu\sigma(\rho) + D(\rho)\nabla\rho)^2}{2\sigma(\rho)}\right\}$$

with the constraint: $\partial_t \rho = -\nabla . j$
Knowing $\mathcal{I}(j,\rho)$ one can deduce (by contraction) the LDF of the current or the profile.
For example

$$\Phi(j) = \min_{\rho}\{\mathcal{I}(j,\rho)\}$$

This variational problem has a Hamiltonian structure and can be expressed by using a pair of conjugate variables (p,q).

MFT Formalism

Mathematically, one has to solve the corresponding Euler-Lagrange equations. After some transformations, one obtains a set of coupled PDE's (here, we take $\nu = 0$):

$$\begin{aligned} \partial_t q &= \partial_x[D(q)\partial_x q] - \partial_x[\sigma(q)\partial_x p] \\ \partial_t p &= -D(q)\partial_{xx} p - \frac{1}{2}\sigma'(q)(\partial_x p)^2 \end{aligned}$$

where $q(x,t)$ is the density-field and $p(x,t)$ is a conjugate field.
The physical content is encoded in the 'transport coefficients' $D(q)(=1)$ and $\sigma(q)(=2q(1-q))$ that contain the information of the microscopic dynamics relevant at the macroscopic scale.
Do note that these equations have a Hamiltonian structure.

- A general framework but these non-linear MFT equations are very difficult to solve in general. By using them one can in principle calculate large deviation functions directly at the macroscopic level.
- The analysis of this new set of 'hydrodynamic equations' has just begun!

Conclusions

The asymmetric exclusion process is a paradigm for the behaviour of systems far from equilibrium in low dimensions. The ASEP is important for the Theory and for its multiple Applications (especially in biophysics).

Large deviation functions (LDF) appear as a generalization of the thermodynamic potentials for non-equilibrium systems. They exhibit remarkable properties such as the Fluctuation Theorem, valid far away from equilibrium. The LDF's are very likely to play a key-role in the future of non-equilibrium statistical mechanics.

Current fluctuations are a signature of non-equilibrium behaviour. The exact results we derived can be used to calibrate the more general framework of fluctuating hydrodynamics (MFT), which is currently being developed.

Supplementary note:

How to write a path integral for a Langevin equation?

X random variable
$Y = F(X)$

Then $P(Y) = \int dX \, P(X) \, \delta(Y - F(X))$
probability

(The usual " $P'(Y)\,dY = P'(X)\,dX$ ")

For N-variables $X_1, \dots X_N$

suppose
$$\begin{cases} Y_1 = \varphi_1(X_1 \dots X_N) \\ Y_2 = \varphi_2(X_1 \dots X_N) \\ \vdots \\ Y_N = \varphi_N(X_1 \dots X_N) \end{cases}$$

Then
$$\text{Prob}(Y_1, \dots Y_N) = \int dX \, P(X_1 \dots X_N) \, \delta(Y_1 - \varphi_1(X_1 \dots X_N)) \times \delta(Y_2 - \varphi_2(\,)) \cdots \times \delta(Y_N - \varphi_N(\,))$$

Then explain that it's the same for A LANGEVIN equation
$$\dot{Y} = F(Y) + \xi \quad \text{whitenoise}$$

Discretise time $0, \Delta t, 2\Delta t, \dots N\Delta t, \dots$

Y_0 given and $Y_{N+1} = Y_N + \Delta t \, [F(Y_N) + \xi_N]$

$$\langle \xi_i \xi_j \rangle = \frac{\sigma}{\Delta t} \delta_{ij}$$

$$P(Y_1, \dots Y_N / Y_0) = \int d\xi_0 \dots d\xi_{N-1} \, e^{-\frac{\Delta t}{2\sigma}[\xi_0^2 + \xi_1^2 + \dots \xi_{N-1}^2]}$$
$$\times \delta\left[\frac{Y_1 - Y_0}{\Delta t} - F(Y_0) - \xi_0\right] \times \delta\left[\frac{Y_2 - Y_1}{\Delta t} - F(Y_1) - \xi_1\right] \times \dots$$
$$\dots \times \delta\left[\frac{Y_N - Y_{N-1}}{\Delta t} - F(Y_{N-1}) - \xi_N\right]$$

$$\underset{\text{limit}}{\rightsquigarrow} \int \mathcal{D}\xi(t') \, e^{-\frac{1}{2\sigma}\int_0^t \xi^2(t')\,dt'} \prod \delta(\dot{Y} - F(Y) - \xi)$$

Personal NOTES : more detailed derivations (1)

(Sept 2013) Equations et formalisme de la MACROSCOPIC FLUCTUATION THEORY

Postulats de départ : systèmes diffusifs (e.g. SEP)

Q_t = charge totale transférée pendant la durée t

si $\rho_1 = \rho_2$, en moyenne $\langle Q_t \rangle = 0$, mais il y a des fluctuations

$$\frac{\langle Q_t^2 \rangle}{t} \xrightarrow[t\to\infty]{} \frac{\sigma(\rho)}{L}$$

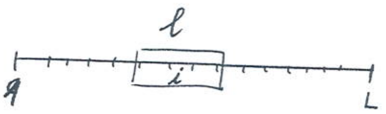
Pour $\rho_1 \neq \rho_2$, il s'écoule un courant stationnaire non nul

$$\frac{\langle Q_t \rangle}{t} \xrightarrow[t\to\infty]{} D(\rho)\,\frac{\rho_1 - \rho_2}{L} \qquad \text{pour } |\rho_1 - \rho_2| \ll 1$$

Si l'on ajoute un champ uniforme et faible de module $\frac{\nu}{2L}$ la moyenne précédente est modifiée en :

$$\frac{\langle Q_t \rangle}{t} \simeq D(\rho)\,\frac{\rho_1-\rho_2}{L} + \nu\,\frac{\sigma(\rho)}{L} \qquad \text{(réponse linéaire)}$$

M.F.T : Scalings



$1 \ll \ell \ll L$

limite continue :
$$\begin{cases} x = \dfrac{i}{L} & 0 \le x \le 1 \\ \tau = \dfrac{t}{L^2} & \text{(diffusif)} \end{cases} \qquad \Delta x = \frac{\ell}{L}$$

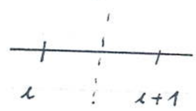
- Densité locale en un site $\rho_i(t)$. A la limite continue on écrit

$$\hat{\rho}(x,\tau) \underset{\text{def}}{\equiv} \hat{\rho}\left(\frac{i}{L},\frac{t}{L^2}\right) = \frac{1}{\ell}\sum_{j=i-\frac{\ell}{2}}^{i+\ell/2} \rho_i(t)$$

la fonction macroscopique

- Flux total à travers le point x (depuis $t=0$) :

au niveau microscopique

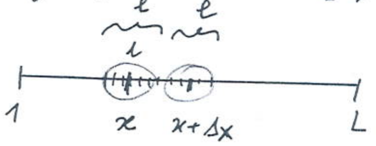


$Q_{i+\frac{1}{2}}(t)$ = nombre total de particules de $i \to i+1$ − nombre total de particules de $i+1 \to i$ } pendant la durée t

Le courant microscopique vaut $j_{i-i+1} = \frac{d}{dt} Q_{i-i+1}$

on écrira $j_{i+\frac{1}{2}}$ pour j_{i-i+1} et $Q_{i+\frac{1}{2}}$ pour Q_{i-i+1}

Echelle macroscopique :



Flux total entre la boite centrée en x et celle centrée en $x+\Delta x$: $Q_{i+\frac{\ell}{2}}(t)$

on a défini des fonctions microscopiques $\rho_i(t)$, $j_{i+\frac{1}{2}}(t)$ et $Q_{i+\frac{1}{2}}(t)$ - on voudrait définir des fonctions macroscopiques $\hat{\rho}(x,\tau)$, $\check{j}(x,\tau)$ et $\check{Q}(x,\tau)$

Façon 1 : on a $j_i(t) \sim \frac{cste}{L}$ et donc $\frac{dQ_i}{dt} \sim \frac{cste}{L}$ i.e $Q_i^{(t)} \sim \frac{t}{L}$

mais on va jusqu'à des temps $t = L^2\tau$ donc

$Q_i(\tau) \sim \frac{L^2\tau}{L} \sim O(L)$ La charge transportée pendant une durée macroscopique $\Delta\tau$ est d'ordre $L\Delta\tau$

Donc les bonnes limites finies sont

$$\hat{Q}(x,\tau) \equiv \hat{Q}\left(\frac{i}{L}, \frac{t}{L^2}\right) = \frac{1}{L} Q_{i+\frac{1}{2}}(t)$$

et
$$\check{j}(x,\tau) \equiv \check{j}\left(\frac{i}{L}, \frac{t}{L^2}\right) = L\, j_{i+\frac{1}{2}}(t)$$

Façon 2 : conservation du # de particules

$$\frac{d\rho_i}{dt} = \dot{j}_{i-\frac{1}{2}} - \dot{j}_{i+\frac{1}{2}} = \frac{d}{dt}\left(Q_{i-\frac{1}{2}} - Q_{i+\frac{1}{2}}\right)$$

i.e $\rho_i(t) - \rho_i(0) = Q_{i-\frac{1}{2}}(t) - Q_{i+\frac{1}{2}}(t)$ $\quad (Q_i(0) \equiv 0)$

macroscopique ↳

$$\hat{\rho}(x,\tau) - \hat{\rho}(x,0) = Q\left(x - \frac{1}{2L}, \tau\right) - Q\left(x + \frac{1}{2L}, \tau\right)$$
$$= -\frac{1}{L}\frac{d}{dx} Q(x,\tau) \qquad \left(\text{i.e. } \frac{1}{di} = \frac{1}{L}\frac{1}{dx}\right.$$

donc la bonne limite continue consiste à prendre

$$\hat{Q}(x,\tau) = \frac{1}{L} Q_{i+\frac{1}{2}}(t)$$

↳ ici le scaling du temps ne joue pas

pour le courant $J_{i+\frac{1}{2}}(t) = \frac{d}{dt} Q_{i+\frac{1}{2}}(t)$ micro

$$\downarrow$$

$$J_{i+\frac{1}{2}}(t) = \frac{1}{L^2} \frac{d}{d\tau} L \hat{Q}(x,\tau)$$

i.e $\frac{d}{d\tau} \hat{Q}(x,\tau) = L J_{i+\frac{1}{2}}(t) \equiv \hat{j}(x,\tau)$

3e façon $\frac{d}{dt} \rho_i = -\frac{d}{di} j_i$ (micro) $\longrightarrow$ $\rho \to \hat{\rho}$, $dt = L^2 d\tau$, $di = L\,dx$

$$\frac{d\hat{\rho}}{d\tau} = -L \frac{dj}{dx}$$

Donc $\hat{j} = L j$

$j_i = \frac{d}{dt} Q_i \longrightarrow \frac{\hat{j}}{L} = \frac{1}{L^2} \frac{dQ}{d\tau}$

Donc $\hat{Q} = \frac{1}{L} Q$

Bilan :

MICRO $\longrightarrow$ MACRO

MICRO	MACRO	
i	$x = \frac{i}{L}$	$(\hat{i})$
t	$\tau = \frac{t}{L^2}$	$(\hat{t})$
ρ	$\hat{\rho}_i = \rho$	
j	$\hat{j} = L j$	
Q	$\hat{Q} = \frac{1}{L} Q$	

(les fonctions ^ dépendent des variables ^)

$$\frac{d\hat{\rho}}{d\tau} = -\frac{d\hat{j}}{dx} \text{ et } \hat{j} = \frac{d\hat{Q}}{d\tau}$$

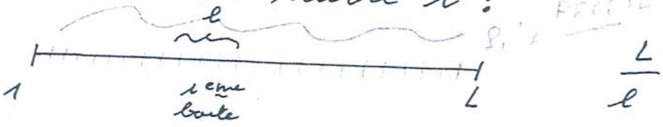
Preuve des équations MFT $\longrightarrow$

cf Bodineau + Derrida "Distribution of current in non-eq. diffusive systems and phase transitions" PRE 72, 066110 (2005)

On discrétise le temps microscopique Δt :

$$t_k = k\Delta t \qquad k \in [0, \frac{t}{\Delta t}]$$

On considère des boîtes de taille l :



$\frac{L}{l}$ boîtes

Dans la boîte numéro i, centrée autour il, on a une densité $\rho(il, k\Delta t)$

On appelle $q_i(k) =$ le flux total entre la boîte n° i et la boîte n° $(i+1)$ entre $k\Delta t$ et $k\Delta t + \Delta t$

avec les notations précédentes, on a

$$q_i(k) = Q_{il+\frac{l}{2}}((k+1)\Delta t) - Q_{il+\frac{l}{2}}(k\Delta t)$$

<u>Equilibre LOCAL</u> – Le profil varie peu pendant Δt (Δt pas TROP GRAND)

– Les variables aléatoires $q_i(k)$ sont indépendantes, Gaussiennes, avec : FIELD STRENGTH

$$\langle q_i(k) \rangle = \left\{ \frac{D(\rho_i)}{l} (\rho_i - \rho_{i+1}) + \frac{\nu}{L} \sigma(\rho_i) \right\} \Delta t$$

et $$\langle q_i^2(k) \rangle = \frac{\sigma(\rho_i)}{l} \Delta t$$ (i.e. Δt assez grand pour que ce soit vrai et l'indépendance aussi)

<u>Relation entre ρ_i et q_i :</u>

Variation du nombre de particules dans la boîte n° i entre $k\Delta t$ et $(k+1)\Delta t$ $= q_{i-1}(k) - q_i(k)$

sachant que la boîte est de taille l, on déduit

$$\rho_i((k+1)\Delta t) - \rho_i(k\Delta t) = \frac{q_{i-1}(k) - q_i(k)}{l}$$

$$\text{Proba}(\{q_i,\rho_i\}) = \exp - \sum_{k=0}^{t/\Delta t}\sum_{i=0}^{L/\ell} \frac{1}{2\frac{\sigma(\rho_i)\Delta t}{\ell}} \left\{ q_i(k) - \frac{D(\rho_i)(\rho_i-\rho_{i+1})}{\ell}\Delta t - \frac{\nu\sigma(\rho_i)}{L}\Delta t \right.$$

on remplace dans ces expressions par la limite continue avec les scalings précédents, avec $\ell = \Delta x$

en particulier $q_i(k) \simeq \Delta t \dfrac{d}{dt} Q_{(i+\frac{1}{2})\ell}(k\Delta t)$ $\quad j_{i+\frac{1}{2}}\ell\,(k\Delta t)$

où $\boxed{q_i(k) = \Delta t\, j_{(i+\frac{1}{2})\ell}(k\Delta t)}$

$$= \Delta t \frac{d}{dt} L\hat{Q}\Big(\underbrace{\frac{(i+\frac{1}{2})\ell}{L}}_{x}, \underbrace{\frac{k\Delta t}{L^2}}_{\tau}\Big)$$

$$= \Delta t \frac{1}{L}\frac{d}{d\tau}\hat{Q}$$

$$q_i(k) = \Delta t \frac{\hat{j}}{L}$$

$$\text{Proba}(\{q_i,\rho_i\}) = \exp - \sum_{k=0}^{t/\Delta t}\sum_{i=0}^{L/\ell} \frac{\ell\Delta t}{2\sigma(\hat{\rho})}\left\{\frac{\hat{j}}{L} + \frac{D(\hat{\rho})}{\cancel{\ell}}\frac{\ell}{L}\frac{\partial\hat{\rho}}{\partial x} - \frac{\nu}{L}\sigma(\hat{\rho})\right\}^2$$

$i \to i+1 \qquad x \to x + \dfrac{\ell}{L}$

$k \to k+1 \qquad \tau \to \tau + \dfrac{\Delta t}{L^2}$

$$= \exp - \underbrace{\frac{\ell}{L^2}\Delta t}\sum_{k=0}^{t/\Delta t}\sum_{i=0}^{L/\ell} \frac{\left\{\hat{j} + D(\hat{\rho})\frac{\partial\hat{\rho}}{\partial x} - \nu\sigma(\hat{\rho})\right\}^2}{2\sigma(\hat{\rho})}$$

$$= -L\frac{\ell}{L}\frac{\Delta t}{L^2}$$

les fonctions $\hat{\rho}$ et $\hat{j}$ étant évaluées en des valeurs discrètes

$t = k\Delta t$ i.e $\tau = k\dfrac{\Delta t}{L^2}$ i.e $\Delta\tau = \dfrac{\Delta t}{L^2}$

$x = i\dfrac{\ell}{L}$ i.e $\Delta x = \ell/L$

$$= \exp - L\,\Delta x\,\Delta\tau \sum_k\sum_i \frac{\left\{\hat{j}(i\Delta x, k\Delta\tau) + D(\hat{\rho})\frac{\partial\hat{\rho}}{\partial x} - \nu\sigma(\hat{\rho})\right\}^2}{2\sigma(\hat{\rho})}$$

$$\boxed{\text{Proba}\{q_i,\rho_i\} \longrightarrow \exp\left[-L\int_0^{t/L^2} d\tau \int_0^1 dx \frac{\left\{\hat{j}(x,\tau) + D(\hat{\rho}(x,\tau))\frac{\partial\hat{\rho}}{\partial x} - \nu\sigma(\hat{\rho})\right\}^2}{2\sigma(\hat{\rho})}\right.}$$

Formalisme MFT à partir d'une dynamique de LANGEVIN

On retrouve la fonctionnelle MFT à partir de l'équation stochastique suivante :

$$\frac{\partial \rho}{\partial t} = -\frac{\partial j}{\partial x}$$

(ici toutes les variables sont MACROSCOPIQUES)

avec

$$j = -D(\rho)\frac{\partial \rho}{\partial x} + \nu\sigma(\rho) + \sqrt{\sigma(\rho)}\,\xi(x,t)$$

où $\xi(x,t)$ est un bruit blanc Gaussien de covariance

$$\langle \xi(x,t)\,\xi(x',t')\rangle = \frac{1}{L}\,\delta(x-x')\,\delta(t-t')$$

On peut alors écrire la probabilité d'une configuration comme une intégrale de chemin.

Pour SEP $\quad D(\rho) \equiv 1 \quad$ et $\quad \sigma(\rho) = 2\rho(1-\rho)$

il y a un champ externe $E = \frac{\nu}{L}$ ce qui va donner une asymétrie $p \propto e^{E}$ et $q \propto e^{-E}$ et aussi $p - q \simeq \frac{2\nu}{L}$

Path integral formalism : [Phase space Path integral]

$$\mathrm{Proba}\left(\rho(x,t), j(x,t) / \rho_0(x), j_0(x)\right) = \int_{\substack{\rho_0 \to \rho_{x,t} \\ j_0 \to j_{x,t}}} \mathcal{D}\rho\, \mathcal{D}j\, \mathcal{D}\xi\; e^{-\frac{L}{2}\int_0^t dt' \int_0^1 dx\, \xi^2}$$
$$\times \prod_{x,t'} \delta\left(\frac{\partial \rho}{\partial t'} + \frac{\partial j}{\partial x}\right)$$
$$\times \prod_{x,t'} \delta\left(j + D(\rho)\frac{\partial \rho}{\partial x} - \nu\sigma(\rho) + \sqrt{\sigma}\,\xi\right)$$

$$= \int \mathcal{D}\rho\, \mathcal{D}j\; \underbrace{\prod_{x,t'} \delta\left(\frac{\partial \rho}{\partial t'} + \frac{\partial j}{\partial x}\right) e^{-\frac{L}{2}\int_0^t dt' \int_0^1 dx\, \frac{\left(j + D(\rho)\frac{\partial \rho}{\partial x} - \nu\sigma\right)^2}{\sigma}}}_{\text{POIDS D'UNE TRAJECTOIRE}}$$

on a "oublié" un Jacobien $\to$ Ito

Les δ de Dirac sont exprimés via un champ auxiliaire H (MSR)

$$\int \mathcal{D}\rho\, \mathcal{D}j\, \mathcal{D}H\; e^{-L\int_0^t dt'\int_0^1 dx\, H(x,t')\left[\frac{\partial \rho}{\partial t'}+\frac{\partial j}{\partial x}\right]}\; e^{-L\iint \frac{\left(j+D\frac{\partial\rho}{\partial x}-\nu\sigma(\rho)\right)^2}{2\sigma}}$$

Rescale $H \to L H$, l'action s'écrit alors (avec $-L$ devant

$$-\frac{S}{L} = \int_0^t dt'\int_0^1 dx\, H\frac{\partial\rho}{\partial t'} + \int_0^t dt'\left\{H(1,t')j(1,t') - H(0,t')j(0,t')\right\} - \iint_0^t dt'dx\, j\,\frac{\partial H}{\partial x}$$
$$+ \iint \frac{\left(j+D\frac{\partial\rho}{\partial x}-\nu\sigma\right)^2}{2\sigma}$$

On veut effectuer l'intégrale Gaussienne sur j

$$\int \mathcal{D}\rho\, \mathcal{D}H \left[\int \mathcal{D}j\; e^{-L\iint\frac{\left[j+D\frac{\partial\rho}{\partial x}-\nu\sigma-\sigma\frac{\partial H}{\partial x}\right]^2}{2\sigma} - L\int_0^t dt'\, H_1(t')j_1(t')-H_0(t')j_0(t')}\right] e^{-L\iint H\frac{\partial\rho}{\partial t'}-\frac{\sigma}{2}\left(\frac{\partial H}{\partial x}\right)^2}$$
$$\times\; e^{-L\iint \frac{\partial H}{\partial x}\left(D\frac{\partial\rho}{\partial x}-\nu\sigma\right)}$$

Si l'on suppose que les termes de bords s'annullent $H_1 = H_0 = 0$
(ou $j_1 = j_0 = 0$, ou $H_1 j_1 - H_0 j_0 = 0$)
on effectue l'intégrale Gaussienne sur j et on obtient

$$\int \mathcal{D}\rho\, \mathcal{D}H\; e^{-L\int_0^t dt'\int_0^1 dx\left[H(x,t')\frac{\partial\rho}{\partial t'} - \frac{\sigma}{2}\left(\frac{\partial H}{\partial x}\right)^2 + \frac{\partial H}{\partial x}\left(D\frac{\partial\rho}{\partial x}-\nu\sigma(\rho)\right)\right]}$$

on fait ensuite du calcul variationnel à $L\to\infty$

Utiliser MFT sur un intervalle fini ? sur un anneau ?

Cas d'un système infini :

on ne fait plus le scaling $x\to\frac{i}{L}$ et $0\le x\le 1$.

On part de l'éq. de Langevin

$$\begin{cases}\frac{\partial\rho}{\partial t} = -\frac{\partial j}{\partial x}\\ j = -D(\rho)\frac{\partial\rho}{\partial x} + \nu\sigma(\rho) + \sqrt{\sigma(\rho)}\,\xi\end{cases}$$

$$\langle \xi(x,t)\,\xi(x',t')\rangle = \delta(x-x')\,\delta(t-t')$$

La même opérat MSR conduit à

$$\int \mathcal{D}\rho\, \mathcal{D}H\; e^{-\int_0^T dt'\int_{-\infty}^{+\infty} dx\left[H\frac{\partial\rho}{\partial t'} - \frac{\sigma}{2}\left(\frac{\partial H}{\partial x}\right)^2 + \frac{\partial H}{\partial x}\left(D\frac{\partial\rho}{\partial x}-\nu\sigma\right)\right]}$$

on prend T = durée de l'expérience
on suppose que le terme de bord : $H_\infty j_\infty - H_{-\infty} j_{-\infty} = 0$
- Soit les courants sont nuls aux BORDS (cas SEP)
- sinon il faut ajuster H à $\pm\infty$.

Rescaling ($\to$ méthode du col) :

Cas $\nu = 0$ (SEP) : $$\begin{cases} t \to \dfrac{t}{T} \\ x \to \dfrac{x}{\sqrt{T}} \end{cases}$$

cela fournit une action avec $\sqrt{T}\int_0^1 dt \int_{-\infty}^{+\infty} dx \, \left(H \partial_t \rho - \frac{\sigma}{2}(\partial_x H)^2 + D\partial_x H \partial_x \rho\right)$

on fera ci-dessous la variation de cette action.

Pour $\nu \neq 0$ (Waseps) : pas de scaling évident.

On peut faire le cas diffusif

un scaling linéaire $x \to \frac{x}{T^x}$ $\quad H \to T^\beta H$

scaling du biais : $\nu \to \dfrac{\nu}{T^\alpha}$ $\quad \rho \to T^\delta \rho$?

(déduire un scaling des résultats connus).

Variation de l'action :

$$\delta S = \int_0^T dt \int_0^\infty dx \Big\{ \delta H \, \partial_t \rho + H \partial_t(\delta\rho) - \frac{\sigma'(\rho)}{2} \delta\rho (\partial_x H)^2 - \sigma(\rho)(\partial_x \delta H)\partial_x H$$
$$+ \partial_x(\delta H)\left(D\partial_x \rho - \nu\sigma(\rho)\right) + \partial_x H \left[D'(\rho)\delta\rho\, \partial_x \rho + D\partial_x \delta\rho - \nu\sigma'(\rho)\right.$$

(on garde les termes de bord)

$$\delta S = \int_0^T dt \int_{-\infty}^{+\infty} dx \, \delta H \left\{ \partial_t \rho + \partial_x(\sigma \partial_x H) - \partial_x(D\partial_x \rho - \nu\sigma) \right\}$$
$$+ \int_0^T dt \left[(-\partial_x H \, \sigma(\rho))\delta H + (D\partial_x \rho - \nu\sigma)\delta H \right]_{x=-\infty}^{x=+\infty}$$

$$+ \int_0^T dt \int_{-\infty}^{+\infty} dx \, \delta\rho \Big\{ -\partial_t H - \frac{\sigma'(\rho)}{2}(\partial_x H)^2 + \underbrace{D'(\rho)\partial_x \rho\, \partial_x H - \partial_x(D(\rho)\partial_x H)}_{= -D(\rho)\partial_{xx} H} - \nu\sigma'\partial_x H \Big\}$$
$$+ \int_{-\infty}^{+\infty} dx \left[\delta\rho(x,\cdot)\, H(x,\cdot)\right]_{t=0}^{t=T} + \int_0^T dt \left[\delta\rho\, D\, \partial_x H\right]_{x=-\infty}^{x=+\infty}$$

il faut garder les termes de bord : ils se combinent avec la moyenne quenched/annealed et avec la définition de la fonction génératrice (courant, surface...) pour donner les conditions aux limites des équations.

Les termes de bulk donnent les équations différentielles :

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \dfrac{\partial}{\partial x}\left\{ D(\rho)\dfrac{\partial \rho}{\partial x} - \nu\sigma(\rho) - \sigma(\rho)\dfrac{\partial H}{\partial x} \right\} \\[2ex] \dfrac{\partial H}{\partial t} = -D(\rho)\dfrac{\partial^2 H}{\partial x^2} - \nu\sigma'(\rho)\dfrac{\partial H}{\partial x} - \dfrac{\sigma'(\rho)}{2}\left(\dfrac{\partial H}{\partial x}\right)^2 \end{cases}$$

$x \in]-\infty, +\infty[\qquad t \in [0, T]$

on peut dans ce cas SEP $\underline{\underline{(\nu = 0)}}$ faire le scaling maintenant $t \to t/T \quad x \to x/\sqrt{T}$ et se ramener à $\underline{\underline{t \in [0, 1]}}$

<u>Termes de bords :</u> $\delta\rho = 0$ pour $x = \pm\infty$
$\delta H = 0$ pour $x = \pm\infty$

① il reste $\displaystyle\int_{-\infty}^{+\infty} dx \left\{ H(x,T)\,\delta\rho(x,T) - H(x,0)\,\underline{\delta\rho(x,0)} \right\}$

ça dépend de si on fixe la condition initiale ou pas

② Terme provenant de la fonction de gde déviation à calculer

$$\langle e^{\lambda Y_T} \rangle = \iint \mathcal{D}\rho\, \mathcal{D}j \; e^{\lambda Y_T} \, e^{\text{Action}(\rho, j)}$$

fonctionnelle du processus

$$Y_T = \int_0^\infty [\rho(x,T) - \rho(x,0)]\, dx \qquad \text{courant à travers l'origine}$$

$$\int_{-\infty}^{+\infty} \theta(x-x_0)[\rho(x,T)-\rho(x,0)] = \int_{x_0}^\infty [\rho(x,T) - \rho(x,0)]\, dx \qquad \text{courant à travers } x_0$$

$$= \int_{-\infty}^{+\infty} x\,[\rho(x,T) - \rho(x,0)]\, dx \qquad \text{surface (courant total)}$$

$$= \int_0^{+\infty} x\,[\rho(x,T) - \rho(x,0)]\, dx \qquad \tfrac{1}{2}\text{ surface (courant total à droite)}$$

$$= \int_{x_0}^{+\infty} x\,[\rho(x,T) - \rho(x,0)]\, dx \qquad \text{"surface à droite de } x_0\text{"}$$

$= \dots$

③ Terme provenant de la moyenne sur la condition initiale à $t = 0$

QUENCHED : $\rho(x,0)$ FIXÉ $\Rightarrow \delta\rho(x,0) = 0$

ANNEALED : $\rho(x,0)$ est choisie selon une loi

$\text{Proba}(\rho(x,0)) \sim e^{-\mathcal{F}_{\text{initiale}}(\rho(x,0))}$

↳ une variable aléatoire

Cela fournit une variation $\dfrac{\delta \mathcal{F}_{\text{initiale}}}{\delta\rho(x,0)}\,\delta\rho(x,0)$

Si l'on rassemble tous les termes de bord, on a :

$$\frac{\delta \mathcal{F}_{\text{init}}}{\delta\rho(x,0)}\,\delta\rho(x,0) + \int_{-\infty}^{+\infty}\left[H(x,T)\,\delta\rho(x,T) - H(x,0)\,\delta\rho(x,0)\right]dx - \lambda\,\delta Y_T$$

par exemple : • courant à travers x_0 : $\delta Y_{T,x_0} = \int_{-\infty}^{+\infty} dx\,\theta(x-x_0)\left[\delta\rho(x,T) - \delta\rho(x,0)\right]$

* terme en $\delta\rho(x,T)$: $H(x,T) = \lambda\,\theta(x-x_0)$

* terme en $\delta\rho(x,0)$ $H(x,0) = \lambda\,\theta(x-x_0) + \dfrac{\delta\mathcal{F}_{\text{init}}}{\delta\rho(x,0)}$ (annealed)

cas quenched $\delta\rho(x,0) = 0$ (car $\rho(x,0)$ fixée)

$\rho(x,0) = \rho_0(x)$ donnée du pb.

• surface à droite de x_0 $\delta Y_{T,x_0} = \int_{-\infty}^{+\infty} dx\; x\,\theta(x-x_0)\left[\delta\rho(x,T) - \delta\rho(x,0)\right]$

* terme en $\delta\rho(x,T)$ $H(x,T) = \lambda x\,\theta(x-x_0)$ $\theta(x_0-x)$

* terme en $\delta\rho(x,0)$ $H(x,0) = \lambda x\,\theta(x-x_0) + \dfrac{\delta\mathcal{F}_{\text{init}}}{\delta\rho(x,0)}$ (annealed)

ou $\rho(x,0) = \rho_0(x)$ (quenched)

choix de $\mathcal{F}_{\text{initiale}}$

$r(x) = (1-\theta(x))\rho_a + \rho_b\,\theta(x)$

On a $\dfrac{\delta\mathcal{F}_{\text{init}}}{\delta\rho(x,0)} = 2\displaystyle\int_{r(x)}^{\rho(x,0)} \frac{D(\rho)}{\sigma(\rho)}\,d\rho$

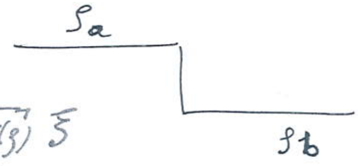
Calcul direct avec Langevin

br.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} J \qquad J = -D\frac{\partial \rho}{\partial x} + \nu\,\sigma(\rho) + \xi\sqrt{\sigma}$$

$D = 1 \qquad \sigma = 2\rho(1-\rho)$

sans bruit
$$\boxed{\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} - 2\nu\frac{\partial}{\partial x}\rho(1-\rho)}$$



$$+\frac{\partial}{\partial x}\sqrt{\sigma(\rho)}\,\xi$$

Hopf-Cole $\rho = \dfrac{1 - \frac{\partial h}{\partial x}}{2}$ $\qquad \dfrac{\partial h}{\partial x} = 1 - 2\rho$

$$\rho(1-\rho) = \frac{1 - \left(\frac{\partial h}{\partial x}\right)^2}{4}$$

$$-\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{\partial h}{\partial x}\right] = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial h}{\partial x}\right) - 2\nu\left(-\frac{1}{4}\right)\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}\right)^2$$

$$\cancel{\frac{\partial}{\partial x}}\left[\frac{\partial h}{\partial t}\right] = \cancel{\frac{\partial}{\partial x}}\left[\frac{\partial^2 h}{\partial x^2} + \nu\left(\frac{\partial h}{\partial x}\right)^2\right] \qquad + \frac{\partial}{\partial x}\sqrt{\frac{1-h_x^2}{2}}\,\xi$$

$\nu\, h(x,t)$ $\qquad + \sqrt{2(1-h_x^2)}\,\xi$

$W(x,t) = e$ $\qquad \dfrac{\partial W}{\partial t} = \alpha\dfrac{\partial h}{\partial t} W$

$\nu h(x,t) = \log W(x,t)$ $\qquad h_x = \dfrac{1}{\nu}\dfrac{\frac{\partial W}{\partial x}}{W}$

$$\frac{\partial W}{\partial x} = \alpha\frac{\partial h}{\partial x} W$$

$$\frac{\partial^2 W}{\partial x^2} = \left(\alpha\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2\alpha^2\right) W$$

$$\frac{1}{W}\,\boxed{\frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial x^2}}\,\frac{1}{W} \qquad W(x, t=0) \text{ connu } \quad W_0 = e^{\nu h_0}$$

à $t=0$
$$\begin{cases} x \le 0 & h(x,0) = (1-2\rho_a)x \\ x \ge 0 & h(x,0) = (1-2\rho_b)x \end{cases}$$

à l'ordre 0 :
$$W(x,t) = \int_{-\infty}^{+\infty} G(x-x', t)\, e^{\nu h_0(x')}\, dx'$$

$$\frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial x^2} + \sqrt{S(W)}\,\xi \times W$$
bruit multiplicatif.