Lecture 5:

The exclusion Process on an infinite line. Connection with Random Matrix Statistics

References: G.M. Schutz J. Stat Phys 88 p427 (1997)

A. Rakos and GM Schutz: ArXiv 2004 (0405464)

Recent papers by Tracy & Widom.

Aldous & Diaconis: Longest increasing subsequences: From patience sorting to the Baik-Deift-Johansson Theorem.

T. Sasamoto, H. Spohn, P. Ferrari: exact solutions of KPZ (recent papers)

Shorter version: For the original version: see the following Appendix

We consider only TASEP (for extensions to ASEP see eg recent papers by Tracy & Widom, T. Sasamoto ...)

Recall the results for TASEP on a ring:

<u>N = 2</u> $\qquad \dfrac{A_{21}}{A_{12}} = -\dfrac{1-z_2}{1-z_1} \qquad \psi(x_1, x_2) \propto \dfrac{z_1^{x_1} z_2^{x_2}}{1-z_2} - \dfrac{z_1^{x_2} z_1^{x_2}}{1-z_1}$

we can write $\psi(x_1, x_2) = \begin{vmatrix} \dfrac{z_1^{x_1}}{1-z_1} & \dfrac{z_1^{x_2}}{(1-z_1)^2} \\ \dfrac{z_2^{x_1}}{(1-z_2)} & \dfrac{z_2^{x_2}}{(1-z_2)^2} \end{vmatrix}$

<u>N = 3</u> Check that the wave-function is given by

$$\psi(x_1, x_2, x_3) = \begin{vmatrix} \dfrac{z_1^{x_1}}{(1-z_1)} & \dfrac{z_1^{x_2}}{(1-z_1)^2} & \dfrac{z_1^{x_3}}{(1-z_1)^3} \\ \dfrac{z_2^{x_1}}{(1-z_2)} & \dfrac{z_2^{x_2}}{(1-z_2)^2} & \dfrac{z_2^{x_3}}{(1-z_2)^3} \\ \dfrac{z_3^{x_1}}{(1-z_3)} & \dfrac{z_3^{x_2}}{(1-z_3)^2} & \dfrac{z_3^{x_3}}{(1-z_3)^3} \end{vmatrix}$$

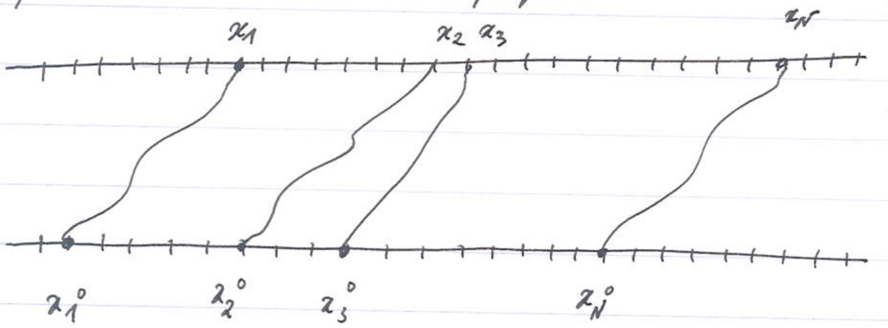
<u>General</u> N: $\psi = \det \left| \dfrac{z_i^{x_j}}{(1-z_i)^j} \right|$ one can verify directly that the 2-body collisions conditions are satisfied

The corresponding eigenvalue is $E = \sum_1^N \dfrac{1}{z_i} - N$

ON <u>A RING</u>: PERIODICITY $\Rightarrow$ Quantification of the z_i's by the Bethe equations $\dfrac{(1-z_i)^N}{z_i^L} = -\prod_j (1-z_j)$

This structure remains valid on the infinite line (only the quantification condition i.e. Bethe equations - do not exist anymore. The z_i's can take any values).

Consider a finite number N of particles on the ∞ lattice $\mathbb{Z}$:



Green-function : $\quad P_t(x_1, x_2 \dots x_N \,/\, x_1^0, \dots, x_N^0)$?

"Fourier Modes" $\quad \Psi_E = \det \left| \dfrac{z_k^{x_\ell}}{(1-z_k)^\ell} \right|_{N\times N} = \det \left| \dfrac{e^{ip_k x_\ell}}{(1-e^{ip_k})^\ell} \right|$

where $z_k = e^{ip_k}$ $\qquad E \equiv \sum_{k=1}^{N} \dfrac{1}{z_k} - N = \sum_1^N e^{-ip_k} - N$

$$\frac{d\Psi_E}{dt} = M\Psi_E = E\Psi_E \quad \text{i.e} \quad \Psi_E(x_1 \dots x_N, t) = e^{Et}\, \Psi_E(x_i \dots, 0)$$

We perform a spectral decomposition

$$P_t(x_1 \dots x_N / x_1^0 \dots x_N^0) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi} \frac{dp_1}{2\pi}\dots\frac{dp_N}{2\pi}\, e^{t(\sum_1^N e^{-ip_k} - N)} \underbrace{\mathcal{A}_{\substack{p_1 \dots p_N \\ x_1^0 \dots x_N^0}}}_{\text{AMPLITUDE = to be determined}} \sum_\sigma \varepsilon(\sigma) \frac{e^{ip_1 x_{\sigma(1)} + \dots + ip_N x_{\sigma(N)}}}{(1-e^{ip_1})^{\sigma(1)} \dots (1-e^{ip_N})^{\sigma(N)}}$$

we want $\quad P_{t=0}(x_1 \dots x_N / x_1^0 \dots x_N^0) = \delta_{x_1 x_1^0} \dots \delta_{x_N x_N^0}$

$\mathcal{A}$: was obtained by Schütz + Tracy-Widom

$$\mathcal{A} = e^{-ip_1 x_1^0 \dots - ip_N x_N^0} \, (1-e^{ip_1}) \dots (1-e^{ip_N})^N$$

in fact a very "natural" outcome of the calculation :

For details, see the Appendix.

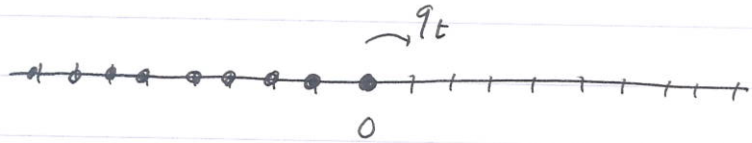
These results have been extended from TASEP → ASEP by TW, Ferrari - Spohn - Sasamoto.

One can use these techniques to study current statistics through a bond.

Be careful: it depends on the initial condition

I shall discuss only the step initial condition here.

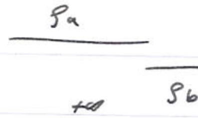
<u>SEP Case</u> : Derrida and Gershenfeld



q_t : current through the 1st bond

$$\langle e^{\lambda q_t} \rangle \simeq e^{\sqrt{t}\, F(w)} \qquad w = e^{\lambda} - 1$$

(for g^al conditions



$w = \rho_a(e^{\lambda}-1) + \rho_b(e^{-\lambda}-1) + \rho_a\rho_a(e^{\lambda}-1)(e^{-\lambda}-1)$)

$$F(w) = \frac{1}{\sqrt{\pi}} \sum_{n \geq 1} (-1)^{n-1} \frac{w^n}{n^{3/2}} \equiv \frac{1}{\pi} \int_{-\infty}^{+\infty} dk \log(1 + w e^{-k^2})$$

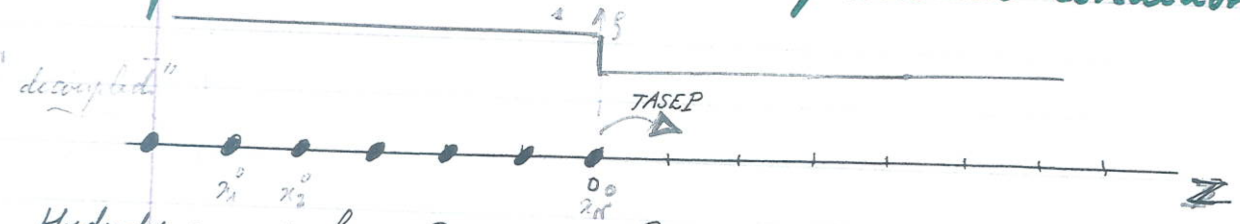
Expand w.r.t. $\lambda \simeq 0$ $\qquad \langle q_t \rangle = \sqrt{\frac{t}{\pi}} \qquad \mathrm{var}(q_t) = \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{\frac{t}{\pi}}$

$$\langle\langle q_t^3 \rangle\rangle_{\text{cumulant}} = \left[1 - 3/\sqrt{2} + 2/\sqrt{3}\right] \sqrt{t/\pi}$$

$$\mathrm{Prob}\left(\frac{q_t}{\sqrt{t}} = q\right) \sim e^{+\sqrt{t}\, G(q)} \qquad G(q) \sim -\frac{\pi^2}{12} q^3 \text{ when } q \to \infty$$

Let's now discuss the TASEP case :

Application: Density and current fluctuations for the TASEP with step initial conditions:



Hydrodynamic scale → Burgers: $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho(1-\rho) = 0 \qquad \rho_{t=0}(x) = \Theta(-x)$

$$\rho(x,t) = \frac{1}{2}\left(1 - \frac{x}{t}\right) \qquad \text{for } |x| \leq t$$

we want a finer description

$P(M,N;t)$ = probability that the Last Nth particle (PARTICLE LABELLED 1) (which was initially at site $1-N$) has jumped at least M times up to time t (i.e. its position at time t is at least $M+1-N$)

$$P(M,N,t) = \sum_{M-N < x_1 < x_2 < \dots x_N} P_t(x_1, \dots x_N \mid \mathring{x}_1 = -N+1,\ \mathring{x}_2 = -N+2, \dots \mathring{x}_N = 0)$$

Starting from the determinantal expression for $P_t(\{x\}|\{\mathring{x}\})$ and using the explicit expressions for the F_m + row/columns manipulations one can show the following result:

$$\boxed{P(M,N,t) = \frac{1}{Z_{M,N}} \int_{[0,t]^N} d^N x \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{j=1}^{N} x_j^{M-N} e^{-x_j}}$$

$(Z_{M,N} = P(M,N,t=\infty))$

This result was first obtained by Johansson, CMP (2000) using combinatorial arguments.

(This derivation: Schütz + Rakos, cond-mat/2004)

The ~~numerator~~ expression on the r.h.s. has direct meaning in random matrix theory: it is the PROBABILITY that the largest eigenvalue of the random matrix AA^* is $\leq t$

GIVEN THAT A is a $N \times M$ matrix of complex Gaussian random variables with mean zero and variance $\frac{1}{2}$. Laguerre ens.

→ relates TASEP to the field of random matrices and TRACY-WIDOM distributions

cf Patrick FERRARI's lectures.

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<u>Another interpretation</u>: we call $J_x(t)$ the total number of particles that have crossed the bond $x-(x+1)$ between time 0 and t (i.e. this is the time integrated current through the bond $x \to x+1$) Then

$$\text{Proba}\{J_x(t) > m\} = \text{Prob}(m+x+1, m+1, t)$$

probability that the $(m+1)$ th particle has jumped at least $m+x+1$ sites.

special case: $\text{Proba}\{J_0(t) \geq m\} = \text{Proba}[m, m, t]$

Using the results of Tracy & Widom ($\to$ asymptotic analysis of the preceding integral formula) one can prove that

$$0 \leq \nu < 1 \qquad J_{\underset{\substack{\downarrow\\ x}}{[\nu t]}}(t) = \frac{t}{4}(1-\nu)^2 + \frac{t^{1/3}(1-\nu^2)^{2/3}}{2^{4/3}}\chi$$

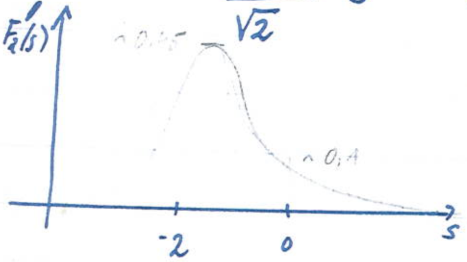
where the distribution of χ is given by:

$$\text{Proba}(\chi \leq s) = 1 - F_2(-s)$$

where $F_2(s) = F_{GUE}(s)$ is the cumulative distribution of λ_{max} in the Gaussian Unitary ensembles i.e (self-adjoint Hermitian matrices)

$$\lambda_{max} = \sqrt{2N} + \frac{N^{-1/6}}{\sqrt{2}}\xi \qquad \text{Proba}(\xi \leq s) = F_2(s)$$

and we have

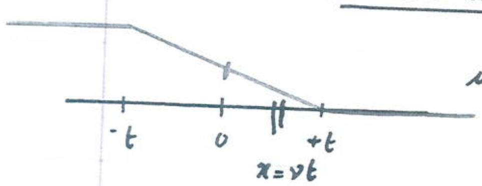


$$\langle s \rangle = \int s F_2'(s)\, ds \simeq -1{,}77109$$

$$1 - F_2(-s) \underset{s \to +\infty}{\sim} 1 - e^{-|s|^3/12}$$

$$\underset{s \to -\infty}{\sim} O\left(e^{-\frac{4}{3}|s|^{3/2}}\right)$$

<u>Remarks</u>:
- $t^{1/3} \to$ the KPZ exponent
- <u>linear contribution</u> is elementary:



initial profile: step $\to$ fan state $\rho(x,t) = \frac{1 - x/t}{2}$ $|x| \leq t$

Local current at time t: (through x) $\rho(x,t)(1-\rho(x,t))$

$$= \frac{1}{4}\left\{1 - \left(\frac{x}{t}\right)^2\right\}$$

Total integrated current through x from $t=0$ to $t=T$

$$\frac{1}{4}\int_x^T dt \left(1 - \left(\frac{x}{t}\right)^2\right)$$

the lower bound $\underline{t_{min} = x}$ $\leadsto$ TSVP thus $\frac{1}{4}\int_x^T dt\left(1-\left(\frac{x}{t}\right)^2\right) = \frac{1}{4}\left\{T - x + x^2\left(-\frac{1}{x}+\frac{1}{T}\right)\right\}$ cours 5 VI

$$= \frac{T}{4}\left(1-\left(\frac{x}{T}\right)\right)^2 = \frac{T}{4}(1-\nu)^2$$

Pour ces questions, consulter :

- D. Aldous & P. Diaconis : Longest increasing Subsequences: from patience Sorting to the Baik-Deift-Johansson Theorem

Appendix to LECTURE V:

Exact NON-STATIONARY probabilities for the TASEP on the infinite lattice.

N: a finite # of particles $\quad x_1^0 < x_2^0 < \ldots < x_N^0$ at $t=0$

$\qquad\qquad\qquad\qquad\qquad\qquad x_1 < x_2 < \quad < x_N$ at t

$$\boxed{\text{Probability } P_t(x_1, \ldots x_N \mid x_1^0, \ldots, x_N^0)\ ?}$$

Same reasoning as above = without the cyclicity condition.

An eigenvector $\Psi_E(x_1, \ldots x_N) = \det\left(\dfrac{z_i^{x_j}}{(1-z_i)^j}\right)_{1 \le i,j \le N}$ with $E = \sum_i 1/z_i - N = \sum_{i=1}^{N}\left(\frac{1}{z_i} - 1\right)$

But now : <u>NO QUANTIFICATION</u> of the pseudo-momenta (i.e. an infinite number of eigenvalues) and the eigenfunction Ψ_E such that $M.\Psi_E = E\,\Psi_E$ evolves with time as $\Psi_E(x_1, \ldots x_N; t) = e^{Et}\,\Psi_E(x_1, \ldots x_N)$

We now want to <u>decompose</u> $P_t(x_1 \ldots x_N \mid x_1^0 \; x_N^0)$ on the infinite set of the eigenvector (as in Quantum Mechanics).

We have $\Psi_E(x_1, \ldots x_N) = \sum_{\sigma \in \mathfrak{S}_N} A_\sigma\, z_{\sigma(1)}^{x_1} \ldots z_{\sigma(N)}^{x_N}$ with

$$A_\sigma = \varepsilon(\sigma)\,\frac{1}{1 - z_{\sigma(1)}}\;\frac{1}{(1 - z_{\sigma(2)})^2}\;\cdots\;\frac{1}{(1 - z_{\sigma(N)})^N}$$

An equivalent expression is :

$$\boxed{\begin{gathered}\Psi_E(x_1 \ldots x_N) = \sum_\sigma \varepsilon(\sigma)\,\frac{z_1^{x_{\sigma(1)}} \ldots z_N^{x_{\sigma(N)}}}{(1-z_1)^{\sigma(1)} \cdot (1-z_N)^{\sigma(N)}} \equiv \sum_\sigma \varepsilon(\sigma)\,\frac{e^{ip_1 x_{\sigma(1)} + \cdot\, ip_N x_{\sigma(N)}}}{(1-e^{ip_1})^{\sigma(1)} \cdots (1-e^{ip_N})^{\sigma(N)}} \\ z_k = e^{ip_k} \\ \text{valeur propre associée} \qquad E = \sum_{k=1}^{N}\left(e^{-ip_k} - 1\right)\end{gathered}}$$

Here $x_i \in \mathbb{Z}$ $\rightsquigarrow$ Ψ_E has to be a bounded function

$\rightsquigarrow$ $|z_i| = 1$ i.e $p_i \in [0, 2\pi[$

($\neq$ from the ring case)

From Q.M. (ie spectral analysis) we expect the linear superposition

$$P_t(x_1 \dots x_N / X_1^\circ \dots X_N^\circ) = \int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \dots \frac{dp_N}{2\pi}\, e^{-Nt + t\sum_{k=1}^{N} e^{-ip_k}}\, f(p_1 \dots p_N)\, \Psi_{p_1 \dots p_N}(x_1 \dots x_N)$$

with $\Psi_{p_1 \dots p_N}(x_1 \dots x_N) = \det \left| \frac{e^{ip_k x_\ell}}{(1-e^{ip_k})^\ell} \right| = \sum_\sigma \varepsilon(\sigma) \frac{e^{ip_1 x_{\sigma(1)} + \dots ip_N x_{\sigma(N)}}}{(1-e^{ip_1})^{\sigma(1)} \dots (1-e^{ip_N})^{\sigma(N)}}$

and the AMPLITUDE $f(p_1 \dots p_N)$ is chosen so that

$$P_{t=0}(x_1 \dots x_N / x_1^\circ \dots x_N^\circ) = \delta_{x_1 x_1^\circ} \dots \delta_{x_N x_N^\circ}$$

The "simplest" choice: $f(p_1, \dots, p_N) = e^{-ip_1 x_1^\circ \dots - ip_N x_N^\circ} (1-e^{ip_1})^1 \dots (1-e^{ip_N})^N$ provides the correct answer:

$$P_t(x_1, \dots x_N / X_1^\circ \dots X_N^\circ) = \int \prod_k \left(\frac{dp_k}{2\pi}\right) e^{-Nt + t\sum_k e^{-ip_k}} \sum_\sigma \varepsilon(\sigma) \frac{e^{ip_1(x_{\sigma(1)} - x_1^\circ) + \dots ip_N(x_{\sigma(N)} - x_N^\circ)}}{(1-e^{ip_1})^{\sigma(1)-1} \dots (1-e^{ip_N})^{\sigma(N)-N}}$$

$$= \det_{k,\ell} \left| \int_{-\pi}^{\pi} \frac{dp_k}{2\pi} \frac{e^{ip_k(x_\ell - x_k^\circ)}}{(1-e^{ip_k})^{\ell-k}}\, e^{-t(1-e^{-ip_k})} \right|$$

By definition we call: $F_m(x;t) = \int_{-\pi}^{\pi} \frac{dp}{2\pi} \frac{e^{ipx}}{(1-e^{ip})^m} e^{-t(1-e^{-ip})}$

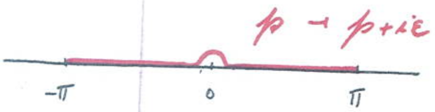
$m, x \in \mathbb{Z}$

Then

$$\boxed{P_t(x_1, \dots x_N / X_1^\circ \dots X_N^\circ) = \det_{k,\ell} \left| F_{\ell-k}(x_\ell - x_k^\circ; t) \right|}$$

(Schütz, 1997) + Sla[illegible] Priezzhev [illegible] 2007

N: the integral that defines F_m is singular at $p = 0$ -
One must define a regularization



The explicit calculation can be done:

$$F_m(x,t) = e^{-t} \sum_{\substack{r=0 \\ r \geq -x}}^{\infty} \frac{m(m+1)\dots(m+r-1)}{r!} \frac{t^{x+r}}{(x+r)!} = e^{-t}\tilde{F}_m(x,t)$$

si $m<0$ la somme stops at: $r = |m|$

$$F_0(x,t) = e^{-t}\frac{t^x}{x!} \qquad \tilde{F}_0(x,t) = \frac{t^x}{x!}$$

(i) we have $\dfrac{d}{dt} F_m(x,t) = F_{m-1}(x-1,t)$

$$= F_m(x-1,t) - F_m(x,t)$$

$$\boxed{\begin{array}{l} \dfrac{d}{dt}\tilde{F}_m(x,t) = \tilde{F}_{m-1}(x,t) \\ \tilde{F}_m(x) + \tilde{F}_{m-1}(x-1) = \tilde{F}_m(x-1) \end{array}}$$

$$\int_{t_1}^{t_2} F_m(x,t)\,dt = F_{m+1}(x+1,t_2) - F_{m+1}(x+1,t_1)$$

(ii) for $t=0$: $F_m(x,0) = 0$ si $x>0$ $\rightarrow \tilde{F}_m(x,0)=0$

$F_m(x=0,0) = 1$ si $x=0$ $= \tilde{F}_m(0,0)$

$$F_m(x,0) = \frac{m(m+1)\dots(m+|x|-1)}{|x|!} = \frac{(m+|x|-1)!}{(m-1)!\,|x|!} = \tilde{F}_m(x,0)$$

The first property allows us to show that the initial condition is well satisfied $= e^{-Nt}\det|\tilde{F}_{\ell-k}(x_\ell - x_k^0, 0)|$

at $t=0$ $\quad \det\limits_{k,\ell}\left|F_{\ell-k}(x_\ell - x_k^0, 0)\right|$ $\quad$ initial condition $x_1^0 < x_2^0 < \dots < x_N^0$

at time t $\quad x_1 < x_2 < \dots < x_N$

we can not have $x_1 > x_1^0$

otherwise $x_N > x_{N-1} > \dots > x_2 > x_1 > x_1^0$

i.e $\forall \ell$ $x_\ell - x_1^0 > 0 \Rightarrow$ the first line of the matrix vanishes

Therefore $x_1 = x_1^0$ and the matrix is of the type $\begin{vmatrix} 1 & 0 & 0 \\ & \boxed{} & \end{vmatrix}$

do a recursion.

<u>Property (i)</u> allows to prove directly that the determinant "ANSATZ" solves the time-dependent master equation (i.e one can remove the "scaffolding" that has led us to this eq.

Each function F_m contains a factor e^{-t}: rewrite it as

$$F_m(x,t) = e^{-t}\tilde{F}_m(x,t)$$

Then $\quad P_t = e^{-Nt} \det\limits_{k,\ell} |\tilde{F}_{\ell-k}(x_\ell - x_k^0; t)|$

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and $\frac{d}{dt} P_t = e^{-Nt} \frac{d}{dt} \det|\tilde{F}| - N P_t$

But
$$\frac{d}{dt} \det|\tilde{F}| = \frac{d}{dt} \begin{vmatrix} \tilde{F}_0(x_1 - x_1^\circ) & \tilde{F}_1(x_2 - x_1^\circ) & \tilde{F}_{N-1}(x_N - x_1^\circ) \\ \tilde{F}_{-1}(x_1 - x_2^\circ) & \tilde{F}_0(x_2 - x_2^\circ) & \tilde{F}_{N-2}(x_N - x_2^\circ) \\ \vdots & \vdots & \vdots \\ \tilde{F}_{1-N}(x_1 - x_N^\circ) & & \tilde{F}_0(x_N - x_N^\circ) \end{vmatrix}$$

$$= \begin{vmatrix} \dot{\tilde{F}}_0 & \tilde{F}_1 & \tilde{F}_{N-1} \\ \dot{\tilde{F}}_{-1} & \tilde{F}_0 & \\ \dot{\tilde{F}}_{1-N} & \tilde{F}_{2-N} & \tilde{F}_0 \end{vmatrix} + \begin{vmatrix} \tilde{F}_0 & \dot{\tilde{F}}_1 & \tilde{F}_{N-1} \\ & & \\ \tilde{F}_{1-N} & \dot{\tilde{F}}_{2-N} & \tilde{F}_0 \end{vmatrix} + \cdots + \begin{vmatrix} \tilde{F}_0 & \tilde{F}_1 \ldots & \dot{\tilde{F}}_{N-1} \\ \vdots & & \\ & & \dot{\tilde{F}}_0 \end{vmatrix}$$

$x_2 - 1$ $\qquad$ $x_N - 1$

now we remark that

$$\frac{d}{dt} \tilde{F}_m(x) = \frac{d}{dt}\left(e^{+t} F_m\right) = \tilde{F}_m(x) + e^t \overbrace{\frac{d}{dt} F_m}^{[F_m(x-1) - F_m(x)]} = \tilde{F}_m(x-1)$$

$$e^{-Nt} \begin{vmatrix} \tilde{F}_0(x_1 - 1 - x_1^\circ) & \cdots \\ & \\ \tilde{F}_{1-N}(x_1 - 1 - x_N^\circ) & \cdots \end{vmatrix} \rightsquigarrow P_t(x_1 - 1, \ldots x_N / x_1^\circ \ldots x_N^\circ)$$

$$\boxed{\frac{d}{dt} P_t(x_1, \ldots x_N) = \sum_i P_t(x_1 \ldots, x_i - 1, \ldots, x_N) - N P_t(x_1 \ldots x_N)}$$

this proves the generic equation.

we now have to prove the annulation (cancellation) of unwanted two-body collision terms i.e

$$P_t(\underbrace{x_1, \ldots x_{i-1}}_{\text{spectators}}, \underset{i}{x}, \underset{i+1}{x}, x_{i+2} \ldots x_N) = P_t(x_1 \ldots x_{i-1}, x, x+1, x_{i+2} \ldots x_N)$$

$$\begin{vmatrix} & \tilde{F}_{i-1}(x - x_1^\circ) & \tilde{F}_i(x - x_1^\circ) & \\ & \tilde{F}_{i-2}(x - x_2^\circ) & \tilde{F}_{i-1}(x - x_2^\circ) & \\ & & & \\ & \underbrace{\tilde{F}_{i-N}(x - x_N^\circ)}_{\text{same}} & \tilde{F}_{i+1-N}(x - x_N^\circ) & \end{vmatrix} \cdot \begin{vmatrix} \cdots & \tilde{F}_{i-1}(x - x_1^\circ) & \tilde{F}_i(x+1-x_1^\circ) \\ \cdots & \tilde{F}_{i-2}(x - x_2^\circ) & \tilde{F}_{i-1}(x+1-x_2^\circ) \\ & \vdots & \vdots \\ & \tilde{F}_{i-N}(x - x_N^\circ) & \tilde{F}_{i+1-N}(x+1-x_N^\circ) \end{vmatrix}$$

add the COLUMNS

+ use $\tilde{F}_m(x) + \tilde{F}_{m-1}(x-1) = \tilde{F}_m(x-1)$

which shows that the two determinants are equal.