Lecture 4

Vertex Models and introduction to Algebraic Bethe Ansatz.

References: R. Baxter: Exactly solvable Models in eq. Stat. Mech (Book)

R. Nepomechie : A SPIN CHAIN PRIMER (arXiv)

P.W. Kasteleyn: Exactly solvable lattice models in "FONDAMENTAL PROBLEMS in STATISTICAL MECHANICS" Vol 3 EDG Cohen Editor (1975, North Holland)

Lectures by L. FADDEEV

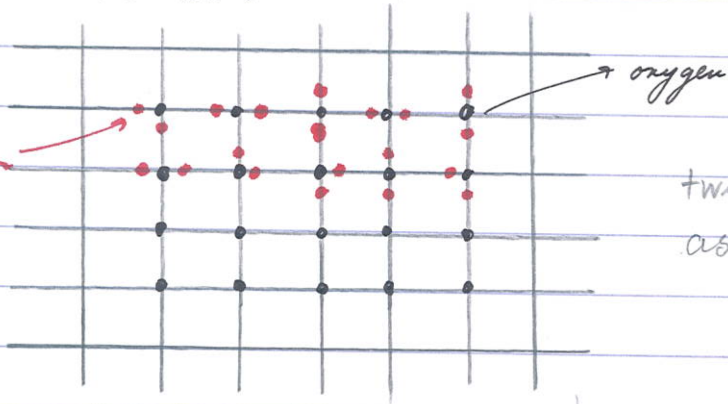
G. Gollnelli + K.M: The asymmetric exclusion process: an integrable model for non-eq. statistical mechanics (review in J. phys A 2006 or 2007)

GOLINELLI

Two-dimensional vertex models

These models of equilibrium statistical mechanics originated from a very concrete problem: "the entropy "of ice".
These systems were originally called ICE MODELS

water H_2O : Put the Oxygen atoms on a lattice; the hydrogens on the edges of the lattice. Each O must have two H's close to it. (one per bond)



how many configurations are there?

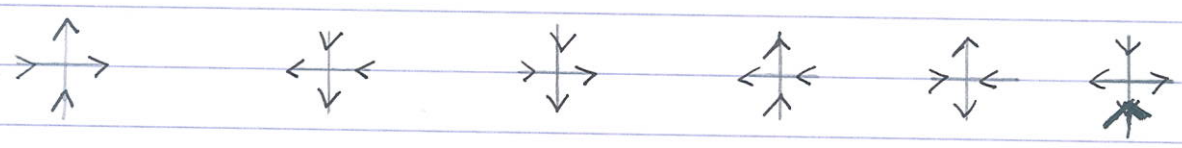
twice as many bonds as vertices

An equivalent drawing = use arrows

- 2 arrows pointing towards a vertex
- 2 arrows going away from a vertex

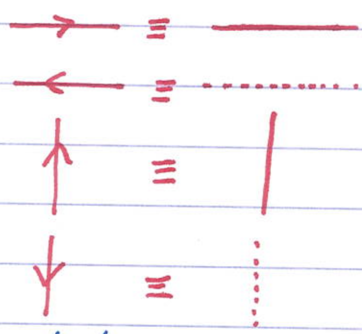
} 6 POSSIBILITIES

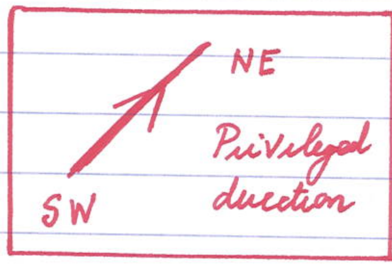
Dipole representation



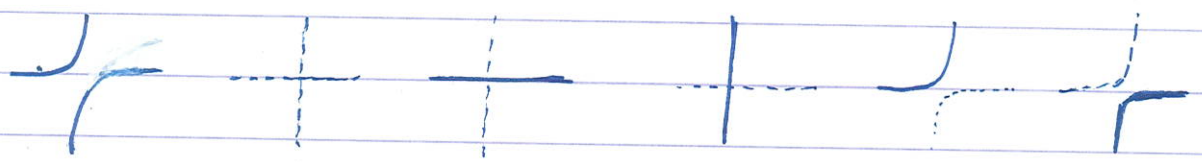
Line configuration

Convention :





Then, the dipoles can be represented as :

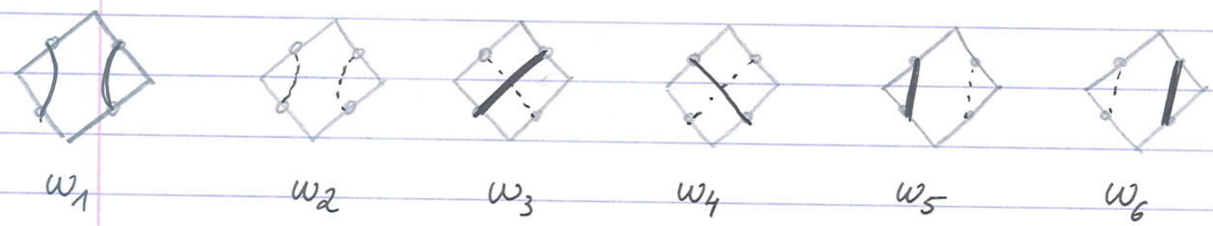


weights w_1 w_2 w_3 w_4 w_5 w_6

Relation with the exclusion process

(Domany, Kandel, Nienhuis)

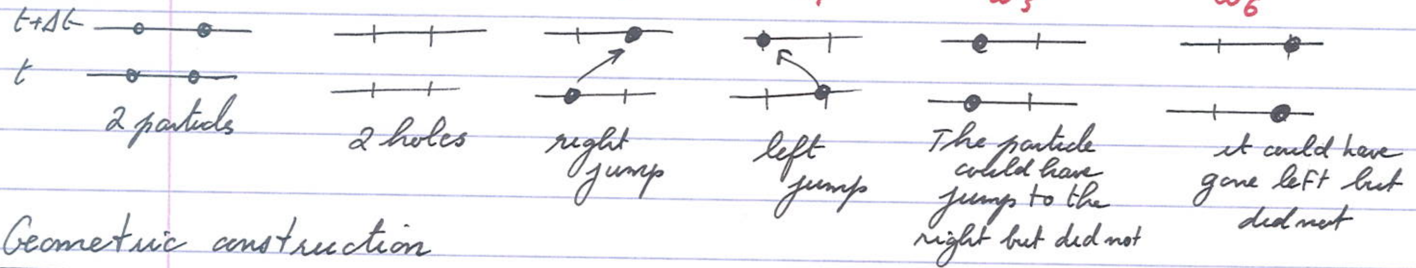
Rotate the vertices by 45°:



This can be viewed as trajectories of particles

WEIGHTS:

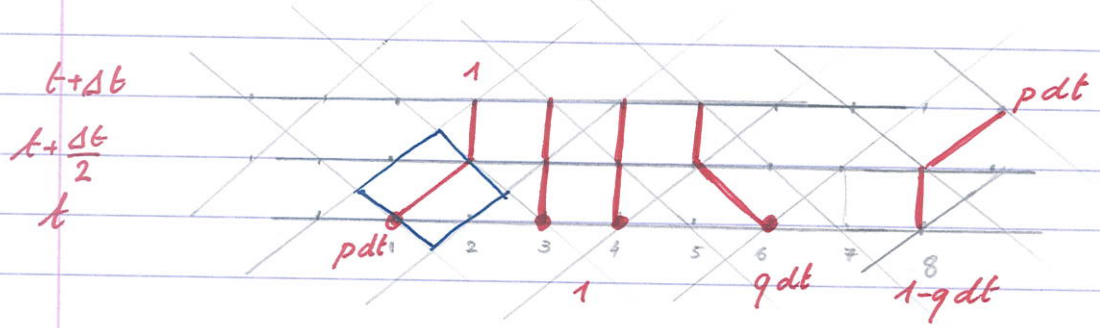
w_1	w_2	w_3	w_4	w_5	w_6
1	1	$p\,dt$	$q\,dt$	$1-p\,dt$	$1-q\,dt$



Geometric construction

6 vertex $\leftrightarrow$ ASEP

Discretize time: Consider a paralell dynamics of ASEP: even sites are updated every other step, odd sites are updated in between: t, $t+\frac{\Delta t}{2}$, $t+\Delta t$, ...



trajectories of ASEP build a vertex model.

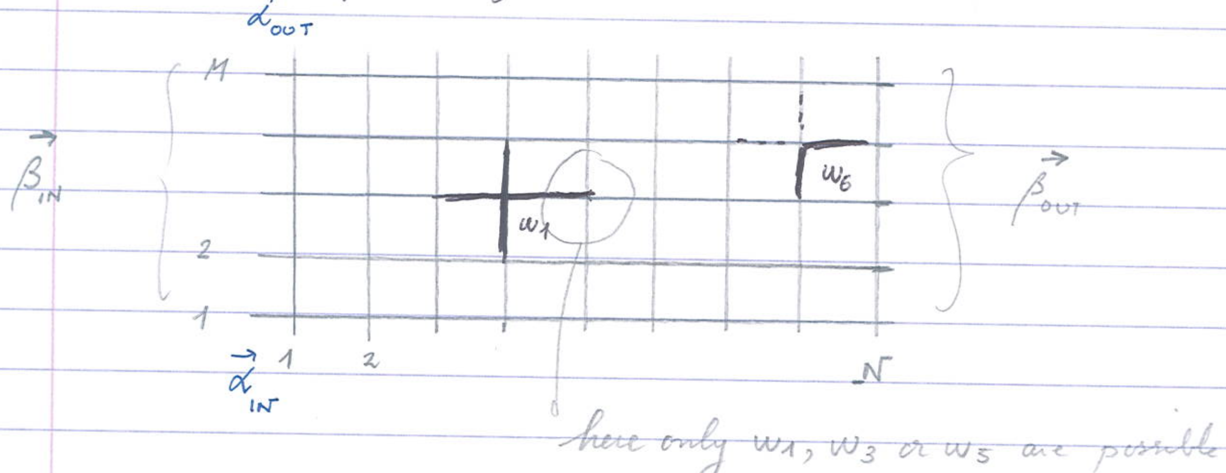
$\Delta t \to 0$: continuous-time sequential ASEP.

We now explain a more formal relation

Statistical mechanics of the 6-vertex model (equilibrium):

each vertex has a weight: $w_1, \dots, w_6$

The model is defined e.g. on a square lattice $M \times N$



A the boundaries: (i) we can fix incoming and outgoing lines and count the weight of the possible configurations

$\vec{\alpha}_{IN} = (\underbrace{|, \;\vdots\;, \;\vdots\;, |, \dots \vdots}_{n \text{ components}})$ $\qquad \vec{\beta}_{IN} = (\cdots, \cdots, -, -, \cdots)$

i.e $\vec{\alpha}_{IN}, \vec{\alpha}_{OUT}, \vec{\beta}_{IN}, \vec{\beta}_{OUT}$ are given

(ii) $\vec{\alpha}_{IN}, \vec{\beta}_{IN}$ fixed & $\vec{\alpha}_{OUT}, \vec{\beta}_{OUT}$ free

(iii) Periodic boundary conditions in β

$\vec{\beta}_{IN} = \vec{\beta}_{OUT}$ [NOT FIXED] , $\vec{\alpha}_{IN}$ & $\vec{\alpha}_{OUT}$ fixed

Mimics a time-evolution (time ↑, → space) : calculates the weight to evolve from $\vec{\alpha}_{IN}$ TO $\vec{\alpha}_{OUT}$.

(iv) Totally periodic b.c. in both directions etc...

ANYWAY: • The weight of a given configuration is the product of the vertices that make it

$$W(\mathcal{C}) = w_1^{n_1} w_2^{n_2} \dots w_6^{n_6} \qquad n_i = \#\text{ vertices of type } i$$

• The partition function Z is given by

$$Z(\{w_i\}, M, N, \vec{\alpha}_{IN}, \vec{\alpha}_{OUT}, \vec{\beta}_{IN}, \vec{\beta}_{OUT})$$

$$= \sum_{\substack{\text{admissible configurations} \\ \text{taking into account b.c.}}} w_1^{n_1} \dots w_6^{n_6}$$

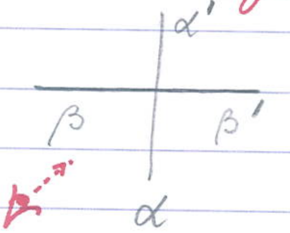
(This corresponds to case ①, the other cases can be reached by summing).

Adjacent vertices must patch → Matrix Products
→ Products of Transfer Matrices.

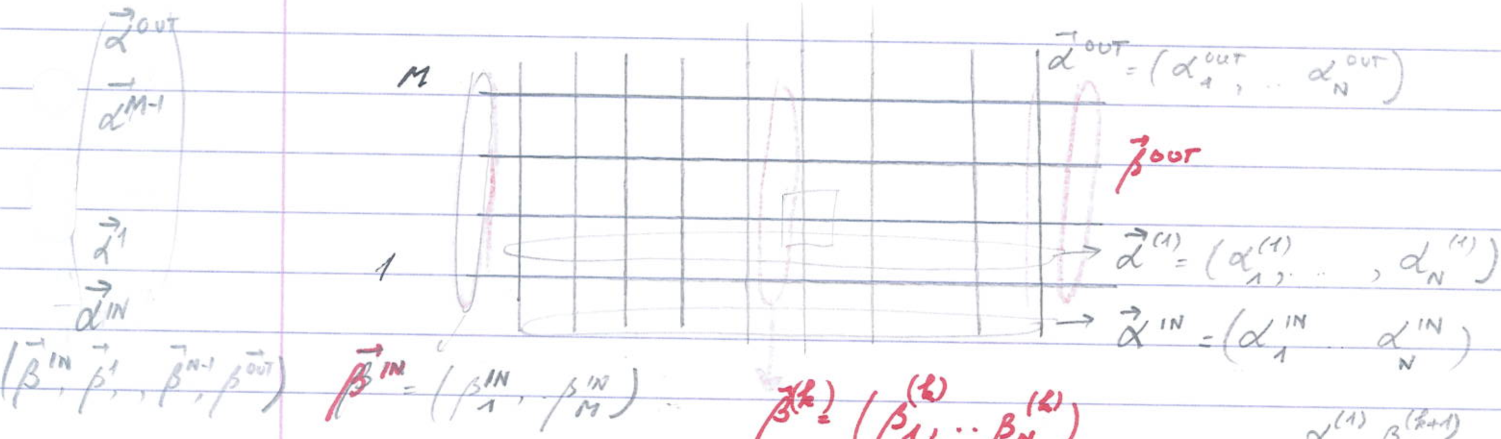
Notation for vertex weights:

look in this direction

$\alpha^{(1)}, \beta^{(1)}$ = binary variables $= 1, \ldots$



WEIGHT: $\boxed{R^{\alpha'\beta'}_{\beta\alpha}}$



$R^{\alpha^{(1)}_{k+1}\beta_1^{(k+1)}}_{\beta_1^{(k)}\alpha^{IN}_{k+1}}$ terme générique

Fully explicit weight of a configuration:

$$W(\mathcal{C}) = R^{\alpha_1^{(1)}\beta_1^{(1)}}_{\beta_1^{IN},\alpha_1^{IN}} R^{\alpha_2^{(1)}\beta_1^{(2)}}_{\beta_1^{(1)},\alpha_2^{IN}} R^{\alpha_3^{(1)}\beta_1^{(3)}}_{\beta_1^{(2)},\alpha_3^{IN}} \ldots R^{\alpha_N^{(1)}\beta_1^{OUT}}_{\beta_1^{(N-1)},\alpha_N^{IN}}$$

organised per line (= row)

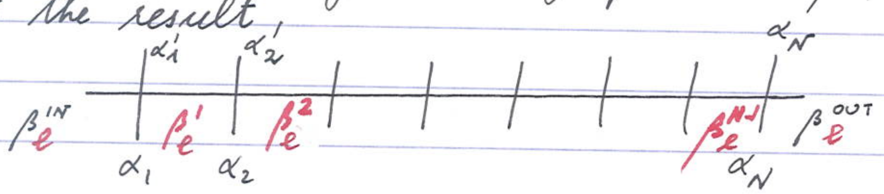
$$\times\; R^{\alpha_1^{(2)}\beta_2^{(1)}}_{\beta_2^{IN}\alpha_1^{(1)}} \ldots R^{\alpha_{k+1}^{(2)}\beta_2^{(k+1)}}_{\beta_2^{(k)}\alpha_{k+1}^{(1)}} \ldots R^{\alpha_N^{(2)}\beta_2^{OUT}}_{\beta_2^{(N-1)}\alpha_N^{(1)}}$$

$$\times\; \ldots$$

$$\times\; R^{\alpha_1^{OUT}\beta_M^{(1)}}_{\beta_M^{IN}\alpha_1^{(M-1)}} \ldots R^{\alpha_{k+1}^{(OUT)}\beta_M^{(k+1)}}_{\beta_M^{(k)}\alpha_{k+1}^{(M-1)}} \ldots R^{\alpha_N^{(OUT)}\beta_M^{(OUT)}}_{\beta_M^{(N-1)}\alpha_N^{(M-1)}}$$

Then we must SUM over all configurations compatible with b.c. i.e sum over all intermediate values of $\alpha_i^{(j)}, \beta_\ell^{(k)}$.

THEN each ~~line~~ product can be interpreted as a row to row transfer matrix.
Let's write the sum, by summing first over the β's and interpret the result,



ligne number ℓ

The transfert matrix from one row to the next one is given by

$$\sum_{\beta_e^1, \beta_e^2, \dots \beta_e^{N-1}} R^{\alpha'_1, \beta_e^1}_{\beta_e^{IN}, \alpha_1} \; R^{\alpha'_2 \beta_e^2}_{\beta_e^1 \alpha_2} \cdots R^{\alpha'_N \beta_e^{OUT}}_{\beta_e^{N-1} \alpha_N}$$

it may be line-dependent because of the index l

$$= \mathcal{T}_{\beta_e^{IN} \to \beta_e^{OUT}} (\vec{\alpha} \to \vec{\alpha}')$$

inputs $\vec{\alpha}, \beta_e^{IN}$

OUTPUTS $\vec{\alpha}', \beta_e^{OUT}$

il faut donc traiter les indices en bas β, α ensemble
en haut α', β' ensemble

$\mathcal{T}_{\beta_e^{IN} \to \beta_e^{OUT}}$ is a $2^L \times 2^L$ matrix. There are 4 such matrices:
$\beta_e^{OUT} = 1, 0$ & $\beta_e^{IN} = 1, 0$

Rappel: A matrix $A_{ij} \leftrightarrow A(e_j \to e_i) = \langle e_i | A | e_j \rangle$ (input: e_j; output: e_i)

INPUT → column index
OUTPUT → ROW INDEX (line index)

The building blocks are $R^{\alpha' \beta'}_{\beta \alpha}$ ($\alpha'\beta'$: OUTPUT: LINE; $\beta\alpha$: input: COLUMN)

Efficient way to write $R^{\alpha' \beta'}_{\beta \alpha}$:

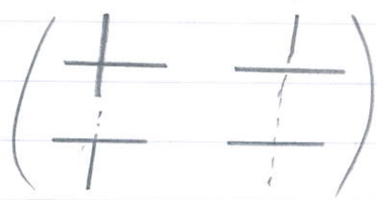
$\beta' = 1, 0$
$\beta = 1, 0$

$$R^{\alpha'}_{\alpha} = \begin{pmatrix} R^{\alpha' 1}_{1 \alpha} & R^{\alpha' 1}_{0 \alpha} \\ R^{\alpha' 0}_{1 \alpha} & R^{\alpha' 0}_{0 \alpha} \end{pmatrix}$$

4 such matrices.

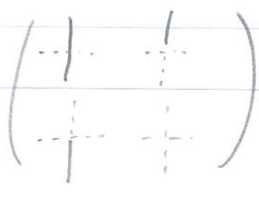
consider this as a Matrix indexed by α, α'
attach auxiliary space indices

$$R_1{}^1 = \begin{pmatrix} R^{1 1}_{1 1} & R^{1 1}_{1 0} \\ R^{0 1}_{1 1} & R^{0 1}_{1 0} \end{pmatrix} =$$



$$= \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_3 \end{pmatrix}$$

$$R_0{}^0 = \begin{pmatrix} R^{1 0}_{0 1} & R^{1 0}_{0 0} \\ R^{0 0}_{0 1} & R^{0 0}_{0 0} \end{pmatrix} =$$



$$= \begin{pmatrix} \omega_4 & 0 \\ 0 & \omega_2 \end{pmatrix}$$

$$R_0{}^1 = \begin{pmatrix} R^{1\,1}_{0\,1} & R^{1\,1}_{0\,0} \\ R^{0\,1}_{0\,1} & R^{0\,1}_{0\,0} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 \\ w_6 & 0 \end{pmatrix}$$

$$R_1{}^0 = \begin{pmatrix} R^{1\,0}_{1\,1} & R^{1\,0}_{1\,0} \\ R^{0\,0}_{1\,1} & R^{0\,0}_{1\,0} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & w_5 \\ 0 & 0 \end{pmatrix}$$

The full matrix can be reorganized as

$$R = \begin{pmatrix} \begin{pmatrix} w_1 & 0 \\ 0 & w_3 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ w_6 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & w_5 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} w_4 & 0 \\ 0 & w_2 \end{pmatrix} \end{pmatrix}$$

If we identify with the ASEP weights :

$$\boxed{R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda p & 1-\lambda q & 0 \\ 0 & 1-\lambda p & \lambda q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

where $\lambda \equiv dt$

λ is called the spectral parameter

Consider now the line to line transfer matrix and let's come back to the partition function

$$Z\{\vec{\alpha}^{IN}, \vec{\alpha}^{OUT}; \vec{\beta}^{IN}, \vec{\beta}^{OUT}\} = \sum_{\vec{\alpha}^{(1)}, \dots \vec{\alpha}^{(M-1)}} T_{\beta_M^{IN} \to \beta_M^{OUT}}(\vec{\alpha}^{(M-1)} \to \vec{\alpha}^{OUT}) \qquad T_{\beta_1^{IN} \to \beta_1^{OUT}}(\vec{\alpha}^{IN} \to \vec{\alpha}^{(1)})$$

$$= \sum_{\vec{\alpha}^{(1)}, \vec{\alpha}^{(M-1)}} \langle \vec{\alpha}^{OUT} | T_{\beta_M^{IN} \to \beta_M^{OUT}} | \vec{\alpha}^{(M-1)} \rangle \qquad \langle \vec{\alpha}^{1} | T_{\beta_1^{IN} \to \beta_1^{OUT}} | \vec{\alpha}^{IN} \rangle$$

product of transfer matrices.

Suppose we have periodic cylinder conditions: 
$\vec{\beta}^{IN} = \vec{\beta}^{OUT}$ and we sum over all possibles $\vec{\beta}^{IN}$'s.

Then, the transfer matrix from one circle to the next takes a simpler form

$$T(\vec{\alpha} \to \vec{\alpha}') = \sum_{\substack{\beta,\, \beta^1 \ldots \beta^{N-1} \\ \beta = \beta^{IN}}} R^{\alpha'_N \beta}_{\beta^{N-1} \alpha_N} R^{\alpha'_{N-1} \beta^{N-1}}_{\beta^{N-2} \alpha_{N-1}} \ldots R^{\alpha'_2 \beta_2}_{\beta_1 \alpha_2} R^{\alpha'_1 \beta^1}_{\beta \alpha_1}$$

product matriciel sur β

(and you observe that it does not depend on the line)

THEN

$$Z_{cylinder} = \sum_{\alpha^{(1)} \, \alpha^{(M-1)}} \langle \alpha^{OUT} | T | \alpha^{(M-1)} \rangle \ldots \langle \alpha^1 | T | \alpha^{IN} \rangle$$

$$= \langle \alpha^{OUT} | T^M | \alpha^{IN} \rangle$$

For fully periodic conditions $\alpha^{IN} = \alpha^{OUT}$ and sum over $\vec{\alpha}$

$$\boxed{Z_{TORUS} = Tr(T^M)}$$

We must diagonalize T. T is a $2^N \times 2^N$ matrix, so are the four matrices $\tau_{\beta \to \beta'}$, but it has <u>a structure</u>: it is built from the elementary blocks R.

Let us investigate how $\tau_{\beta^{IN} \to \beta^{OUT}}$ is built. Take the case of $N = 2$

$$\tau_{\beta^{IN} \to \beta^{OUT}} \left((\alpha_1, \alpha_2) \to (\alpha'_1, \alpha'_2) \right) = \sum_\beta \underbrace{R^{\alpha'_2, \beta^{OUT}}_{\beta \, \alpha_2}}_{\text{colonne}} \overbrace{R^{\alpha'_1 \beta}_{\beta^{IN} \alpha_1}}^{\text{ligne}}$$

4 different 4×4 matrices.

Matrix product over β

"Concatenation over" α, α': tensor product

To understand the details let's fix $\beta^{IN} = 0$, $\beta^{OUT} = 0$

$$\mathcal{T}_{00}(\vec{\alpha} \mapsto \vec{\alpha}') = \sum_\beta R^{\alpha_2' \, 0}_{\beta \, \alpha_2} R^{\alpha_1' \, \beta}_{0 \, \alpha_1} = R^{\alpha_2' \, 0}_{0 \, \alpha_2} R^{\alpha_1' \, 0}_{0 \, \alpha_1} + R^{\alpha_2' \, 0}_{1 \, \alpha_2} R^{\alpha_1' \, 1}_{0 \, \alpha_1}$$

$$\mathcal{T}_{00} = \mathcal{T}^0_0$$

Take two matrices $A = \left(A^{\alpha_1'}_{\alpha_1}\right)$ $2\times2 \to$ 4 entries $\begin{pmatrix} A^1_1 & A^1_0 \\ A^0_1 & A^0_0 \end{pmatrix}$

$B = \left(B^{\alpha_2'}_{\alpha_2}\right)$ 2×2

$C^{\alpha_1'\alpha_2'}_{\alpha_1\alpha_2} = A^{\alpha_1'}_{\alpha_1} B^{\alpha_2'}_{\alpha_2}$ has 16 entries, it is a 4×4 matrix

$$C = \begin{pmatrix} A^1_1 B^1_1 & A^1_1 B^1_0 & \overbrace{\cdot \quad\quad \cdot}^{A^1_0(B)} \\ A^1_1 B^0_1 & A^1_1 B^0_0 & \cdot \quad\quad \cdot \\ \cdot & \cdot & \cdot \quad\quad \cdot \\ \underbrace{\cdot \quad\quad\quad \cdot}_{A^0_1(B)} & & \cdot \quad A^0_0 B^0_0 \end{pmatrix} \equiv A \otimes B$$

It is the tensor product of A by B

Now: $\mathcal{T}_{00} = R^0_0 \otimes R^0_0 + R^0_1 \otimes R^1_0$

Similarly

$$\mathcal{T}_{0\to1} \equiv \mathcal{T}^1_0 = R^1_0 \otimes R^0_0 + R^1_1 \otimes R^1_0$$

($\alpha_2'\alpha_2$, $\alpha_1'\alpha_1$, (2) space, (1) space)

$$\mathcal{T}_{1\to0} = \mathcal{T}^0_1 = \overset{(2)}{R}{}^0_0 \otimes \overset{(1)}{R}{}^0_1 + \overset{}{R}{}^0_1 \otimes \overset{(1)}{R}{}^1_1$$

$$\mathcal{T}^1_1 = \overset{(2)}{R}{}^1_0 \otimes \overset{(1)}{R}{}^0_1 + \overset{(2)}{R}{}^1_1 \otimes \overset{(1)}{R}{}^1_1$$

4×4 $\vec{\alpha}\to\vec{\alpha}'$ — $\alpha_2\alpha_2'$ space (2) — space (1) $\alpha_1\alpha_1'$

$$\mathcal{T} \equiv \begin{pmatrix} \mathcal{T}^1_1 & \mathcal{T}^1_0 \\ \mathcal{T}^0_1 & \mathcal{T}^0_0 \end{pmatrix} = \begin{pmatrix} \overset{(2)}{R}{}^1_1 & \overset{(2)}{R}{}^1_0 \\ \overset{(2)}{R}{}^0_1 & \overset{(2)}{R}{}^0_0 \end{pmatrix} \dot{\otimes} \begin{pmatrix} \overset{(1)}{R}{}^1_1 & \overset{(1)}{R}{}^1_0 \\ \overset{(1)}{R}{}^0_1 & \overset{(1)}{R}{}^0_0 \end{pmatrix}$$

we put the 4 matrices in one big matrix

$$= \begin{pmatrix} \overset{(2)}{R}{}^1_1 \otimes \overset{(1)}{R}{}^1_1 + \overset{(2)}{R}{}^1_0 \otimes \overset{(1)}{R}{}^0_1 & \cdot \\ \overset{(2)}{R}{}^0_1 \otimes \overset{(1)}{R}{}^1_1 + \overset{(2)}{R}{}^0_0 \otimes \overset{(1)}{R}{}^0_1 & \cdot \end{pmatrix}$$

Formally, we are multiplying the matrices as usual, but the matrix elements are "concatenated" i.e. combined by tensor-product.

If we had $N=3$ $\qquad \mathcal{T} = \begin{pmatrix} \mathcal{T}_1^1 & \mathcal{T}_0^1 \\ \mathcal{T}_1^0 & \mathcal{T}_0^0 \end{pmatrix}$

8×8 matrices $(\alpha_1,\alpha_2,\alpha_3)\to(\alpha_1',\alpha_2',\alpha_3)$

$$\mathcal{T} = R^{(3)} \dot{\otimes} R^{(2)} \dot{\otimes} R^{(1)}$$

For arbitrary N $\qquad \mathcal{T} = \begin{pmatrix} \mathcal{T}_1^1 & \mathcal{T}_0^1 \\ \mathcal{T}_0^1 & \mathcal{T}_0^0 \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$2^N \times 2^N$ matrices

$$\boxed{\mathcal{T} = \dot{\bigotimes_{k=1,N}} R^{(k)} = R^{(N)} \ldots\ldots R^{(1)}}$$

For periodic boundary conditions

$$\boxed{T = \mathcal{T}_1^1 + \mathcal{T}_0^0 = A + D}$$

$2^N \times 2^N$ operator

acts on a "chain" of N sites

Remember we had a "spectral parameter" λ:

$$\boxed{\text{We have constructed a family of } 2^N \times 2^N \text{ operators: } T(\lambda)}$$

CRUCIAL PROPERTY:

$$\boxed{[T(\lambda), T(\lambda')] = 0}$$

Family of commuting operators

They can be co-diagonalized

Cours 4 (X)

For $\lambda = 0$, one can check that:

$T(0) =$ shift operator (Look at R) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$(\alpha_1, \alpha_2, \ldots \alpha_N) \longmapsto (\alpha_2, \alpha_3, \ldots \alpha_N, \alpha_1)$

(commutes with every one)

Besides: $T(0)^{-1} \left.\dfrac{dT(\lambda)}{d\lambda}\right|_{\lambda=0} = M_{ASEP}$ Markov Matrix of ASEP

$$\boxed{M_{ASEP} = \left.\frac{d}{d\lambda} \operatorname{Log} T(\lambda)\right|_{\lambda=0}}$$

Solving the 6-vertex model is akin to solving the Exclusion process.

The commutation of the $T(\lambda)$'s can be proved by using the Yang-Baxter eq.

The eigenstates can be built algebraically (cf Golinelli & K.M: review in J Phys A.)