Lecture 3

Large deviations : basic concepts.

Fluctuations of the current in the exclusion process with periodic boundaries.

[Functional Bethe Ansatz]

Ref

Hugo Touchette : Phys. Reports

B. Derrida & J. L. Lebowitz PRL 80 (1997)

S. PROLHAC & K. M. J Phys A 41 (2008)

Rare Events and Large Deviations

Let $\epsilon_1, \ldots, \epsilon_N$ be N independent binary variables, $\epsilon_k = \pm 1$, with probability p (resp. $q = 1 - p$). Their sum is denoted by $S_N = \sum_1^N \epsilon_k$.

$\langle \varepsilon_i \rangle = p - q$

$\langle \varepsilon_i^2 \rangle - \langle \varepsilon_i \rangle^2 = 1 - (p-q)^2$ $= (p+q)^2 - (p-q)^2$ $= 4pq$

- The Law of Large Numbers implies that $S_N/N \to p - q$ a.s.
- The Central Limit Theorem implies that $[S_N - N(p-q)]/\sqrt{N \times 4pq}$ converges towards a Gaussian Law.

One can show that for $-1 < r < 1$, in the large N limit,

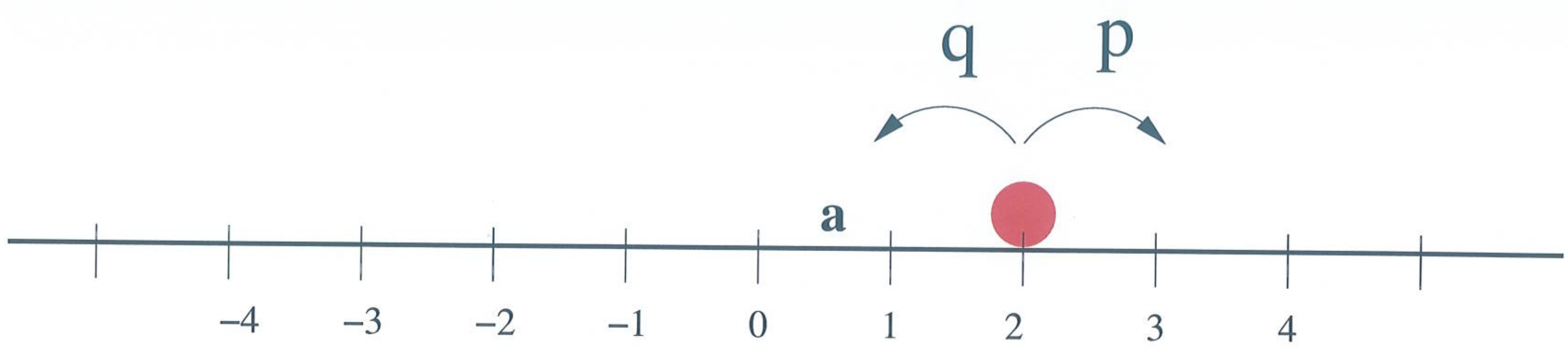
$$\mathrm{Proba}\left(\frac{S_N}{N} = r\right) \sim \mathrm{e}^{-N\,\Phi(r)}$$

where the positive function $\Phi(r)$ vanishes for $r = (p-q)$.

The function $\Phi(r)$ is a Large Deviation Function: it encodes the probability of rare events (*use Stirling's Formula*)

$$\Phi(r) = \frac{1+r}{2}\ln\left(\frac{1+r}{2p}\right) + \frac{1-r}{2}\ln\left(\frac{1-r}{2q}\right)$$

Large Deviations of an asymmetric walk



$$\frac{d\,P(n)}{dt} = q\,P(n+1) + p\,P(n-1) - (p+q)\,P(n)$$

The asymmetric walker has an average speed given by $p-q$. If X_t is its position at time t we have

$$\frac{X_t}{t} \to p - q \quad (a.s.)$$

We can again define a large deviation function as

$$\text{Proba}\left(\frac{X_t}{t} = v\right) \sim e^{-t\,G(v)}$$

This function can be calculated explicitly. It is given by:

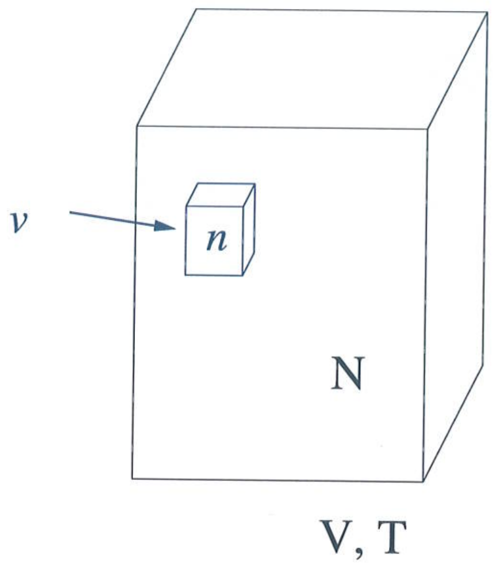
$$G(v) = q + \frac{v}{2} \log \frac{q}{p} - \sqrt{v^2 + 4pq} - |v| \log \frac{2\sqrt{pq}}{|v| + \sqrt{v^2 + 4pq}}$$

Note that

- $G(v)$ is a positive function that vanishes at $v = p - q$.
- $G(v)$ is convex.
- $G(v) - G(-v) = v \log \frac{q}{p}$.
- In the large time limit, the previous identity means that:

$$\frac{\mathrm{Proba}\left(\frac{X_t}{t} = v\right)}{\mathrm{Proba}\left(\frac{X_t}{t} = -v\right)} = \mathrm{e}^{t\, v \log \frac{p}{q}}$$

Local density fluctuations in a gas at thermal equilibrium



Mean Density $\rho_0 = \frac{N}{V}$

In a volume v s. t. $1 \ll v \ll V$
$\langle \frac{n}{v} \rangle = \rho_0$

The local density varies around ρ_0 . Typical fluctuations scale as $\sqrt{v/V}$.

The probability of observing large fluctuations is given by

$$\mathrm{Proba}\left(\frac{n}{v} = \rho\right) \sim \mathrm{e}^{-v\,\Phi(\rho)} \text{ with } \Phi(\rho_0) = 0$$

Large Deviations of the Density fluctuations

How can we calculate the Large Deviation Function $\Phi(\rho)$ using elementary statistical mechanics?

We must count the fraction of the configurations of the gas that have $n = \rho v$ particles in the small volume v and $N - n$ particles in the rest of the volume $V - v$.

Suppose that the interactions of the gas molecules are local. Then, neglecting surface effects, this number is given by

$$\mathrm{Proba}\left(\frac{n}{v} = \rho\right) \simeq \frac{Z(v,n,T)Z(V-v,N-n,T)}{Z(V,N,T)}$$

Use that, by definition,

$$Z(v,n,T) = \mathrm{e}^{-v\beta f(\rho,T)}$$

where $\beta = 1/k_B T$ is the inverse temperature and $f(\rho,T)$ is the free energy per unit volume and perform an expansion for $1 \ll v \ll V$.

The Large Deviation Function for density fluctuations is given by

$$\Phi(\rho) = \beta\left(f(\rho, T) - f(\rho_0, T) - (\rho - \rho_0)\frac{\partial f}{\partial \rho_0}\right)$$

We can ask the more general question of the large deviation of a density profile: cover the large box with $K = V/v$ small boxes and calculate the probability of having a density ρ_1 in the first box, ρ_2 in the second box ...

$$\mathrm{Proba}\left(\rho_1, \rho_2, \dots \rho_K\right) \simeq \mathrm{e}^{-V\,\mathcal{F}(\rho_1,\rho_2,\dots\rho_K)}$$

A reasoning similar to the one above allows us to show that

neglecting surface terms

$$\mathrm{Proba}\left(\rho_1, \rho_2, \dots \rho_K\right) \sim \frac{\prod_k Z(n_k, v, T)}{Z(V, N, T)}$$

local interactions

Taking the infinite volume limit we obtain / utilisa juste l'extensivité

$$\mathcal{F}(\rho_1, \rho_2, \dots \rho_K) = \frac{\beta}{K}\sum_{k=1}^{K}\left(f(\rho_i, T) - f(\rho_0, T)\right)$$

The Free Energy as a L. D. F.

If we let the number K of boxes go to infinity, then the question we are asking is the probability of observing a given density profile $\rho(x)$ in the big volume V. The large deviation function $\mathcal{F}$ becomes a functional of the density profile: Proba $\{\rho(x)\} \sim e^{-V\mathcal{F}[\rho(x)]}$

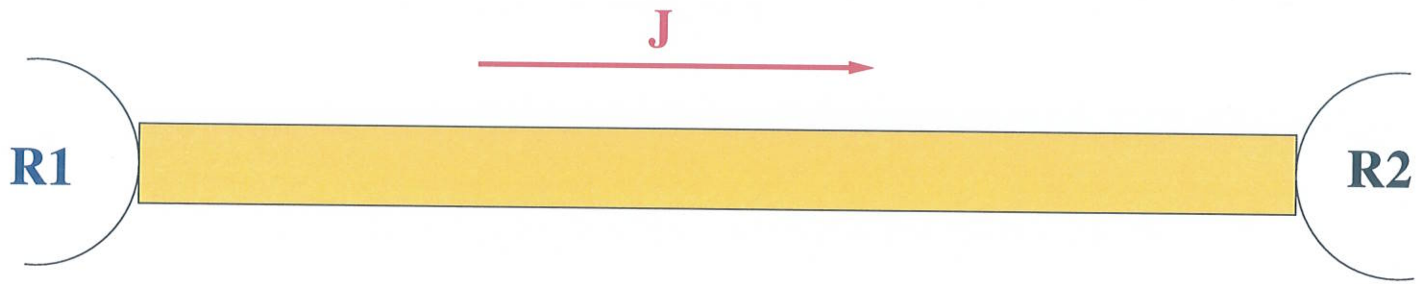
$$\mathcal{F}[\rho(x)] = \beta \int dx \, (f(\rho(x), T) - f(\rho_0, T))$$

$f = -\log Z(\rho, T)$ being, as above, the *free energy per unit volume.*

The Free Energy of Thermodynamics can be viewed as a Large Deviation Function

Conversely, large deviation functions *may* play the role of potentials in non-equilibrium statistical mechanics. Indeed, they can be defined for very general processes, even far from equilibrium.

Large Deviations of the Total Current



Let Y_t be the total charge transported through the system (total current) between time 0 and time t.

In the stationary state: a non-vanishing mean-current $\frac{Y_t}{t} \to J$

The fluctuations of Y_t obey a Large Deviation Principle:

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

$\Phi(j)$ being the *large deviation function* of the total current.

Note that $\Phi(j)$ is positive, vanishes at $j = J$ and is convex (in general).

Cumulant generating function

Equivalently, we can consider the moment-generating function defined as the average value $\left\langle e^{\mu Y_t} \right\rangle$. Expanding with respect of μ, we get

$$\log \left\langle e^{\mu Y_t} \right\rangle = \sum_k \frac{\mu^k}{k!} \langle\langle Y^k \rangle\rangle_c$$

where $\langle\langle Y^k \rangle\rangle_c$ is the k-th cumulant of Y_t.

In many problems, one can show that in the long time limit, we have

$$\left\langle e^{\mu Y_t} \right\rangle \simeq e^{E(\mu)t} \qquad \text{when} \quad t \to \infty$$

The function $E(\mu)$ is the cumulant generating function.

Hence, all cumulants of Y_t grow linearly with time and their values are given by the successive derivatives of $E(\mu)$.

Legendre Transform

The cumulant generating function $E(\mu)$ and the large deviation function $\Phi(j)$ are related by Legendre transform:

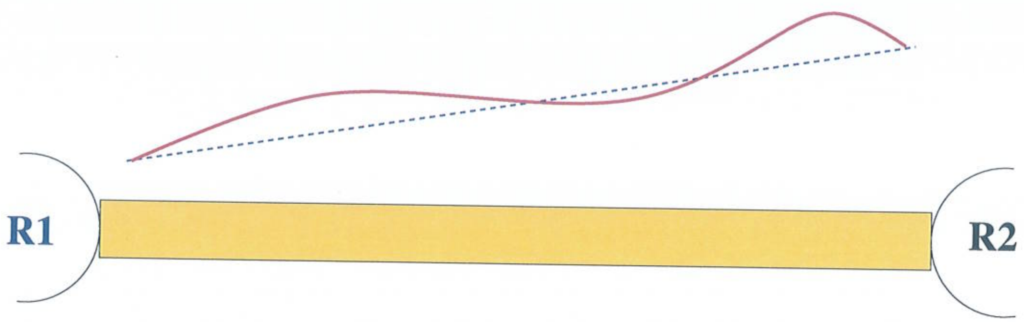
$$\boxed{E(\mu) = \max_j \left(\mu j - \Phi(j)\right)}$$

Indeed, using saddle-point,

$$\langle e^{\gamma Y_t} \rangle = \int \Pr(Y_t) e^{\gamma Y_t} dY_t = t \int \Pr\left(\frac{Y_t}{t} = j\right) e^{\gamma t j} dj \underset{t \to \infty}{\sim} \int e^{\gamma t j - t\Phi(j)}$$

A natural question is to calculate the large deviation function or the cumulant generating function of the current.

Large Deviations of a Non-Equilibrium profile



What is the probability of observing an atypical density profile in the steady state? What does the functional $\mathcal{F}(\{\rho(x)\})$ look like for such a non-equilibrium system? Recall that in the equilibrium case, this functional is the free energy.

More generally, the probability to observe an atypical current $j(x,t)$ and the corresponding density profile $\rho(x,t)$ during $0 \leq s \leq L^2\,T$ (L being the size of the system) is given by

$$\Pr\{j(x,t), \rho(x,t)\} \sim \mathrm{e}^{-L\mathcal{I}(j,\rho)}$$

Symetric
weakly asymmetric
regime

Can one calculate this large deviation functional for systems out of equilibrium?

Large Deviations of the Density Profile in ASEP

The probability of observing an **atypical density profile in the steady state** was calculated starting from the exact microscopic solution of the exclusion process (B. Derrida, J. Lebowitz E. Speer, 2002): / Matrix Ansatz
The Large Deviation Functional for the symmetric case $q = 0$ is given by

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left(B(\rho(x), F(x)) + \log \frac{F'(x)}{\rho_2 - \rho_1} \right)$$

where $B(u, v) = (1 - u) \log \frac{1-u}{1-v} + u \log \frac{u}{v}$ and $F(x)$ satisfies

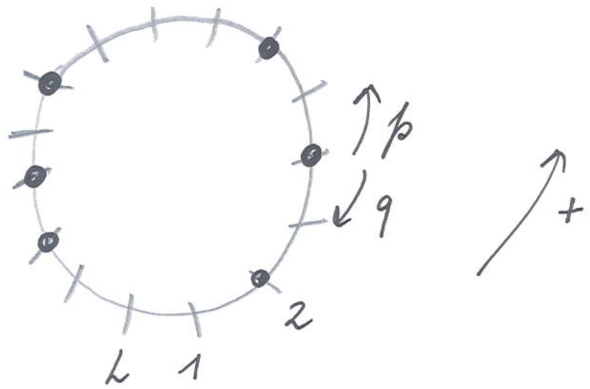
$$F\left(F'^2 + (1 - F)F''\right) = F'^2 \rho \quad \text{with} \quad F(0) = \rho_1 \text{ and } F(1) = \rho_2 \,.$$

This functional is non-local as soon as $\rho_1 \neq \rho_2$. ← densities of the two reservoirs

Note that in the case of equilibrium, for $\rho_1 = \rho_2 = \bar{\rho}$, we recover

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left\{ (1 - \rho(x)) \log \frac{1 - \rho(x)}{1 - \bar{\rho}} + \rho(x) \log \frac{\rho(x)}{\bar{\rho}} \right\}$$

We shall now apply these concepts to current fluctuations in the exclusion process on a ring



Y_t : total number of jumps between 0 and t [with signs]

Rescale time $\quad p \to 1$

$$q \to \frac{q}{p} \equiv x$$

Large Deviations of the Current

Total current Y_t, *total distance covered by all the N particles, hopping on a ring of size L, between time 0 and time t.*

WHAT IS THE STATISTICS of Y_t?

Let $P_t(\mathcal{C}, Y)$ be the joint probability of being at time t in configuration $\mathcal{C}$ with $Y_t = Y$. The time evolution of this joint probability can be deduced from the original Markov equation, by splitting the Markov operator

$$M = M_0 + M_+ + M_-$$

into transitions for which $\Delta Y = 0$, $+1$ or -1.

$$\frac{dP_t(\mathcal{C}, Y)}{dt} = \sum_{\mathcal{C}'} M_0(\mathcal{C}, \mathcal{C}')P_t(\mathcal{C}', Y) + \sum_{\mathcal{C}'} M_+(\mathcal{C}, \mathcal{C}')P_t(\mathcal{C}', Y-1) + \sum_{\mathcal{C}'} M_-(\mathcal{C}, \mathcal{C}')P_t(\mathcal{C}', Y+1)$$

The Laplace transform of $P_t(\mathcal{C}, Y)$ with respect to Y, defined as

$$\hat{P}_t(\mathcal{C}, \mu) = \sum_Y \mathrm{e}^{\mu Y} P_t(\mathcal{C}, Y),$$

satisfies a dynamical equation governed by the deformation of the Markov Matrix M, obtained by adding a jump-counting *fugacity* μ:

$$\frac{d\hat{P}_t}{dt} = M(\mu)\hat{P}_t$$

with

$$M(\mu) = M_0 + \mathrm{e}^{\mu} M_+ + \mathrm{e}^{-\mu} M_-$$

The Matrix $M(\mu)$ is not a Markov Matrix in general (it does not conserve probability). But it is a matrix with positive off-diagonal entries and the Perron-Frobenius Theorem can still be applied: $M(\mu)$ has a unique dominant eigenvalue, denoted by $E(\mu)$, with eigenvector $F_\mu(\mathcal{C})$

$$M(\mu).F_\mu = E(\mu)F_\mu$$

When $t \to \infty$, we have

$$\hat{P}_t(\mathcal{C}, \mu) \sim \mathrm{e}^{E(\mu)t} F_\mu(\mathcal{C})$$

Cumulant generating function

From the previous result, one deduces that when $t \to \infty$:

$$\boxed{\langle e^{\mu Y_t} \rangle \simeq e^{E(\mu) t}}$$

$\langle e^{\mu Y_t} \rangle = \sum_{\mathcal{C}} \hat{P}_t(\mathcal{C}, \mu)$

The cumulant generating function $E(\mu)$ is the eigenvalue with maximal real part of the deformed operator $M(\mu)$

$$M(\mu) = M_0 + e^{\mu} M_+ + e^{-\mu} M_-$$

corresponding to splitting the Markov operator $M = M_0 + M_+ + M_-$ according to the increments of the total current.

The large deviation function $\Phi(j)$ of the current is defined as

$$P\left(\frac{Y_t}{t} = j\right) \sim e^{-t\Phi(j)}$$

Bethe Ansatz for current statistics

The current statistics is reduced to an eigenvalue problem, solvable by Bethe Ansatz.

The Bethe Equations are given by

Here $\boxed{x = \frac{q}{p}}$

asymmetry parameter

$$z_i^L = (-1)^{N-1} \prod_{j=1}^{N} \frac{x e^{-\mu} z_i z_j - (1+x) z_i + e^{\mu}}{x e^{-\mu} z_i z_j - (1+x) z_j + e^{\mu}}$$

The eigenvalues of $M(\mu)$ are

$$E(\mu; z_1, z_2 \ldots z_N) = e^{\mu} \sum_{i=1}^{N} \frac{1}{z_i} + x e^{-\mu} \sum_{i=1}^{N} z_i - N(1+x) \,.$$

Discuter de Gallavotti - Cohen ici ?

Note that here we have taken $p = 1$ and $q/p = x$.
The Bethe equations do not decouple unless $x = 0$
(This TASEP case was solved by B. Derrida and J. L. Lebowitz, 1998).

Functional Bethe Ansatz for the General Case

After a change of variable, $y_i = \frac{1-e^{-\mu} z_i}{1-xe^{-\mu} z_i}$, the Bethe equations read

$$e^{L\mu}\left(\frac{1-y_i}{1-xy_i}\right)^L = -\prod_{j=1}^{N} \frac{y_i - xy_j}{xy_i - y_j} \quad \text{for} \quad i = 1 \dots N\,.$$

Let T be auxiliary variable playing a symmetric role w.r.t. all the y_i:

$$e^{L\mu}\left(\frac{1-T}{1-xT}\right)^L = -\prod_{j=1}^{N} \frac{T - xy_j}{xT - y_j} \quad \text{for} \quad i = 1 \dots N\,.$$

$$i.e.\ P(T) = e^{L\mu}(1-T)^L \prod_{j=1}^{N}(xT - y_j) + (1-xT)^L \prod_{j=1}^{N}(T - xy_j) = 0.$$

But $P(y_i) = 0$ (Bethe Eqs.). Thus, $Q(T) = \prod_{i=1}^{N}(T - y_i)$ divides $P(T)$:

$Q(T)$ DIVIDES $e^{L\mu}(1-T)^L Q(xT) + (1-xT)^L x^N Q(T/x)$.

Functional Bethe Ansatz

There exist two polynomials $Q(T)$ and $R(T)$ such that

$$\boxed{Q(T)R(T) = e^{L\mu}(1-T)^L Q(xT) + x^N (1-xT)^L Q(T/x)}$$

where $Q(T)$ of degree N vanishes at the Bethe roots.
Functional Bethe Ansatz (Baxter's TQ equation): Restatement of the Bethe Ansatz as a purely algebraic problem. This equation is solved perturbatively w.r.t. μ.

Knowing $Q(T)$, we obtain an expansion of $E(\mu)$. This provides the full statistics of the current and its large deviations.

From the functional Bethe Ansatz to Large Deviations :

$$\boxed{Q(T)\,R(T) = e^{L\mu}(1-T)^L Q(xT) + x^N Q\left(\frac{T}{x}\right)(1-xT)^L}$$

Baxter's functional B.A eq

$Q(T)$ is of degree N and it vanishes at the Bethe roots
$R(T)$ is of degree L.
The eigenvalue is given by

$p=1$
$q=x$

$$E(\mu) = e^{\mu}\sum_{i=1}^{N}\frac{1}{z_i} + x e^{-\mu}\sum_{i=1}^{N} z_i - N(1+x)$$

$$\boxed{E(\mu) = (1-x)\sum_{i=1}^{N}\left(\frac{1}{1-y_i} - \frac{1}{1-xy_i}\right) = (1-x)\left\{\frac{Q'(1)}{Q(1)} - \frac{1}{x}\frac{Q'(\frac{1}{x})}{Q(\frac{1}{x})}\right\}}$$

Using the FBA eq at $T=1$ and taking its derivative ($T=1$)
leads to $E(\mu) = -(1-x)\dfrac{R'(1)}{R(1)} - Lx$

$Q(1)R(1) = x^N Q\left(\frac{1}{x}\right)(1-x)^L$
$Q'(1)R(1) + Q(1)R'(1)$
$= x^{N-1}Q'\left(\frac{1}{x}\right)(1-x)^L$
$+ Lx^{N}Q\left(\frac{1}{x}\right)(1-x)^{L-1}$
Take the ratio.

Selection of the Perron-Frobenius eigenvalue
Perturbative solution of the F.B.A. eq. :

Conservation of momentum :

T commutes with M (translations) or directly from the Bethe equations : $(z_1 \cdots z_N)^L = 1 \;\rightarrow\; z_1 \cdots z_N = e^{\frac{2k\pi}{L}}$

discrete

When $\mu \to 0$, all the $z_i(\mu)$ converge towards the degenerate solution $z_i = 1$ of the non-deformed TASEP
Therefore, for μ small enough, we still have

$$z_1 \cdots z_N = e^{N\mu}\prod_{i=1}^{N}\frac{1-y_i}{1-xy_i} = 1$$

$k=0$
translation invariance.

for the Perron-Frobenius eigenv.

This equation can be recasted as

$$e^{N\mu}\,\frac{Q(1)}{x^N Q(\frac{1}{x})} = 1 \quad \text{or, equivalently,} \quad R(1) = e^{N\mu}(1-x)^L$$

~~à faire plus tard: la perturbative expansion~~

This condition, together with the fact that $y_i \xrightarrow[]{3i \to 1} 0$ as $\mu \to 0$ will allow us to develop a perturbative scheme to calculate $Q(T)$ and $R(T)$

Indeed we can write

$$\boxed{Q(T) = \prod_{j=1}^{N}(T - y_i) = T^N + \mu Q_1(T) + \mu^2 Q_2(T) + \ldots}$$

where $Q_i(T)$ is of degree $\leq N-1$

Similarly

$$\boxed{R(T) = R_0(T) + \mu R_1(T) + \mu^2 R_2(T) + \ldots}$$

with $R_0(T) = x^N(1-T)^L + (1-xT)^L$

By substituting these expansions into the FBA equation we obtain a hierarchical set of polynomial equations that can be solved recursively.

Before doing that, we shall fully solve the TASEP case $x=0$.

- SOLUTION of the FBA for TASEP ($x=0$); Large deviations:

for $x=0$, the FBA eq becomes

$$\boxed{Q(T)R(T) = (-1)^{N-1}(1-T)^L B + T^N}$$

Here: Q of degree N, R of degree $L-N$

where we have defined $B = (-1)^{N-1} e^{L\mu} Q(0)$

The eigenvalue becomes • $E(\mu) = -\frac{R'(1)}{R(1)} \quad \left(= \frac{Q'(1)}{Q(1)} - N \right)$

The 0-momentum condition leads to $R(1) = e^{N\mu}$ i.e,

• $\mu = \frac{1}{N} \operatorname{Log} R(1)$

$R_0(1) = 1$ for TASEP

We use a nice trick : separation of $\begin{smallmatrix} >0 \\ <0 \end{smallmatrix}$ degrees.

$$\frac{Q(T)}{T^N} R(T) = 1 + (-1)^{N-1} \frac{(1-T)^L}{T^N} B$$

$R(T)$: polynomial in T, degree $L-N$ (TASEP)

$\frac{Q(T)}{T^N}$ is a polynomial in $\frac{1}{T}$: $1, \frac{1}{T}, \dots \frac{1}{T^N}$ $Q(0)=0$ negative degrees

Taking the logarithm, we have:

$$\underbrace{\log\left(\frac{Q(T)}{T^N}\right)}_{\text{only negative powers of } T} + \log(R(T)) = \log\left(1 + (-1)^{N-1} B \frac{(1-T)^L}{T^N}\right)$$

Recall $B \propto Q(0) = 6(\mu) \ll 1$

we can expand the log on the r.h.s. wrt B

for TASEP we have
$$\begin{cases} \frac{Q(T)}{T^N} = 1 + \mu \frac{Q_1(T)}{T^N} + \mu^2 \frac{Q_2(T)}{T^N} & d(Q_i) \leq N-1 \\ R(T) = 1 + \mu R_1(T) + \mu^2 R_2(T) \dots \end{cases}$$

$$\log \frac{Q(T)}{T^N} + \log R(T) = \sum_{k=1}^{\infty} \frac{(-1)^{Nk-1}}{k} B^k \frac{(1-T)^{kL}}{T^{kN}}$$

By selecting positive powers of T on the r.h.s, we obtain an explicit expression for $\log R(T)$!

This will give us parametric expansions of $E(\mu) = -\frac{R'(1)}{R(1)}$ and of $\mu = \frac{1}{N} \log R(1)$, as functions of B. $= -\frac{d}{dT} \log R(T)\Big|_{T=1}$

$$\log R(T) = \sum_{k=1}^{\infty} \frac{(-1)^{Nk-1}}{k} B^k \left(\sum_{j=kN}^{kL} (-1)^j \binom{kL}{j} T^{j-kN}\right)$$

($\log \frac{Q(T)}{T^N}$ can be obtained in a similar fashion)

$$\mu = \frac{1}{N} \log R(1) = \sum_{k=1}^{\infty} \frac{(-1)^{Nk-1} B^k}{N k} \underbrace{\left(\sum_{j=kN}^{kL} (-1)^j \binom{kL}{j}\right)}_{\text{alternating sum of binomials}}$$

We have the following identity

$$\sum_{p=A}^{B} (-1)^p \binom{N}{p} = (-1)^A \binom{N}{A} + (-1)^{A+1} \binom{N}{A+1} + \dots + (-1)^B \binom{N}{B}$$

$$= (-1)^A \left[\cancel{\binom{N-1}{A}} + \binom{N-1}{A-1} \right] + (-1)^{A+1} \left[\binom{N-1}{A+1} + \cancel{\binom{N-1}{A}} \right] + \cdots (-1)^B \left[\binom{N-1}{B} + \cancel{\binom{N-1}{B-1}} \right]$$

$$= (-1)^B \binom{N-1}{B} + (-1)^A \binom{N-1}{A-1}$$

Here
$$\boxed{\mu = -\frac{1}{N} \sum_{k=1}^{\infty} \frac{B^k}{k} \binom{kL-1}{kN-1} = -\frac{1}{L} \sum_{k=1}^{\infty} \binom{kL}{kN} \frac{B^k}{k}}$$

Similarly
$$\boxed{E = -\sum_{k=1}^{\infty} \frac{(kL-2)!}{(kN-1)!\,(k(L-N)-1)!} \frac{B^k}{k} = -\sum_{k=1}^{\infty} \binom{kL-2}{kN-1} \frac{B^k}{k}}$$

By eleminating B order by order, between these two eqs we can calculate any cumulant of the total integrated current

Recall
$$\lim_{t\to\infty} \frac{\log \langle e^{\mu Y_t} \rangle}{t} = E(\mu) = \mu \frac{\langle Y_t \rangle}{t} + \frac{\mu^2}{2} \frac{(\text{var}\, Y_t)}{t} + \frac{\mu^3}{3!} \frac{\langle\langle Y_t^3 \rangle\rangle_c}{t}$$

$$J = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = \frac{N(L-N)}{L-1} \sim L\rho(1-\rho) \text{ (elementary)}$$

$$\Delta = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = \frac{LN(L-N)}{(L-1)(2L-1)} \frac{\binom{2L}{2N}}{\binom{L}{N}^2} \sim L^{3/2} \frac{\sqrt{\pi}}{2} [\rho(1-\rho)]^{3/2}$$

D - Ev - Mukamel (Matrix Ansatz)

Higher moments can be obtained also.

These parametric expressions for E and μ, were first obtained by Derrida & Lebowitz, PRL 80 p209 (1998), by coordinate Bethe Ansatz, followed by contour integrals analysis of the eqs. The functional B.A. allows us to shortcut the complex analysis part. More importantly, it will allows us to study the general PASEP case which was not accessible by coord. B.A.

But first we discuss some physical properties of the cumulant generating functn and the L.D.F.

Calculating the first few cumulants, we observe that

$$\langle\langle Y^k \rangle\rangle_c \sim L^{3/2(k-1)} \; ; \quad k \geq 2$$

Cours 3

IV BIS

<u>Summary</u>

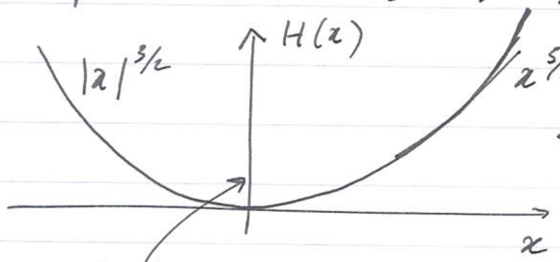
$$\begin{cases} \mu = -\frac{1}{L} \sum_{1}^{\infty} \binom{kL}{kN} \frac{B^k}{k} \\ E = - \sum_{1}^{\infty} \binom{kL-2}{kN-1} \frac{B^k}{k} \end{cases}$$

$$P\left(\frac{Y_t}{t} = J\right) \sim e^{-tG(J)} \qquad \begin{cases} J = dE/d\mu \\ G(J) = \mu J - E(\mu) \end{cases}$$ (Legendre Transform)

Take $L, N \to \infty$, $\frac{N}{L} = \rho$ finite

$$G(J) = \sqrt{\frac{\rho(1-\rho)}{2\pi L^3}}\, H\left(\frac{J - L\rho(1-\rho)}{\rho(1-\rho)}\right)$$

large Deviations of order 1



SKEW : much easier to reduce total current than to increase it

quadratic: variance $\propto L^{3/2}$

[Recall: local current → height in KPZ → universality]

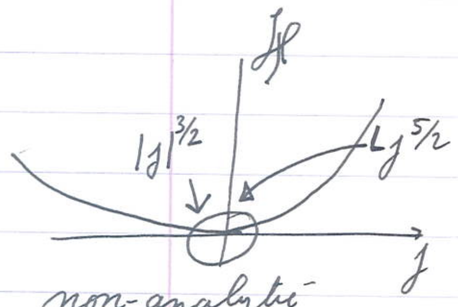
From total current Y_t → local current $q_t = \frac{Y_t}{L}$ (q_t: through a bond)

$$\text{Proba}\left(\frac{q_t}{t} = \rho(1-\rho) + j\right) \equiv \text{Proba}\left(\frac{Y_t}{t} = \underbrace{L\rho(1-\rho) + Lj}_{J}\right)$$

large deviations of ORDER $\underline{L}$

IT IS NOT THE SAME!

$$\sim e^{-t\mathcal{H}_\pm(j)}$$



non-analytic behaviour

$\mathcal{H}_+(j) \sim Lj^{5/2}$ $j>0$ → collective effect needed to increase the current

$\mathcal{H}_-(j) \sim |j|^{3/2}$ $j<0$

NOW Compare with the ∞-system $\xrightarrow{\text{GOTO}}$ PAGE VIII

Take the limit $L \to \infty$, $\frac{N}{L} = \rho$ in the expressions for μ and E: + Stirling

$$E(\mu) - \mu L \rho(1-\rho) = \sqrt{\frac{\rho(1-\rho)}{2\pi L^3}}\ \Phi\left[\mu\sqrt{2\pi\rho(1-\rho)L^3}\right]$$

where Φ is obtained by eliminating C:

same formulae as for the quantum gas

$$\boxed{\begin{aligned}\beta &= \sum_{q=1}^{\infty}(-1)^{q-1}\frac{C^q}{q^{3/2}}\\ \Phi(\beta) &= \sum_{q=1}^{\infty}(-1)^{q-1}\frac{C^q}{q^{5/2}}\end{aligned}}$$

new parameter (Bona-Fede)

$$C = \frac{B}{[\rho^\rho(1-\rho)^{1-\rho}]^L}$$

$|C| < 1$

Although these series are singular when $|C| \simeq 1$, the function $\Phi(\beta)$ remains well-defined for all values of $\beta \in]-\infty, +\infty[$

- For $C > 1$, we can write

$$\begin{cases}\beta = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^\infty d\varepsilon\,\sqrt{\varepsilon}\,\frac{Ce^{-\varepsilon}}{1+Ce^{-\varepsilon}}\\ \Phi(\beta) = \dfrac{4}{3\sqrt{\pi}}\displaystyle\int_0^\infty d\varepsilon\,\varepsilon^{3/2}\,\frac{Ce^{-\varepsilon}}{1+Ce^{-\varepsilon}}\end{cases}$$

(polylogs)

valid for $C \in]-1, +\infty[$ $[\beta_-, +\infty[$

i.e. $\beta \in [\beta_- = \zeta(3/2) = \sum \frac{1}{q^{3/2}}, +\infty[$

- To analytically continue $\Phi(\beta)$ for $\beta < \beta_- = \zeta(3/2)$, we have to replace the above series by their analytic continuations

$$\begin{cases}\beta = -4\sqrt{\pi}\,(-\ln(-C))^{1/2} + \displaystyle\sum_{q\geq 1}(-1)^{q-1}\frac{C^q}{q^{3/2}}\\ \Phi(\beta) = \dfrac{8}{3}\sqrt{\pi}\,(-\ln(-C))^{3/2} + \displaystyle\sum_{q\geq 2}(-1)^{q-1}\frac{C^q}{q^{5/2}}\end{cases}$$

8?

$[-\infty, \beta_-]$

Here C varies from -1 to 0 and β varies from β_- to $-\infty$. The complete range of β has been covered.

We can use these formulae the extract the asymptotic behaviour of $\Phi(\beta)$:

- β close to 0 $\quad \Phi(\beta) \simeq \beta + \frac{\sqrt{2}}{8}\beta^2 + \frac{27-16\sqrt{3}}{216}\beta^3 + \ldots$
- $\beta \to +\infty$ $\quad \Phi(\beta) \simeq \frac{1}{5}\left(\frac{9\pi}{2}\right)^{1/3}\beta^{5/3}$ (Large C behaviour)

Cours 3 VI

- $\beta \to -\infty$ ($C \to 0^-$ in the last formulae)

$$\Phi(\beta) \simeq -\frac{1}{24\pi}\beta^3$$

From this analysis, the behaviour of the large deviation function can be deduced.

recall:
$$\boxed{P\left(\frac{Y_t}{t}=j\right) \sim e^{-tG(j)}}$$

By Legendre transform, we know that:

$$\begin{cases} j = \dfrac{dE}{d\mu} \\ G(j) = \mu j - E(\mu) \end{cases}$$

$$\int P\left(\frac{Y_t}{t}=j\right) e^{\mu Y_t} dY_t = e^{tE(\mu)}$$
$$e^{t[\mu j - G(j)]}$$

Using the scalings discussed above, we can write:

$$\begin{cases} j = L\rho(1-\rho) + \rho(1-\rho)\,\Phi'\left[\mu\sqrt{2\pi\rho(1-\rho)L^3}\right] \\ G(j) = \mu\left[j - L\rho(1-\rho)\right] - \sqrt{\dfrac{\rho(1-\rho)}{2\pi L^3}}\,\Phi\left[\mu\sqrt{2\pi\rho(1-\rho)L^3}\right] \end{cases}$$

again a parametric representation:

Let us call $\zeta = \mu\sqrt{2\pi\rho(1-\rho)L^3}$

and $v = \dfrac{j - L\rho(1-\rho)}{\rho(1-\rho)}$

$$\boxed{\begin{aligned} v &= \Phi'(\zeta) \\ G &= \sqrt{\frac{\rho(1-\rho)}{2\pi L^3}}\left\{\zeta v - \Phi(\zeta)\right\} \end{aligned}}$$

eliminating ζ allows us to calculate the function $G(v)$

we can deduce from that: $G(j) = \sqrt{\dfrac{\rho(1-\rho)}{2\pi L^3}}\,H\left(\dfrac{j - L\rho(1-\rho)}{\rho(1-\rho)}\right)$

with
$$H(v) \sim + \frac{2\sqrt{6}}{5\sqrt{\pi}} v^{5/2} \quad \text{for } v \to +\infty$$

$\zeta \to +\infty$

$\zeta = \left(\frac{6v^3}{\pi}\right)^{1/2}$

$$H(v) \sim + \frac{4\sqrt{\pi \times 2}}{3} |v|^{3/2} \quad \text{for } v \to -\infty$$

$\zeta \to -\infty$

$$H(v) \sim \sqrt{2}(v-1)^2 \quad |v-1| \ll 1$$

$$\begin{cases} \zeta = 0 \\ v = 1 \end{cases}$$

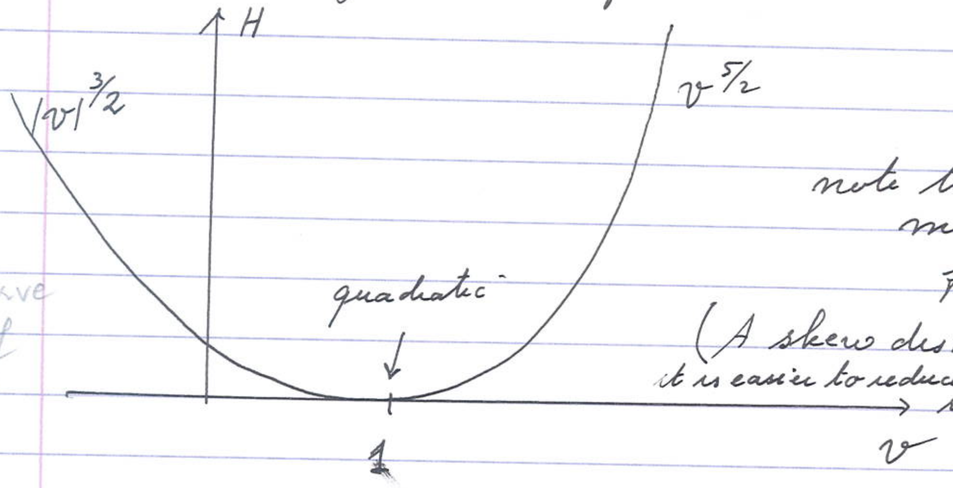
$$\text{Proba}\left(\frac{Y_t}{t} = L\rho(1-\rho) + v\right) = e^{-t\sqrt{\frac{\rho(1-\rho)}{2\pi L^3}}H(v)}$$

finite

Cours 3 VII

We can draw the large deviation function

universal curve for KPZ-type of behaviour



note that the minimum is for $v=1$

(A skew distribution)

it is easier to reduce the current rather than to increase it

Indeed, we know that the average current is $\bar{j} = \frac{N(L-N)}{L-1} = \frac{L^2}{L-1}\rho(1-\rho)$

$$\bar{v} = \frac{\bar{j} - L\rho(1-\rho)}{\rho(1-\rho)} = \frac{L\rho(1-\rho)}{(L-1)\rho(1-\rho)} = \frac{L}{L-1} \xrightarrow[L\to\infty]{} 1 \quad (\text{offset})$$

This large deviation function is valid for $j - L\rho(1-\rho) = v$ taking finite values.

We may also be interested in the case $v = O(L)$ i.e the deviation of the current is of the order of the total current itself.

Let us write $\boxed{v = Ly}$. Then, there are 2 types of behaviour:

Proba $\left(\frac{q_t}{t} = \rho(1-\rho)+y\right) \sim$ Proba $\left(\frac{Q_t}{t} = L(\rho(1-\rho)+y)\right) \sim e^{-t f(y)}$

single-bond current

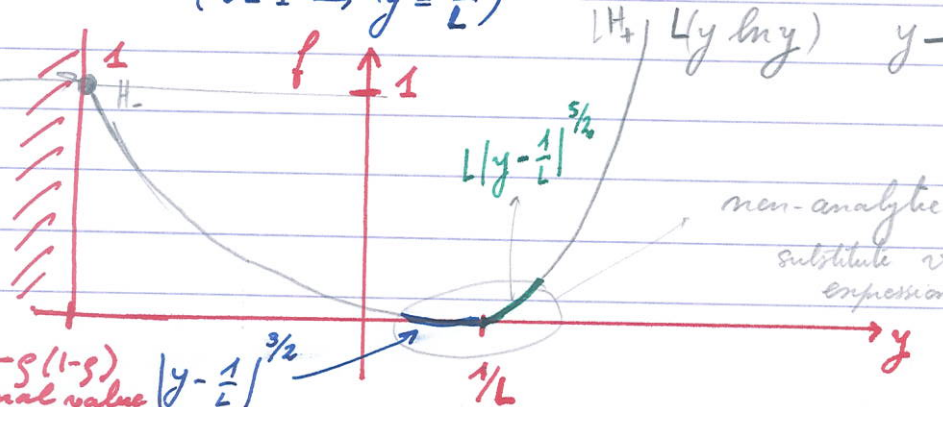
with

for $y>0$ $\quad f(y) \simeq L\, H_+(y)$ collective effect to increase the total current

for $y<0$ $\quad f(y) \simeq H_-(y)$

for $y = O\left(\frac{1}{L}\right)$ it has to match the previous results. $\left(v=1 \to y = \frac{1}{L}\right)$

e^{-t} probability to obtain a 0 current



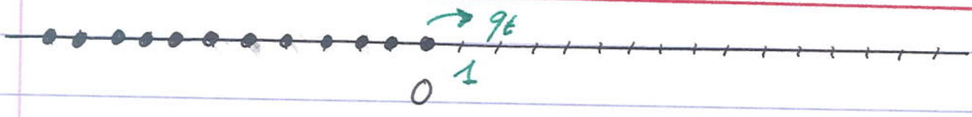
Essayer d'obtenir ces résultats par fonctional Bethe Ansatz.

NOTE THAT $q_t = \frac{Q_t}{L}$ current through a single-bond

We shall jump ahead and show how to compare these results with Airy - Tracy-Widom Laws:

Non-commutativity of $\begin{pmatrix} L \to \infty \\ t \to \infty \end{pmatrix}$

INFINITE SYSTEM



q_t = integrated current through the first bond.

NOTE that the initial-condition (STEP) is given

$$\boxed{q_t = \frac{t}{4} + t^{\frac{1}{3}} \xi}$$

→ KPZ scaling

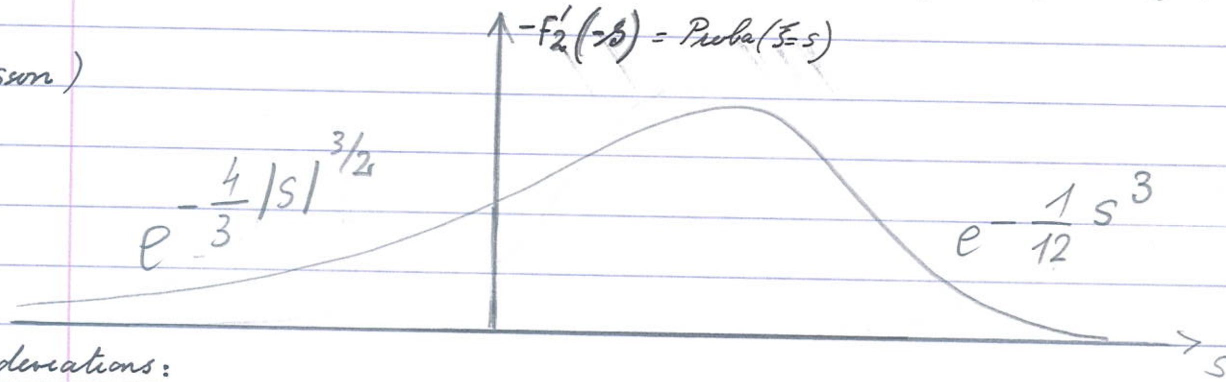
ξ Airy random variable

Random Matrix Theory

$$\text{Proba}(\xi \leq s) = F_2(-s)$$

$\text{Proba}(\xi = s) = -F_2'(-s)$

(Johansson)



Large-deviations:

$$\text{Proba}\left(\frac{q_t}{t} = \frac{1}{4} + j\right) \equiv \text{Proba}\left(\xi = j t^{2/3}\right) \qquad (t \to \infty)$$

$$\left(-\frac{1}{4} <\right) j < 0 \qquad \sim e^{-\frac{4}{3} t |j|^{3/2}} \equiv e^{-t G_-(j)}$$

$$j > 0 \qquad \sim e^{-\frac{t^2}{12} |j|^3} \equiv e^{-t^2 G_+(j)}$$

A layman's explanation of these asymptotic Behaviours (Krapivsky - Redner - Ben Naim)

$\xi \to -\infty$: $\text{Proba}(q_t = 0) \sim e^{-t}$ $\qquad 0 = \frac{t}{4} + t^{1/3}\xi \to |\xi| \sim t^{+2/3}$

$\sim e^{-|\xi|^{3/2}}$

$\xi \to +\infty$ Create a large current of order t

Variant of ASEP: discrete time $t \to t+1$ each particle SYNCHRONOUSLY hops with probability $\frac{1}{2}$ - To have a current of value N (of order t)

The first particle must hop N sites   Cours 3 IX
second one : $(N-1)$ sites etc $\sim N^2$ hops

with probability $\left(\frac{1}{2}\right)^{N^2} \sim e^{-cN^2} \sim e^{-ct^2}$ $\xi \sim t^{2/3}$

$\sim e^{-c\xi^3}$

Match these infinite size results with the ones for finite systems, explained above?

$y<0$ $\quad e^{-t|y|^{3/2}}$ and $e^{-t|y|^{3/2}}$: agree

dynamical exponent

$y>0$ $\quad e^{-tL|y|^{5/2}}$ vs $e^{-t^2 j^3}$ $\quad t \sim L^{3/2}$

$$e^{-\underbrace{tL^{3/2}}_{t^2}\underbrace{L^{-\frac{1}{2}}|y|^{5/2}}_{y^3 \text{ or } j^3}} \qquad y \sim j \sim t^{-2/3} \sim 1/L$$

This is non-unique but coherent

An argument is missing

Some results for the general ASEP case:

<u>Voir les transparents</u> : (voir pages suivantes)

- $J = (1-x)\dfrac{(L-N)N}{L-1}$

- $D = (1-x)\dfrac{2L}{L-1}\displaystyle\sum_{k>0} k^2 \frac{\binom{L}{N+k}\binom{L}{N-k}}{\binom{L}{N}^2}\frac{1+x^k}{1-x^k}$

One can derive asymptotics for $E(\mu)$ and plot the results, especially in the weak asymmetry limit

GENERAL STRUCTURE derived from the FBA.

See slides next pages.

Cumulants of the Current (General Case)

- Mean Current: $J = (1-x)\frac{N(L-N)}{L-1} \sim (1-x)L\rho(1-\rho)$ for $L \to \infty$

- Diffusion Constant: $D = (1-x)\frac{2L}{L-1}\sum_{k>0} k^2 \frac{C_L^{N+k}}{C_L^N}\frac{C_L^{N-k}}{C_L^N}\left(\frac{1+x^k}{1-x^k}\right)$

$$D \sim 4\phi L\rho(1-\rho)\int_0^\infty du \frac{u^2}{\tanh \phi u} e^{-u^2}$$

when $L \to \infty$ and $x \to 1$ with *fixed value* of $\phi = \frac{(1-x)\sqrt{L\rho(1-\rho)}}{2}$.

- Third cumulant (Skewness):

$$\frac{E_3}{\phi(\rho(1-\rho))^{3/2}L^{5/2}} \simeq -\frac{4\pi}{3\sqrt{3}} +$$

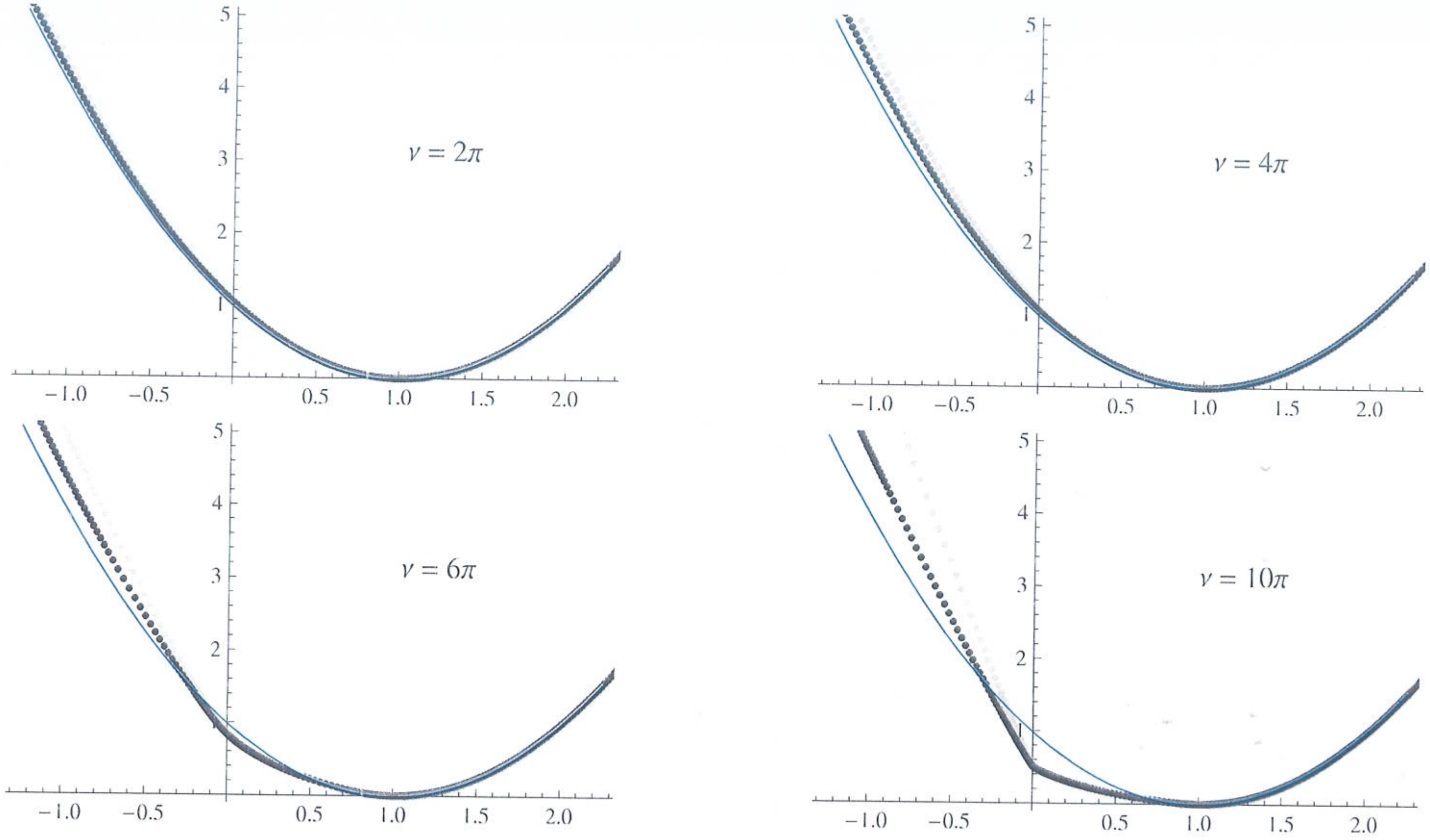
$$12\int_0^\infty dudv \frac{(u^2+v^2)e^{-u^2-v^2} - (u^2+uv+v^2)e^{-u^2-uv-v^2}}{\tanh \phi u \tanh \phi v}$$

$\to$ Non Gaussian fluctuations. TASEP limit for $\phi \to \infty$:

$$E_3 \simeq \left(\frac{3}{2} - \frac{8}{3\sqrt{3}}\right)\pi(\rho(1-\rho))^2 L^3$$

$$
\begin{aligned}
\frac{E_3}{6L^2} &= \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N-i}C_L^{N+j}C_L^{N-j}}{(C_L^N)^4}(i^2+j^2)\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N+i}C_L^{N+j}C_L^{N-i-j}}{(C_L^N)^3}\frac{i^2+ij+j^2}{2}\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\sum_{j>0}\frac{C_L^{N-i}C_L^{N-j}C_L^{N+i+j}}{(C_L^N)^3}\frac{i^2+ij+j^2}{2}\frac{1+x^i}{1-x^i}\frac{1+x^j}{1-x^j} \\
&- \frac{1-x}{L-1}\sum_{i>0}\frac{C_L^{N+i}C_L^{N-i}}{(C_L^N)^2}\frac{i^2}{2}\left(\frac{1+x^i}{1-x^i}\right)^2 \\
&+ (1-x)\frac{N(L-N)}{4(L-1)(2L-1)}\frac{C_{2L}^{2N}}{(C_L^N)^2} \\
&- (1-x)\frac{N(L-N)}{6(L-1)(3L-1)}\frac{C_{3L}^{3N}}{(C_L^N)^3}
\end{aligned}
$$

Full large deviation function (weak asymmetry)



$$E\left(\frac{\mu}{L}\right) \simeq \frac{\rho(1-\rho)(\mu^2+\mu\nu)}{L} - \frac{\rho(1-\rho)\mu^2\nu}{2L^2} + \frac{1}{L^2}\psi[\rho(1-\rho)(\mu^2+\mu\nu)]$$

with $\psi(z) = \sum_{k=1}^{\infty} \frac{B_{2k-2}}{k!(k-1)!} z^k$

The General Case (S. Prolhac, 2010)

The function $E(\mu)$ is again obtained in a parametric form:

$$\mu = -\sum_{k\geq 1} C_k \frac{B^k}{k} \quad \text{and} \quad E = -(1-x)\sum_{k\geq 1} D_k \frac{B^k}{k}$$

C_k and D_k are combinatorial factors enumerating some tree structures.
There exists an auxiliary function

$$W_B(z) = \sum_{k\geq 1} \phi_k(z) \frac{B^k}{k}$$

such that C_k and D_k are given by complex integrals along a small contour that encircles 0 :

$$C_k = \oint_C \frac{dz}{2i\pi} \frac{\phi_k(z)}{z} \quad \text{and} \quad D_k = \oint_C \frac{dz}{2i\pi} \frac{\phi_k(z)}{(z+1)^2}$$

The function $W_B(z)$ contains the full information about the statistics of the current.

n $W_B(z)$ is the solution of a functional Bethe equation:

$$\boxed{W_B(z) = -\ln\left(1 - BF(z)e^{X[W_B](z)}\right)}$$

$$\boxed{F(z) = \frac{(1+z)^L}{z^N}}$$

or X is a integral operator

$$X[W_B](z_1) = \oint_C \frac{dz_2}{\imath 2\pi\, z_2} W_B(z_2)K(z_1, z_2)$$

rnel

$$\boxed{K(z_1, z_2) = 2\sum_{k=1}^{\infty} \frac{x^k}{1-x^k}\left\{\left(\frac{z_1}{z_2}\right)^k + \left(\frac{z_2}{z_1}\right)^k\right\}}$$

Solving this Functional Bethe Ansatz equation to all orders enables us to calculate cumulant generating function. For $x = 0$, the TASEP result is readily retrieved.

The function $W_B(z)$ also contains information on the 6-vertex model associated with the ASEP.

From the Physics point of view, the solution allows one to

- Classify the different universality classes (KPZ, EW).
- Study the various scaling regimes.
- Investigate the hydrodynamic behaviour.