Lecture 2

Coordinate Bethe Ansatz for ASEP.

- Special TASEP case : a study of the Bethe equations and Gap calculation

Coordinate Bethe Ansatz for the exclusion process

The evolution of the process is encoded in the Markov Matrix M

$$\boxed{\frac{dP_t}{dt} = M\,P_t}$$

$P_t(\mathcal{C}/\mathcal{C}_0)$ probab. vector

Perron-Frobenius: Ker M : dim 1
all other eigenvalues have Re part < 0 → relaxation modes

Let us call Ψ_E an eigenvector of M : $\boxed{M\Psi_E = E\,\Psi_E}$

$$P_t = \sum_E \underbrace{\alpha_E}_{\text{components of } P \text{ on } \Psi_E} \Psi_E \, e^{Et} = \alpha_0 \underset{\substack{\uparrow \\ \text{Null-eigenv}}}{\Psi_0} + \sum_{Re(E)<0} \alpha_E \Psi_E \, e^{Et}$$

(we assume M is diagonalisable)

i.e $$|P_t\rangle = \sum e^{Et} \langle \underset{\substack{\nearrow \\ \text{left-eigenvector}}}{\Psi_E} | P_t \rangle \, |\Psi_E\rangle$$

$\langle \Psi_E | M = E \langle \Psi_E |$ left-eig ⇝ M is not-hermitian
$M|\Psi_{E'}\rangle = E'|\Psi_{E'}\rangle$ right-eig

$$\boxed{\langle \Psi_{E_i} | \Psi_{E_j} \rangle = \delta_{ij}}$$

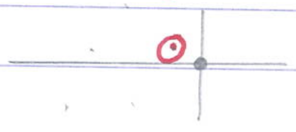
Note that $\langle \Psi_0 | = (1, \dots 1)$ left-null vector of M
$\equiv \sum_{\mathcal{C}} \langle \mathcal{C} |$

i.e $\alpha_0 = \langle \Psi_0 | P_0 \rangle = \sum_{\mathcal{C}} P_0(\mathcal{C}) = 1$

we have $P_t \longrightarrow |\Psi_0\rangle$ convergence to the stationary state

Relaxation times: $\frac{1}{|Re(E_i)|}$ for eigenvalues $E_i \neq 0$

The largest relaxation-time ⟷ eigenvalue E_1 which has the smallest real part in absolute value



i.e. the one "closest" to 0 (along the real part).

Cours 2 ⓪bis

$$P_t \approx \psi_0 + d_1 e^{-\frac{t}{T_1}} e^{i\omega_1 t} \psi_1 + \ldots$$

$$\boxed{T_1 = \frac{-1}{Re(E_1)}}$$ largest relaxation time

$$\boxed{\omega_1 = Im(E_1)}$$ oscillations

Can we determine T_1 and ω_1?
What about the other eigenvalues.

In order to fully resolve the dynamics, we should diagonalize
M → calculate transiant behaviour
unequal-time correlation functions.

→ THIS CAN BE DONE: Using the Bethe Ansatz.

Probably the simplest example to learn the B.A.

Historical Perspective (+ cf M. Batchelor) Feynman '88: "I got really fascinated by these (1+1) dim. models that are solved by the Bethe A. and how mysteriously they jump out at you and work and you don't know why. I am trying to understand all this better".

Bethe ANSATZ → a trick, a "trial", a beginning that H. Bethe used in 1931, solving the Quantum Heisenberg spin chain, by postulating A TRIAL FORM for the WAVEFUNCTION. → 1d magnetic system with nearest neighbourgh interacts

Some LANDMARKS: → a method to solve low-dim. quantum systems with interactions (N-body Pbs)

BOSE GAS (Lieb-Liniger '63) $\hat{H}\psi = -\sum \frac{\partial^2}{\partial x_i^2}\psi + \underbrace{2g \sum_{i<j} \delta(x_i - x_j)}_{\text{localized interact}^{\circ}} \psi = E\psi$ $g<0$ ATTRACTIF, $g>0$ REPULSIF

$\psi(x_1, .. x_N)$ N-body wave function

Today: applications to Bose-Einstein condensates

Various steps → symmetric wave-function (bosons)
→ ψ transforms as a Young Tableau ($\hat{H}_N$ invariant by σ_N)
C.N. YANG (1967)

2d CLASSICAL MODELS VERTEX MODELS → 6 vertex, → 8 vertex Generalizations of the Ising Model

Eq. Stat. Mech E LIEB, CN YANG, R. BAXTER + M. Gaudin

"Exactly solved models in stat Mech."

When does it work? THE YANG-BAXTER Eq.

RUSSIAN SCHOOL: INTEGRABLE SYSTEMS a branch of mathematical φ and ALGEBRA

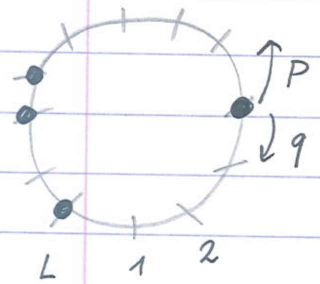
L Faddeev, L. Takhtajan, E Sklyanin, Kulish, Reshetikhin ... '70, '80

→ QUANTUM GROUPS (DRINFELD) new symmetries that generalize LIE ALG.

Today: a crucial tool in COMBINATORICS, SOLID STATE Φ KONDO EFFECT N Andrei, P Wiegmann, Tsvelik; HUBBARD MODEL

STRING THEORY / Super Sym. YANG-MILLS

COORDINATE BETHE ANSATZ for the ASEP on a ring



L sites
N particles $\binom{L}{N}$ configurations

$$\boxed{x \equiv \frac{q}{p}}$$ asymmetry parameter

Rescaling time ($p \to 1$; $q \to x = \frac{q}{p}$)

periodicity
$L+1 \equiv L$

a configuration $\mathcal{C}$: $1 \leq x_1 < x_2 < .. < x_N \leq L$ state-space

The eigenvalue equation can be written as:

$$E\,\psi(x_1 \ldots x_N) = p \sum_i{}' \left[\psi(x_1, .., x_i - 1, , x_N) - \psi(x_1, .., x_i, x_N)\right]$$
$$+ q \sum_j{}' \left[\psi(x_1, . x_j + 1, . x_N) - \psi(x_1, .., x_j, x_N)\right]$$

Sums are restricted on indices i such that $x_{i-1} < x_i - 1$
j such that $x_j + 1 < x_{j+1}$

- These conditions allow that the corresponding jumps are allowed.

It is a kind of a discrete Laplacian with some special b.c.

Note that ψ is defined on the simplex:

$$\boxed{1 \leq x_1 < x_2 .. < x_N \leq L}$$

How do the periodic b.c. on the ring manifest themselves?

$$\psi(x_1, x_2, .. x_{N-1}, L+1) = \psi(1, x_1, x_2, .., x_{N-1})$$
$$1 < x_1 < x_2 \quad < x_{N-1} \leq L$$

"making a full circle around the ring"

Now we must diagonalize M: we use Bethe ANSATZ.

The key idea is that if the particles were not interacting, we know that products of plane waves would diagonalize M.

Here: there <u>are interactions</u>. Bethe's idea was to use linear combination of plane waves. By suitably choosing the amplitude of these plane waves, the interactions can be taken into account.

We shall proceed steps by steps: considering first a single particle on a ring, then $N=2$ particles, then $N=3$. Finally, we'll do the general N case.

<u>FIRST STEP $N=1$</u> A single particle on a ring: $1 \leq x \leq L$

$$E \psi_E(x) = p \psi_E(x-1) + q \psi_E(x+1) - (p+q) \psi_E(x)$$

2 modes z_+ and z_- with $E = \frac{p}{z_\pm} + q z_\pm - (p+q)$

$\psi_E(x) = A z_+^{x} + B z_-^{x}$ $\qquad z_+ z_- = p/q \geqslant 1$

$\qquad z_+ + z_- = \frac{p+q+E}{q}$

z_+ and z_- solve the characteristic equation $q Z^2 - (p+q+E) Z + p = 0$

Periodicity condition: $\psi_E(x+L) = \psi_E(x)$

$$A z_+^{x} + \underbrace{B \left(\frac{p}{q}\right)^{x} z_+^{-x}}_{\equiv z_-^{x}} = A z_+^{L} + B \left(\frac{p}{q z_+}\right)^{x+L} \quad \forall x$$

i.e $\quad 0 = A(1 - z_+^{L}) z_+^{x} + B\left(1 - \frac{1}{z_-^{L}}\right) z_-^{x}$ $\qquad$ take $\frac{p}{q} > 1$ and choose $|z_+| > 1$ and $|z_-| = \frac{p}{q|z_+|}$

this implies $\quad A = 0$ and $z_-^{L} = 1$

$\hookrightarrow$ quantification

Une manière plus efficace de présenter ce calcul :

Le modèle est invariant par translation T (i.e ajouter 1 à chaque position) i.e $MT = TM$ on les co-diagonalise donc on cherche $\psi_E(x_1, \dots x_N)$ telle que

$$\psi_E(x_1+1, x_2+1, \dots, x_N+1) = \zeta\, \psi_E(x_1, x_2, \dots x_N)$$

avec $\zeta^L = 1$ (par périodicité)

Pour le cas à une particule $\psi(x+1) = \zeta\psi(x) \Rightarrow$

$$\psi(x) = A z^x \qquad (\text{avec } z = \zeta)$$

et $\psi(x+L) = \psi(x) \Rightarrow \boxed{z^L = 1}$

$$\boxed{E = p/z + qz - (p+q)}$$

Second CASE, $N=2$:

2 particles $1 \leq x_1 < x_2 \leq L$

GENERIC EQUATION $|x_2 - x_1| > 1 \pmod L$

$$E\psi(x_1, x_2) = p\left[\psi(x_1-1, x_2) + \psi(x_1, x_2-1)\right] + q\left[\psi(x_1+1, x_2) + \psi(x_1, x_2+1)\right] - 2(p+q)\psi(x_1, x_2)$$

SPECIAL CASE of adjacency : $x_2 = x_1 + 1$

$$E\psi(x_1, x_1+1) = p\psi(x_1-1, x_1+1) + q\psi(x_1, x_1+2) - 1(p+q)\psi(x_1, x_1+1)$$

comparing with the generic case we observe that there are some « missing terms » :

$$p\psi(x_1, x_1) + q\psi(x_1+1, x_1+1) - (p+q)\psi(x_1, x_1+1) \rightarrow 0$$

If there were NO special cases (i.e the particles do not interact) then a factorized solution $\psi(x_1, x_2) = A z_1^{x_1} z_2^{x_2}$ would solve the generic equation

with $\boxed{E = p\left(\frac{1}{z_1} + \frac{1}{z_2}\right) + q(z_1 + z_2) - 2(p+q)}$

Plutôt que de résoudre à part le cas spécial d'adjacence, il est plus efficace de résoudre le cas générique et d'imposer une CONTRAINTE supplémentaire, une condition d'annulation :

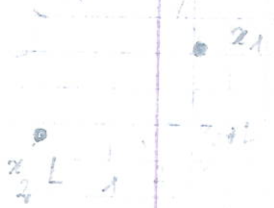
$$p\,\psi(x,x) + q\,\psi(x+1,x+1) - (p+q)\,\psi(x,x+1) = 0$$

Remarque : On peut vérifier que la solution générique $\psi(x_1,x_2) = A\,z_1^{x_1} z_2^{x_2}$ ne permet pas de satisfaire la condition d'annulation avec z_1 et z_2 indépendants $\rightsquigarrow$ $z_2 = \frac{p}{p+q(1-z_1)}$

pour $p=1$, $q=0$ cela donne $z_2 = 1$: aucune dépendance en x_2 !

pour $(p,q) \neq (1,0)$ c'est au niveau de la condition de recollement périodique (voir ci-dessous) que cela coince.

Périodicité → Quantification des z :



$$\boxed{\psi(x_1, x_2+L) = \psi(x_2, x_1)} \qquad 1 \leq x_2 < x_1 < x_2+L$$

en fait, plus simplement $\psi(x, L+1) \equiv \psi(1, x)$

Remarque fondamentale : E est invariant $z_1 \leftrightarrow z_2$

i.e. $A z_1^{x_1} z_2^{x_2}$ et $B z_2^{x_1} z_1^{x_1}$ correspondent à la même v. p. de l'équation générique

on cherche donc le vecteur propre sous la forme :

$$\boxed{\psi(x_1,x_2) = A_{12}\, z_1^{x_1} z_2^{x_2} + A_{21}\, z_2^{x_1} z_1^{x_2}}$$ (Bethe) ANSATZ

L'éq. générique est bien satisfaite avec

$$E = p\left(\frac{1}{z_1} + \frac{1}{z_2}\right) + q(z_1+z_2) - 2(p+q)$$

CONTRAINTE D'ANNULATION :

$$p(A_{12}+A_{21}) + q z_1 z_2 (A_{12}+A_{21}) = (p+q)\left\{ A_{12} z_2 + A_{21} z_1 \right\}$$

⇒ Une relation entre les amplitudes

$$\boxed{\frac{A_{21}}{A_{12}} = -\frac{p-(p+q)z_2+q z_1 z_2}{p-(p+q)z_1+q z_1 z_2}}$$

L'équation aux valeurs propres est maintenant satisfaite dans tous les cas.

PÉRIODICITÉ ⇒ QUANTIFICATION des z :

$\Psi(x, L+1) = \Psi(1, x)$ $\forall x$ $\underset{\sim\sim\sim}{A_{12} z_1^x z_2^{L+1}} + \underline{A_{21} z_2^x} z_1^{L+1} = \underline{A_{12} z_1 z_2^x} + \underset{\sim\sim\sim}{A_{21} z_2 z_1^x}$

indépendance z_1^x et z_2^x :

$$\begin{cases} A_{12} z_2^L = A_{21} \\ A_{12} = A_{21} z_1^L \end{cases}$$

(Notez $(z_1 z_2)^L = 1$ invariance globale par Translation)

Cela mène aux équations de Bethe :

$$\boxed{\begin{aligned} z_1^L &= -\frac{q z_1 z_2-(p+q)z_1+p}{q z_1 z_2-(p+q)z_2+p} \\[2ex] z_2^L &= -\frac{q z_1 z_2-(p+q)z_2+p}{q z_1 z_2-(p+q)z_1+p} \end{aligned}}$$

Ici l'Ansatz de Bethe conduit à 2 éq. polynômiales couplées de degré $\simeq L$ alors que la matrice de Markov (et donc le polynôme caractéristique) sont de taille $\sim L^2$.

Note for TASEP : $p=1,\ q=0$

$$z_1^{(-L)}\{-z_1+1\}^2 = z_2^{-L}(-z_2+1)^2 = -(1-z_1)(1-z_2)$$

i.e

$$\frac{(1-z_i)^2}{z_i^L} = -\prod_{k=1}^{2}(1-z_k) \qquad i=1,2$$

$$\frac{A_{21}}{A_{12}} = -\frac{1-z_2}{1-z_1} \quad \text{et} \quad \Psi(x_1,x_2) = \begin{vmatrix} \frac{z_1^{x_1}}{1-z_1} & \frac{z_1^{x_2}}{(1-z_1)^2} \\ \frac{z_2^{x_1}}{1-z_2} & \frac{z_2^{x_2}}{(1-z_2)^2} \end{vmatrix}$$

FONCTION D'ONDE
=
un
déterminant

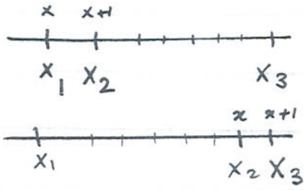
$\boxed{N=3}$ (Three particles on a ring) AND $N \geqslant 3$

The generic equation $x_1 \ll x_2 \ll x_3$ is

$$E\Psi(x_1,x_2,x_3) = p\,\{\Psi(x_1-1,x_2,x_3)+\Psi(x_1,x_2-1,x_3)+\Psi(x_1,x_2,x_3-1)\}$$
$$+q\,\{\Psi(x_1+1,x_2,x_3)+\Psi(x_1,x_2+1,x_3)+\Psi(x_1,x_2,x_3+1)\}$$
$$-3^{(p+q)}\Psi(x_1,x_2,x_3)$$

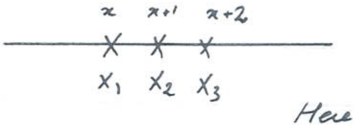
Adjacency conditions :

Two particle collisions
"2" + 1



same type of CONSTRAINTS as above

Triple collisions :



A new CONSTRAINT?
Here is the rub....

Two-body collisions : élimination of the unwanted terms

→ spectator

$$p\Psi(x,x,x_3) + q\Psi(x+1,x+1,x_3) - (p+q)\Psi(x,x+1,x_3) = 0 \quad ●$$

here x_3 : spectator

and, similarly

$$p\Psi(x_1,x,x) + q(x_1,x+1,x+1) - (p+q)\Psi(x_1,x,x+1) = 0 \quad 〰$$

SPECTATOR

here x_1 : spectator

Triple collision : $x_1=x,\ x_2=x+1,\ x_3=x+2$

there are SIX UNWANTED terms

$$p\{\underline{\Psi(x,x,x+2)}+\underline{\Psi(x,x+1,x+1)}\} + q\{\underline{\Psi(x+1,x+1,x+2)}+\underline{\Psi(x,x+2,x+2)}\}$$
$$-\cancel{2}(p+q)\,\underline{\Psi(x,x+1,x+2)} = 0$$
$$-(p+q)\,\underline{\Psi(x,x+1,x+2)}$$

This is not a new constraint : just a linear combination of the previous-two ones

If the double-collision constraints are fulfilled then the 3-body constraint is automatically SATISFIED.

This "factorisation" property of multiple collisions into 2-body interacts lies at the very heart of the Bethe Ansatz.

Bethe Ansatz for the eigenfunction:

$$\underline{\underline{\Psi(x_1,x_2,x_3)}} = A_{123}\, z_1^{x_1} z_2^{x_2} z_3^{x_3} + A_{132}\, z_1^{x_1} z_3^{x_2} z_2^{x_3} + A_{213}\, z_2^{x_1} z_1^{x_2} z_3^{x_3} + A_{231}\, z_2^{x_1} z_3^{x_2} z_1^{x_3}$$

$$+ A_{312}\, z_3^{x_1} z_1^{x_2} z_2^{x_3} + A_{321}\, z_3^{x_1} z_2^{x_2} z_1^{x_3} \equiv \underline{\underline{\sum_{\sigma\in\mathfrak{S}_3} A_\sigma\, z_{\sigma(1)}^{x_1} z_{\sigma(2)}^{x_2} z_{\sigma(3)}^{x_3}}}$$

$$= \sum_{\sigma\in\mathfrak{S}_N} A_\sigma \prod_{i=1}^{N} z_{\sigma(i)}^{x_i}$$

The generic eq. is satisfied with the eigenvalue

$$\boxed{E = p\sum_{i=1}^{3(N)} z_i + q\sum_{i=1}^{3(N)} \frac{1}{z_i} - \underset{\substack{\downarrow\\ N}}{3}(p+q)}$$

Constraints: Let us write the general N case

The 2-body collision cancellation condition for the N particle case reads

$$x_1 \ll x_2 \lll \ldots x_i,\ x_{i+1} = x_i+1 \ll x_{i+2} \ll x_N$$

$$p\,\Psi(\underbrace{x_1,\ldots,}_{\text{spectators}}x_i,x_i,\underbrace{\ldots\, x_N}) + q\,\Psi(x_1,\ldots,x_i+1,x_i+1,\ldots,x_N)$$

$$= (p+q)\,\Psi(x_1,\ldots,x_i,(x_i)+1,\ldots,x_N)$$

Let us substitute the B.A. in this equation $(x_i \equiv x)$

$$0 = p\sum_{\sigma\in\mathfrak{S}_N} A_\sigma\, z_{\sigma(1)}^{x_1}\cdots z_{\sigma(i)}^{x} z_{\sigma(i+1)}^{x}\cdots z_{\sigma(N)}^{x_N} + q\sum_{\sigma\in\mathfrak{S}_N} A_\sigma\, z_{\sigma(1)}^{x_1}\cdots z_{\sigma(i)}^{x+1} z_{\sigma(i+1)}^{x+1}\cdots z_{\sigma(N)}^{x_N}$$

$$-(p+q)\sum_{\sigma\in\mathfrak{S}_N} A_\sigma\, z_{\sigma(1)}^{x_1}\cdots z_{\sigma(i)}^{x} z_{\sigma(i+1)}^{x+1}\cdots z_{\sigma(N)}^{x_N}$$

$\Rightarrow$ this has to vanish identically for generic values of

$x_1 \quad x_2 \qquad x_{i-1},\ x_i \equiv x,\ x_{i+2},\ \ldots\ x_N$

theses monomials of the type $z_{\sigma_1}^{x_1}\,(z z)_{\sigma_i\sigma_{i+1}}^{x}\cdots z_{\sigma_N}^{x_N}$ are independent as soon as the z_k's are $\neq$ But $z_{\sigma(i)}\, z_{\sigma(i+1)}$ appear with the same exponent. In other words, consider the constraint as a function of $x_1, x_2, \ldots x_{i-1}, x, x_{i+2} \ldots x_N$ and identify independent terms.

$$p\left(A_\sigma + A_{\sigma\tau_{i,i+1}}\right) + q\, z_{\sigma(i)} z_{\sigma(i+1)}\left(A_\sigma + A_{\sigma\tau_{i,i+1}}\right) = (p+q)\left\{A_\sigma z_{\sigma(i+1)} + A_{\sigma\circ\tau_{i,i+1}} z_{\sigma(i)}\right\}$$

$\tau_{i,i+1}$ = transposition $i \leftrightarrow i+1$

$$\boxed{\frac{A_{\sigma\tau_{i,i+1}}}{A_\sigma} = -\frac{q\, z_{\sigma(i)} z_{\sigma(i+1)} - (p+q)\, z_{\sigma(i+1)} + p}{q\, z_{\sigma(i)} z_{\sigma(i+1)} - (p+q)\, z_{\sigma(i)} + p}}$$

A_σ known $\Rightarrow$ $A_{\sigma\tau_{i,i+1}}$ KNOWN

ALL A_σ are KNOWN (up to A_{Id})

CYCLICITY and QUANTIZATION : $\boxed{\psi(x_1, x_2, \dots, x_N + L) = \psi(x_N, x_1, x_2, \dots, x_{N-1})}$

$$\sum_\sigma A_\sigma z_{\sigma(1)}^{x_1} \cdots z_{\sigma(N)}^{x_N + L} = \sum_{\sigma \in \mathfrak{S}_N} A_\sigma z_{\sigma(1)}^{x_N} z_{\sigma(2)}^{x_1} \cdots z_{\sigma(N)}^{x_{N-1}}$$

circular permutation π $\begin{cases} \pi(1) = N \\ \pi(2) = 1 \\ \vdots \\ \pi(N) = N-1 \end{cases}$ $\quad \mathfrak{S}_N \longrightarrow \mathfrak{S}_N$, $\sigma \longmapsto \sigma \circ \pi$ bijection

$$= \sum_{\mathfrak{S}_N} A_{\sigma\circ\pi} z_{\sigma(N)}^{x_N} z_{\sigma(1)}^{x_1} \cdots z_{\sigma(N-1)}^{x_{N-1}}$$

compare r.h.s with l.h.s and use idependence again :

$$\boxed{\frac{A_{\sigma\circ\pi}}{A_\sigma} = z_{\sigma(N)}^{L}}$$

But $\pi = \tau_{N-1\,N}\, \tau_{N-2\,N-1} \cdots \tau_{i\,i+1} \cdots \tau_{23}\, \tau_{12}$ i.e

$$\frac{A_{\sigma\circ\pi}}{A_\sigma} \left(= \frac{A_{\sigma\tau_{N-1N}\cdots\tau_{12}}}{A_{\sigma\tau_{N-1,N}\tau_{N-2\,N-2}\cdots\tau_{23}}} \times \frac{A_{\sigma\cdots\tau_{23}}}{A_{\sigma\cdots\tau_{34}}} \times \cdots\cdots \frac{A_{\sigma\tau_{N-1N}}}{A_\sigma} \right)$$

$$= \frac{A_{\sigma\tau_{N-1N}}}{A_\sigma} \times \frac{A_{\sigma\tau_{N-1N}\tau_{N-2N-1}}}{A(\sigma\tau_{N-1N})} \times \cdots \times \frac{A_{\sigma}(\overbrace{\tau_{N-1N}\cdots\tau_{12}}^{\pi})}{A(\sigma\tau_{N-1N}\cdots\tau_{23})}$$

for the intermediate $\tilde\sigma$ $\quad \tilde\sigma = \sigma\tau_{N-1,N}\cdots\tau_{i+1,i+2}$ $\quad \tilde\sigma(i) = i$, $\tilde\sigma(i+1) = N$ $\quad \left(\frac{A_{\tilde\sigma\tau_{i,i+1}}}{A_{\tilde\sigma}}\right)$

and we apply the previous relation to obtain

$$z_{\sigma(N)}^{L} = (-1)^{N-1} \frac{q z_{\sigma(N-1)} z_{\sigma(N)} - (p+q) z_{\sigma(N)} + p}{q z_{\sigma(N-1)} z_{\sigma(N)} - (p+q) z_{\sigma(N-1)} + p} \times \frac{q z_{\sigma(N-2)} z_{\sigma(N)} - (p+q) z_{\sigma(N)} + p}{q z_{\sigma(N-2)} z_{\sigma(N)} - (p+q) z_{\sigma(N-2)} + p}$$

$$\times \cdots\cdots \times \frac{q z_{\sigma(1)} z_{\sigma(N)} - (p+q) z_{\sigma(N)} + p}{q z_{\sigma(1)} z_{\sigma(N)} - (p+q) z_{\sigma(1)} + p}$$

$$z_{\sigma(N)}^{L} = (-1)^{N-1} \prod_{k \neq N} \frac{q z_{\sigma(k)} z_{\sigma(N)} - (p+q) z_{\sigma(N)} + p}{q z_{\sigma(k)} z_{\sigma(N)} - (p+q) z_{\sigma(k)} + p}$$

change notations $\sigma(N) = i$ $\quad \sigma(k) = j \neq i$

$$\boxed{z_i^{L} = (-1)^{N-1} \prod_{j \neq i} \frac{q z_i z_j - (p+q) z_i + p}{q z_i z_j - (p+q) z_j + p}}$$

Bethe equations.

Remarks (i) Translation operator : $x_i \to (x_i)+1$: τ
add 1 to all coordinates. The dynamics commutes with τ → ψ must also be an eigenvector of τ

indeed : $\psi(x_1+1, \dots x_N+1) = \underbrace{(z_1 \cdots z_N)}_{\text{eigenvalue of translation operator}} \psi(x_1, \dots x_N)$

$$\tau^L = 1 \implies \underline{(z_1 \cdots z_N)^L = 1}$$

this is indeed the case as you can check from the Bethe Eqs.

(ii) The TASEP case : $\boxed{\begin{matrix} p=1 \\ q=0 \end{matrix}}$

$$z_i^{-L}(1-z_i)^N = -\prod_j (1-z_j) = \text{CONSTANT}$$

The wave-function is a determinant

$$\boxed{\psi(x_1, \dots x_N) = \det\left(\frac{z_i^{x_j}}{(1-z_i)^j}\right)} \quad 1 \le i, j \le N$$

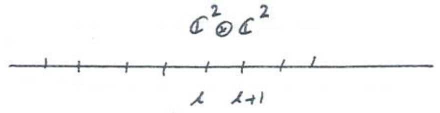
(iii) The SEP case : $p = q = \frac{1}{2}$

$$z_i^L = \prod_{j \neq i} - \frac{z_i z_j - 2z_i + 1}{z_i z_j - 2z_j + 1} \quad \text{identical to Bethe '31}$$

indeed one can write : $M = \sum_{i=1}^{L} M_{i,i+1}^{Loc} \rightsquigarrow \mathbb{1}^{i-1} \otimes M_{i,i+1}^{Loc} \otimes \mathbb{1}^{L-i-1}$

$$\underbrace{M_{i,i+1}}_{4\times4} = \begin{array}{c} \text{basis:} \\ \\ \\ \\ \end{array} \begin{array}{cccc} 11 & 10 & 01 & 00 \\ \end{array}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -p & q & 0 \\ 0 & p & -q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



state space : $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \quad \otimes \mathbb{C}^2$ (L ten)

Rewrite it as a spin-chain operator : $S^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $S^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $S^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

spin basis $(|\uparrow\rangle, |\downarrow\rangle) \equiv (1, 0)$ acts on the local space $(\mathbb{C}^2)_i$

Pauli-matrices : $S_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ creates a particle at site i

$S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ destroys a particle $\quad \begin{cases} S^+ = \frac{S^x + iS^y}{2} \\ S^- = \frac{S^x - iS^y}{2} \end{cases}$

$$M_{i,i+1} = \underbrace{p\, S_i^- S_{i+1}^+ + q\, S_i^+ S_{i+1}^-}_{\text{hopping term}} + \frac{1}{4} S_i^z S_{i+1}^z - \frac{1}{4}$$

$M = \sum M_{i,i+1}$ special case $p = q = \frac{1}{2}$ $\quad M = \sum_i \frac{S_i^- S_{i+1}^+ + S_i^+ S_{i+1}^-}{2} + \frac{1}{4} S_i^z S_{i+1}^z - \frac{1}{4}$

$$H = M = -\frac{L}{4} + \frac{1}{4} \sum \underbrace{\vec{S}_i \cdot \vec{S}_{i+1}}_{\text{nearest neighbour interact of QUANTUM SPINS}} \qquad \vec{S}_i = \begin{pmatrix} S_i^x \\ S_i^y \\ S_i^z \end{pmatrix}$$

this exactly HEISENBERG'S MODEL of Quantum Magnetism – SOLVED BY H. BETHE.

$p = q = \frac{1}{2}$ Hermitian operator $i\hbar\partial_t\psi = H.\psi$ real eigenvalues. $D \neq 0$ NON-HERMITIAN ↗ oscillations

EXACT CALCULATIONS IN THE CASE of TASEP :

Extracting exact results from the Bethe Eqs : a daunting task ! Fortunately, for the TASEP case, analytical progress can be made in a rather elementary way. We shall show how to extract the spectral gap & give some results on spectral degeneracies in the Markov Matrix.

Analysis of the TASEP Bethe Ansatz equations :

Pour TASEP, $p=1, q=0$ the Bethe Eqs become $z_i^{-L}(-1+z_i)^N = -\prod_j(-1+z_j)$

change variables to $Z_i = \frac{2}{z_i} - 1$ i.e $z_i = \frac{2}{Z_i+1}$

$$\boxed{(1+Z_i)^{L-N}(1-Z_i)^N = -2^L\prod_{j=1}^{N}\frac{Z_j-1}{Z_j+1}}$$

r.h.s. does not depend on i

Effective decoupling to a 1 variable pb.

Eigenvalue $\boxed{2E = -N + \sum_{i=1}^{N} Z_i}$

Procedure for solving these equations ⇝ Self-consistently

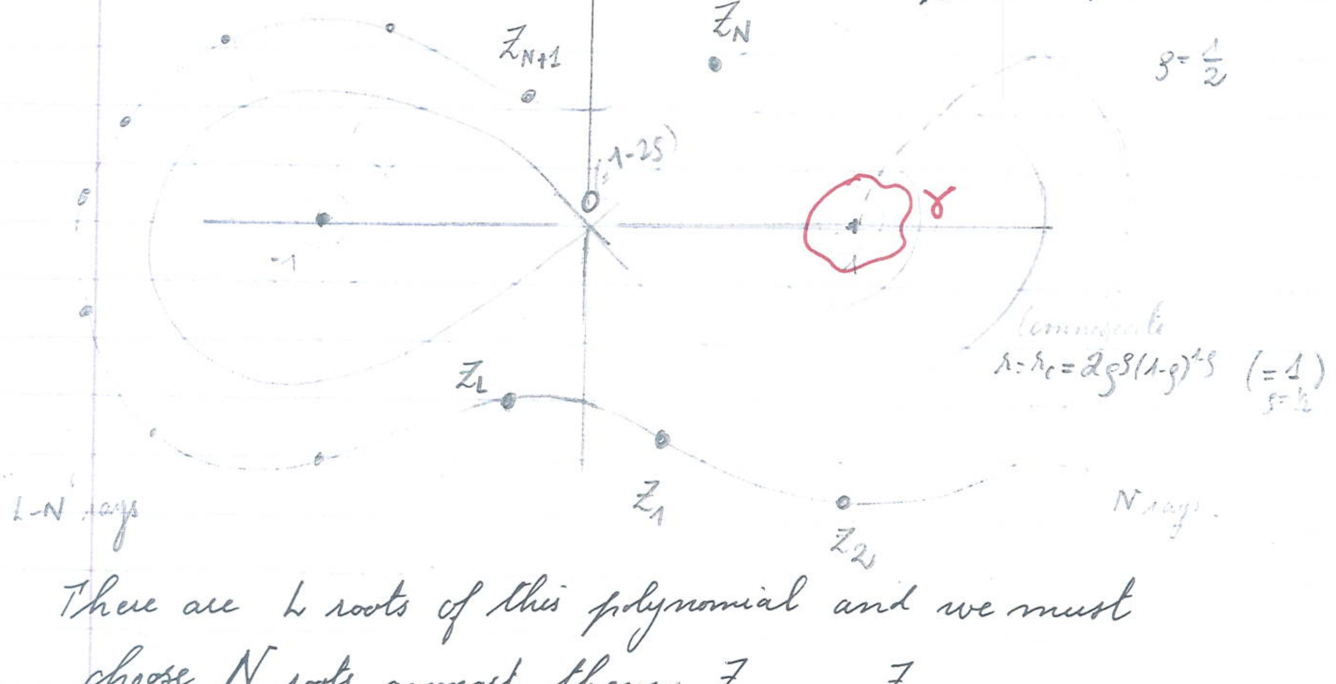
CONSIDER $Y \in \mathbb{C}$ and the polynomial eq $(1+Z)^{L-N}(1-Z)^N = Y$

$\Rightarrow$ $|1-Z|^\rho\,|1+Z|^{1-\rho} = r$ where $r = |Y|^{1/L}$ $\quad\rho = N/L$ (typ. $\rho = 1/2$)

defines a LOCUS in the complex plane [for $\rho = \frac{1}{2}$

$d_1 d_2 =$ cste]

CASSINI OVALS



There are L roots of this polynomial and we must choose N roots amongst them: $Z_{c(1)}, \dots Z_{c(N)}$

$c : \{1, \dots N\} \longmapsto \{1, \dots L\}$ is a choice function

Now, solve the self-consistent equation $\boxed{A_c(Y) = Y}$ where

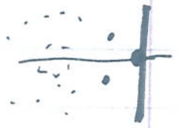
$$A_c(Y) = -2^L \prod_{j=1}^{N} \frac{Z_{c(j)}(Y) - 1}{Z_{c(j)}(Y) + 1}$$

From this fixed point equation, determine the value Y_c of Y and the energy corresponding to the choice set c: $2E_c(Y_c) = -N + 2\sum_j Z_{c(j)}(Y_c)$

Claim (not proved): this procedure determines the spectrum of M
(Note that there are $\binom{L}{N}$ possible choice functions which is precisely the dimension of M)

Calculation of the GAP:

i.e λ eigenvalue $\neq 0$ with $Re(\lambda)$ largest



it determines $\left(= \frac{1}{Re(\lambda)}\right)$ the largest relaxat° time T to stationarity

$\boxed{T \sim L^{z}}$ z dynamical factor here $z = \frac{3}{2}$ $p \neq q$

First, consider the choice function $c_0(j) = j$ which selects the roots $Z_1, \dots Z_N$ (that $\to +1$ when $r \to 0$). The A function and the eigenvalue associated to this choice are given by

$$\boxed{A_0(Y) = -2^L \prod_{j=1}^{N} \frac{Z_j - 1}{Z_j + 1}} \quad \text{and} \quad \boxed{2E_0 = -N + \sum_{j=1}^{N} Z_j}$$

These functions can be calculated exactly in the limit $Y \to 0$
we consider a contour γ that encircles $+1$ such that for sufficiently small values of Y the roots $(Z_1, \dots Z_N)$ are inside γ whereas $(Z_{N+1}, \dots Z_L)$ are outside γ.

Let $h(Z)$ a function analytic in a domain containing γ
et we call $P(Z) = (1-Z)^N (1+Z)^{L-N}$

Then we have

$$\sum_{k=1}^{N} h(Z_k) = \frac{1}{2i\pi} \oint_\gamma \frac{P'(Z)}{P(Z) - Y} h(Z)\, dZ$$

$$= \frac{1}{2i\pi} \sum_{k=0}^{\infty} Y^k \oint \frac{P'(Z)}{P^{k+1}(Z)} h(Z)\, dZ = \underbrace{\frac{1}{2i\pi} \oint \frac{P'}{P} h}_{N\, h(1)} + \frac{1}{2i\pi} \sum_{k=1}^{\infty} \frac{Y^k}{k} \int \frac{h'(Z)}{P^k(Z)}$$

we take $h(Z) = \ln \frac{1+Z}{2}$ and $h(Z) = Z - 1$

to deduce that

$$\ln \frac{A_0(Y)}{Y} = \sum_{k=1}^{\infty} \binom{kL}{kN} \frac{Y^k}{k\, 2^{kL}}$$

and

$$2E_0 = -\sum_{k=1}^{\infty} \binom{kL-2}{kN-1} \frac{Y^k}{k\, 2^{kL-1}}$$

NB| The expression for $\ln \frac{A_0(Y)}{Y}$ was obtained by calculating

$$\frac{A_0(Y)^N}{Y^N} = (-1)^N 2^{LN} \prod_{j=1}^{N} \frac{(Z_j-1)^N}{(Z_j+1)^N} \Big/ \prod_j (1+Z_j)^{L-N}(1-Z_j)^N$$

$$= 2^{LN} \prod_j \frac{1}{(1+Z_j)^L} \cancel{\times(-1)^{N^2+N}}$$

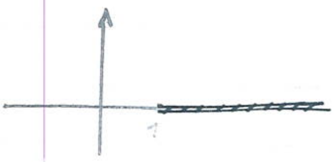
i.e $$\ln\left(\frac{A_0(Y)}{Y}\right)^N = -\sum_j \ln\left(\frac{1+Z_j}{2}\right)^L$$ no constant $2ik\pi$ because when $Y \to 0$ $Z_j \to 0$ $A_0(Y) \to 1$

we now ~~continue~~ take in these expressions $L \to \infty$, $N \to \infty$, $\frac{N}{L} = \rho$

$$\ln \frac{A_0(Y)}{Y} \to \frac{1}{\sqrt{2\pi\rho(1-\rho)}} \frac{1}{\sqrt{L}} Li_{3/2}\left(\frac{Y}{r_c^L}\right)$$ with $r_c = 2\rho^\rho(1-\rho)^{1-\rho}$

and $$Li_{3/2}(z) = \sum_{k\geq 1} \frac{z^k}{k^{3/2}} \equiv \frac{z}{\Gamma(3/2)} \int_0^\infty \frac{t^{\frac{3}{2}-1}}{e^t - z} dz$$

defined in $\mathbb{C} \setminus [1, \infty[$



similarly: $E_0(Y) \propto Li_{5/2}$

From the series expansion above, we observe that $A_0(Y) = Y$ has the solution $Y = 0$ which yields $Z_j = 1$ $j = 1, \ldots N$

This corresponds to the ground state of the Markov Matrix.

The first excited state:

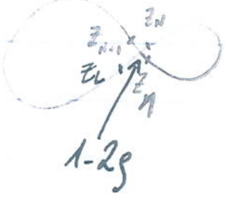
it will be obtained via the choice $Z_1, \ldots Z_{N-1}$ and Z_{N+1}

i.e. $c_1(j) = j$ for $j = 1, 2, \ldots N-1$ and $c_1(N) = N+1$.

Then $$A_1(Y) = A_0(Y) \frac{Z_{N+1}-1}{Z_{N+1}+1} \frac{Z_N+1}{Z_N-1}$$ we want to solve $A_1(Y) = Y$ and then

calculate $2E_1 = 2E_0 + (Z_{N+1} - Z_N)$

+ define $\frac{Y}{r_c^L} = -e^{u\pi}$ $\rightsquigarrow$ $|Y|^{1/L} \simeq r_c$ the limiting curve is the lemniscate

$\rightsquigarrow$ 

Z_N, Z_{N+1} are close to the double point of the lemniscate and can be calculated perturbatively

$$Z_k = 1 - 2\rho + \xi_k \begin{array}{l} \nearrow Z_1, \ldots Z_2; Z_N, Z_{N-1} \ldots \\ \searrow Z_{N+1}\ Z_{N+2} \ldots; Z_L, Z_{L-1} \ldots \end{array}$$

$$Z_N = 1 - 2\rho + 2i\frac{\sqrt{2\pi\rho(1-\rho)}}{\sqrt{L}}(u-i)^{1/2} + \mathcal{O}(1/L)$$

$$Z_{N+1} = 1 - 2\rho + 2i\cdot\frac{\sqrt{2\pi\rho(1-\rho)}}{\sqrt{L}}(u+i)^{1/2} + \mathcal{O}\left(\frac{1}{L}\right)$$

$$A_1(Y) = Y \implies \boxed{Li_{3/2}\left(-e^{u\pi}\right) = 2i\pi\left\{(u+i)^{1/2} - (u-i)^{1/2}\right\}}$$

$$u = 1{,}119\,068\;802\;804\;474\ldots$$

$$\boxed{E_1 = -2\sqrt{\rho(1-\rho)}\,\frac{6.509\,189\,337\ldots}{L^{3/2}} \pm \frac{2i\pi(2\rho-1)}{L}}$$

dynamical exponent

oscillation relaxation
→ kinematic wave.

A DISCUSSION of the DEGENERACIES:

If you diagonalize the Markov Matrix of TASEP on a ring you'll notice that many eigenvalues are degenerate. E.g. $L = 2N$ $\rho = 1/2$, one can draw a degeneracy table

L	N	$\binom{L}{N}$	singlets $d = 1$	$d = 2$	6	20	70 etc... multiplets
2	1	2	2				
4	2	6	4	1			
6	3	20	8	6			
8	4	70	16	24	1		
10	5	252	32	80	10		
12	6	924	64	240	60	1	
14	7	3432	128	672	280	14	
16	8	12870	256	1792	1120	112	1
18	9	48620	512	4608	4032	672	18

values of d : $d_r = \binom{2r}{r}$

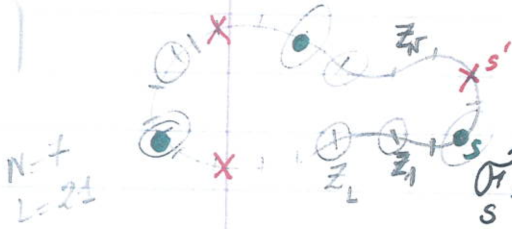
associated multiplicity in a system $L = 2N$ $m(d_r) = \binom{N}{2r} 2^{N-2r}$

sum rule $\sum m(d_r)\, d_r = \binom{2N}{N}$

Why do such degeneracies appear?

There is an equivalence relation amongst choice sets

L, N ; Call $\delta = \gcd(L, N)$



L Bethe roots

They form δ families of L/δ roots

$$\mathcal{F}_s = \{ Z_s, Z_{s+\delta}, Z_{s+2\delta}, \dots, Z_{s+L-\delta} \} \qquad 1 \leq s \leq \delta$$

Consider a choice set $\quad c : \{1, 2, \dots N\} \hookrightarrow \{1, 2, \dots L\}$

that selects N roots amongst L.

Suppose that the family $\mathcal{F}_s \subset c$.

But there is an s' such that $\mathcal{F}_{s'} \cap c = \emptyset$

We can build a different choice set $\underline{\hat{c} = (c \setminus \mathcal{F}_s) \cup \mathcal{F}_{s'}}$

CLAIM: c and $\hat{c}$ correspond to the same eigenvalue

$Z_1, \dots Z_L$ are the roots of $\quad (1-Z)^N (1+Z)^{L-N} = Y$

$\mathcal{F}_s = \{Z_s, Z_{s+\delta}, \dots Z_{s-\delta+L}\}$ are roots of $\quad (1-Z)^{\frac{N}{\delta}} (1+Z)^{\frac{L-N}{\delta}} = y_s$

where y_s is a δ-th root of Y $\quad 1 \leq s \leq \delta$

$y_s = |Y|^{\frac{1}{\delta}} e^{i \, Arg(y_s)}$ $\qquad \arg(y_1) < \arg(y_2) < \dots$

we deduce $\displaystyle \sum_{Z_i \in \mathcal{F}_s} Z_i = \frac{2N-L}{\delta}$ $\qquad \displaystyle \underbrace{\prod_{Z_i \in \mathcal{F}_s} \frac{Z_i - 1}{Z_i + 1} = 1}$

$E_c = E_{\hat{c}}$

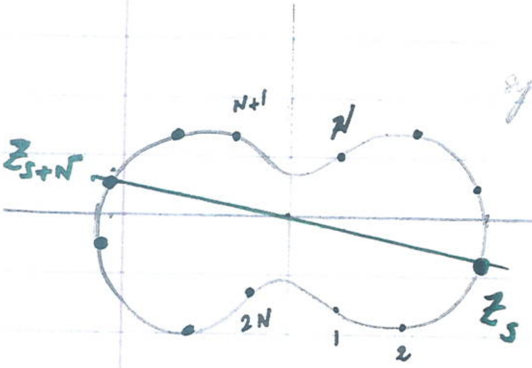
this implies that $\mathcal{F}_s$ does not contribute to A_c

i.e $A_c = A_{\hat{c}}$ → same Bethe equation

→ same value of Y.

If we believe that the B.A. is complete (works by LANGLANDS?) and that a choice function uniquely determines an eigenvalue, then counting the degeneracies reduces to a purely combinatorial problem.

Simple example: $L = 2N \rightsquigarrow (1-Z)^N(1+Z)^N = Y$



symetric oval

$Z \leftrightarrow -Z$ symmetry

$\delta = N$

N families $\{Z_S, Z_{S+N}\} \equiv \{Z_S, -Z_S\}$

a choice set c of N roots is equivalent to $\hat{c}$ obtained by replacing a diameter by another one

i.e the eigenvalue depends only on unpaired roots.

degree of degeneracy : # of $\hat{c}$ having the same single roots as c. Suppose c has $\boxed{N-2r}$ single roots $\overset{2r}{\underset{\text{i.e } r \text{ PAIRS}}{}}$ paired roots

then $\hat{c}$ has the same single roots but different paired roots.

N - 2r singles NO CHOICE

there are 2r pair possibles we must choose r amongst them : $\binom{2r}{r} = d_r$

Number of multiplets of a given degree: they differ by the choice of the singles which are N-2r

$$m(d_r) = \frac{2N(2N-2)(2N-4)\,..\,(2N-2(N-2r-1))}{(N-2r)!}$$

$$= 2^{N-2r}\binom{N}{2r}$$

The general case $(L, N) = \delta$ is more complicated but can also be done.

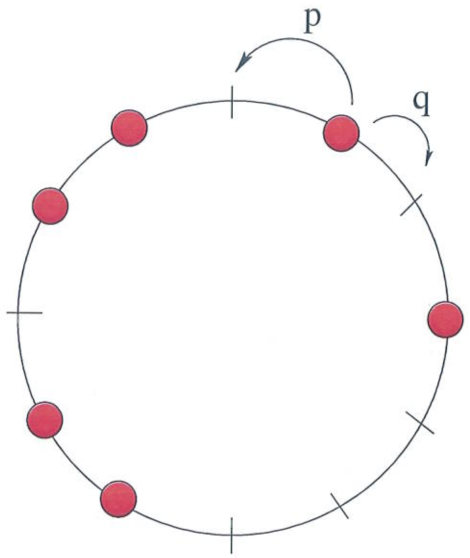
APPENDIX
to COURSE 2

Alternative set of Notes taken from lectures given in Leuven (June 2013).

These slides may be easier to read (but the general case is not given, and many details are missing).

The Periodic ASEP

We consider the asymmetric exclusion process on a homogeneous ring: jumps in the positive (trigonometric) direction occur with rate p, jumps in the negative direction occur with rate q.



L SITES

N PARTICLES

$$\Omega = \binom{L}{N}$$

CONFIGURATIONS

By rescaling time we can always make $p \to 1$ and $q \to x = \frac{q}{p}$. We shall perform this rescaling at the end of our calculations.

The Eigenvalue Problem for the Markov Matrix

A configuration of the system at time t can be specified by the position of the N particles on the ring of size L:

$$1 \leq x_1 < \ldots < x_N \leq L\,.$$

With this representation, the eigenvalue equation becomes:

$$\begin{aligned} & E\psi(x_1, \ldots, x_N) = \\ & p\sum_i{}' [\psi(x_1, \ldots, x_i - 1, \ldots, x_N) - \psi(x_1, \ldots, x_i, \ldots x_N)] \\ + \quad & q\sum_i{}' [\psi(x_1, \ldots, x_i + 1, \ldots, x_N) - \psi(x_1, \ldots, x_i, \ldots x_N)] \end{aligned}$$

where the sum are restricted over the indices i such that $x_{i-1} < x_i - 1$ and over the indices j such that $x_j + 1 < x_{j+1}$: These conditions ensure that the corresponding jumps are allowed.

This equation looks like a discrete Laplacian but with special boundary conditions.

ASEP: An Integrable Spin Chain

MAPPING TO A NON-HERMITIAN SPIN CHAIN

$$M = \sum_{l=1}^{L} \left(q\mathbf{S}_l^+\mathbf{S}_{l+1}^- + p\mathbf{S}_l^-\mathbf{S}_{l+1}^+ + \frac{p+q}{4}\mathbf{S}_l^z\mathbf{S}_{l+1}^z - \frac{p+q}{4} \right)$$

Exercice : Map it into an
XXZ Spin chain

Complex Eigenvalues $M\psi = E\psi$:

- Ground State : $E = 0$, $P = \Omega^{-1}$ (non-degenerate).
- Excited States : $\Re(E) < 0$ (Perron-Frobenius).

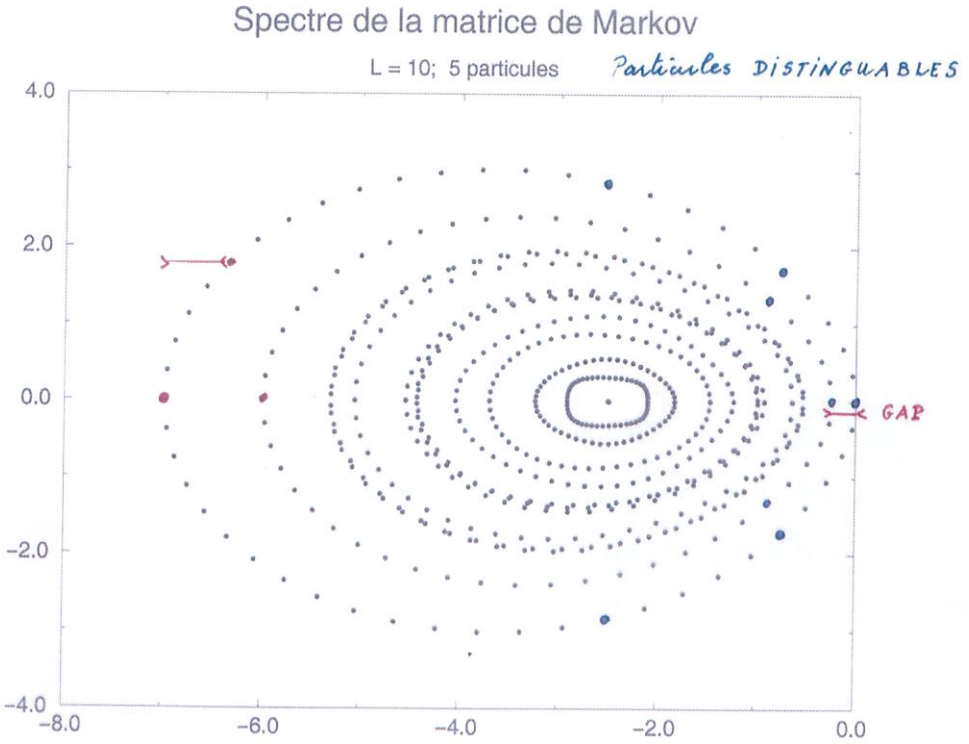
Excitations correspond to relaxation times.

$$\boxed{\text{TASEP} : p = 1, q = 0}$$

Spectrum

Complex Eigenvalues $M\psi = E\psi$ with $\Re(E) \leq 0$ (Perron-Frobenius)

- Ground State $E = 0$ corresponds to the stationary state.
- Excited States → relaxation times.



The single particle case

For $N = 1$, the eigenvalue equation reads

$$E\psi(x) = p\psi(x-1) + q\psi(x+1) - (p+q)\psi(x)\,,$$

with $1 \leq x \leq L$ and where periodicity is assumed: $\psi(x+L) = \psi(x)\,.$
This is a linear recursion of order 2. Thus

$$\psi(x) = Az_+^x + Bz_-^x\,,$$

where $r = z_\pm$ are the two roots of the characteristic equation

$$qr^2 - (E+p+q)r + p = 0\,.$$

Because of the periodicity condition at least one of the two characteristic values is a L-th root of unity (Since $z_+z_- = p/q$, only one of them can be a root of unity when $p \neq q$).

The general solution is

$$\psi(x) = Az^x \qquad \text{with} \qquad z^L = 1$$

This is a *plane wave* with momentum $2k\pi/L$ and with eigenvalue

$$E = \frac{p}{z} + qz - (p+q)$$

The two particles case

When $N = 2$, the exclusion condition begins to play a role and the general eigenvalue equation has to be be split into two different cases.

- Generic case: x_1 and x_2 are separated by at least one empty site

$$\begin{aligned} E\psi(x_1, x_2) &= p\left[\psi(x_1 - 1, x_2) + \psi(x_1, x_2 - 1)\right] \\ &+ q\left[\psi(x_1 + 1, x_2) + \psi(x_1, x_2 + 1)\right] \\ &- 2(p + q)\psi(x_1, x_2) \end{aligned}$$

- Adjacency case: Here $x_2 = x_1 + 1$, some jumps are forbidden and the eigenvalue equation becomes

$$E\psi(x_1, x_1 + 1) = p\psi(x_1 - 1, x_1 + 1) + q\psi(x_1, x_1 + 2) - (p + q)\psi(x_1, x_1 + 1)$$

This equation differs from the generic equation for $x_2 = x_1 + 1$: There are missing terms. Equivalently, one can impose the generic equation everywhere and add the *cancellation boundary condition:*

$$p\psi(x_1, x_1) + q\psi(x_1 + 1, x_1 + 1) - (p + q)\psi(x_1, x_1 + 1) = 0$$

Bethe Wave Function for N=2

In the generic case, particles jump totally independently: the solution of the generic equation can be written as as a product of plane waves

$$\psi(x_1, x_2) = Az_1^{x_1} z_2^{x_2}$$

with the eigenvalue

$$E = p\left(\frac{1}{z_1} + \frac{1}{z_2}\right) + q\,(z_1 + z_2) - 2(p + q)$$

However, the cancellation condition will not be satisfied in general. ●
Crucial Observation: The eigenvalue E is invariant by the permutation $z_1 \leftrightarrow z_2$: there are **two** plane waves $Az_1^{x_1} z_2^{x_2}$ and $Bz_2^{x_1} z_1^{x_2}$ with the **same** eigenvalue E.
One should try a linear combination of plane-waves of the form:

$$\psi(x_1, x_2) = A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_2^{x_1} z_1^{x_2}$$

where the amplitudes A_{12} and A_{21} are yet arbitrary but can be chosen to fulfill the adjacency cancellation condition: **Bethe Ansatz** (Bethe, 1931)

Calculation of the Amplitudes Ratio

The adjacency cancellation condition will be fulfilled if the amplitudes satisfy

$$(p+qz_1z_2)\,(A_{12}+A_{21}) = (p+q)(A_{12}z_2 + A_{21}z_1)$$

Equivalently

$$\frac{A_{21}}{A_{12}} = -\frac{qz_1z_2-(p+q)z_2+p}{qz_1z_2-(p+q)z_1+p}$$

The eigen-equation is now satisfied in all the cases.

We must now impose the boundary conditions (here periodicity): this will **quantify** the Bethe roots z_1 and z_2.

Periodicity condition. The Bethe Equations

We now implement the periodicity condition that takes into account the fact that the system is defined on a ring. This constraint can be written as follows for $1 \leq x_1 < x_2 \leq L$:

$$\psi(x_1, x_2) = \psi(x_2, x_1 + L)$$

i.e., $A_{12} z_1^{x_1} z_2^{x_2} + A_{21} z_2^{x_1} z_1^{x_2} = A_{12} z_1^{x_2} {z_2}^{x_1+L} + A_{21} z_2^{x_2} {z_1}^{x_1+L}$

This leads to a new relation between the amplitudes:

$$\frac{A_{21}}{A_{12}} = z_2^L = \frac{1}{z_1^L}$$

Using the known value of the amplitudes-ratio, we deduce

$$\begin{aligned} z_1^L &= -\frac{qz_1z_2 - (p+q)z_1 + p}{qz_1z_2 - (p+q)z_2 + p} \\ z_2^L &= -\frac{qz_1z_2 - (p+q)z_2 + p}{qz_1z_2 - (p+q)z_1 + p} \end{aligned}$$

These are the Bethe Ansatz Equations for $N = 2$.

N=3 (and larger)

For a system containing three particles, located at $x_1 \leq x_2 \leq x_3$, the generic equation can be written from as above. But now, the special adjacency cases are more complicated.

- **Two-Body collisions:** *Two particles are next to each other and the third one is 'far apart'.* This reduces to $N = 2$ (with a spectator).

There are now two equations that correspond to the two cases $x_1 = x \leq x_2 = x + 1 \ll x_3$ and $x_1 \ll x_2 = x \leq x_3 = x + 1$:

$$p\psi(x, x, x_3) + q\psi(x+1, x+1, x_3) - (p+q)\psi(x, x+1, x_3) = 0$$
$$p\psi(x_1, x, x) + q\psi(x_1, x+1, x+1) - (p+q)\psi(x_1, x, x+1) = 0$$

- **Triple collision:** the three particles are adjacent, with $x_1 = x$, $x_2 = x + 1$ and $x_3 = x + 2$. The cancellation condition becomes

$$\begin{array}{ll} p & [\psi(x, x, x+2) + \psi(x, x+1, x+1)] + \\ q & [\psi(x+1, x+1, x+2) + \psi(x, x+2, x+2)] \\ - & (p+q)\psi(x, x+1, x+2) - (p+q)\psi(x, x+1, x+2) = 0 \end{array}$$

Not a new constraint, just a linear combination of the Two-Body collisions.

Bethe Ansatz for N=3

The fact that 3-body collisions 'factorise' into 2-body collisions is the *crucial property* at the very heart of the Bethe Ansatz.
The plane wave $\psi(x_1, x_2, x_3) = A z_1^{x_1} z_2^{x_2} z_3^{x_3}$ is a solution of the generic equation with the eigenvalue

$$E = p\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) + q\,(z_1 + z_2 + z_3) - 3(p+q)$$

However, collision conditions are not satisfied.
Note that E is invariant (degenerate) by permuting z_1, z_2, z_3.

- **TRY the Bethe Wave function:**

$$\begin{aligned}\psi(x_1, x_2, x_3) \quad = \quad & A_{123}\, z_1^{x_1} z_2^{x_2} z_3^{x_3} + A_{132}\, z_1^{x_1} z_3^{x_2} z_2^{x_3} + A_{213}\, z_2^{x_1} z_1^{x_2} z_3^{x_3} \\ & + A_{231}\, z_2^{x_1} z_3^{x_2} z_1^{x_3} + A_{312}\, z_3^{x_1} z_1^{x_2} z_2^{x_3} + A_{321}\, z_3^{x_1} z_2^{x_2} z_1^{x_3}\end{aligned}$$

i.e., $\boldsymbol{\psi(x_1, x_2, x_3) = \sum_{\sigma \in S_3} A_\sigma z_{\sigma(1)}^{x_1} z_{\sigma(2)}^{x_2} z_{\sigma(3)}^{x_3}}$ where σ is a 3-permutation.

- Fix all amplitude ratios by the 2-collision conditions.
- Quantize the Bethe roots z_1, z_2 and z_3 via the periodicity condition

$$\psi(x_1, x_2, x_3) = \psi(x_2, x_3, x_1 + L)$$

(This yields the Bethe equations).

The general N case

For general values of N, one can have k-body collisions with k=2,3...N. However, all multi-body collisions ‘factorize’ into 2-body collisions. *ASEP can be diagonalized by Bethe Ansatz.*

- **Bethe Wave function:**

$$\psi(x_1, x_2, \ldots, x_N) = \sum_{\sigma \in S_N} A_\sigma \, z_{\sigma(1)}^{x_1} z_{\sigma(2)}^{x_2} \cdots z_{\sigma(N)}^{x_N}$$

- Eigenvalue: $E = p \sum_{i=1}^{N} \frac{1}{z_i} + q \sum_{i=1}^{N} z_i - N(p+q)$
- Periodicity Condition (for $1 \leq x_1 < x_2 < \ldots < x_N \leq L$):

$$\psi(x_1, x_2, \ldots, x_N) = \psi(x_2, x_3, \ldots, x_N, x_1 + L)$$

The Bethe Ansatz Equations

$$z_i^L = (-1)^{N-1} \prod_{j \neq i} \frac{q z_i z_j - (p+q) z_i + p}{q z_i z_j - (p+q) z_j + p}$$

for $i = 1, \ldots N$.

Comments

- The Bethe equations are a system of N algebraic equations of order L whereas the characteristic polynomial of the Markov Matrix is of order 2^L.

- The translation operator T commutes with the dynamics. Indeed, for the Bethe wave function

$$\psi(x_1+1, x_2+1, \dots, x_N+1) = (z_1 \dots z_N)\ \psi(x_1, x_2, \dots, x_N)$$

Because $T^L = 1$ we have $(z_1 \dots z_N)^L = 1$ as seen directly from the Bethe equations.

- In the symmetric case ($p = q = 1$), the Bethe equations are identical to those derived by H. Bethe for the Heisenberg XXX chain, in 1931.

- For the TASEP case ($p = 1$ and $q = 0$), the wave function has the structure of a determinant:

$$\psi(x_1, \dots, x_N) = \det\left(\frac{z_i^{x_j}}{(1-z_i)^j}\right)$$

By expanding this determinant the generic form for the Bethe wave function is recovered. *It can also be shown directly that this determinant satisfies the eigenvalue equation and all the collision conditions.*

Bethe Equations for TASEP

For TASEP, the Bethe equations take a simpler form.
Making the change of variable $\zeta_i = \frac{2}{z_i} - 1$, these equations become

$$(\mathbf{1} - \zeta_\mathbf{i})^\mathbf{N}(\mathbf{1} + \zeta_\mathbf{i})^\mathbf{L-N} = -\mathbf{2}^\mathbf{L} \prod_{\mathbf{j=1}}^{\mathbf{N}} \frac{\zeta_\mathbf{j} - \mathbf{1}}{\zeta_\mathbf{j} + \mathbf{1}} \quad \text{for} \quad i = 1, \ldots N$$

Note that the r.h.s. is a constant independent of i: There is an effective DECOUPLING.

The corresponding eigenvalue is

$$\mathbf{E} = \frac{\mathbf{1}}{\mathbf{2}}(-\mathbf{N} + \sum_\mathbf{j} \zeta_\mathbf{j})$$

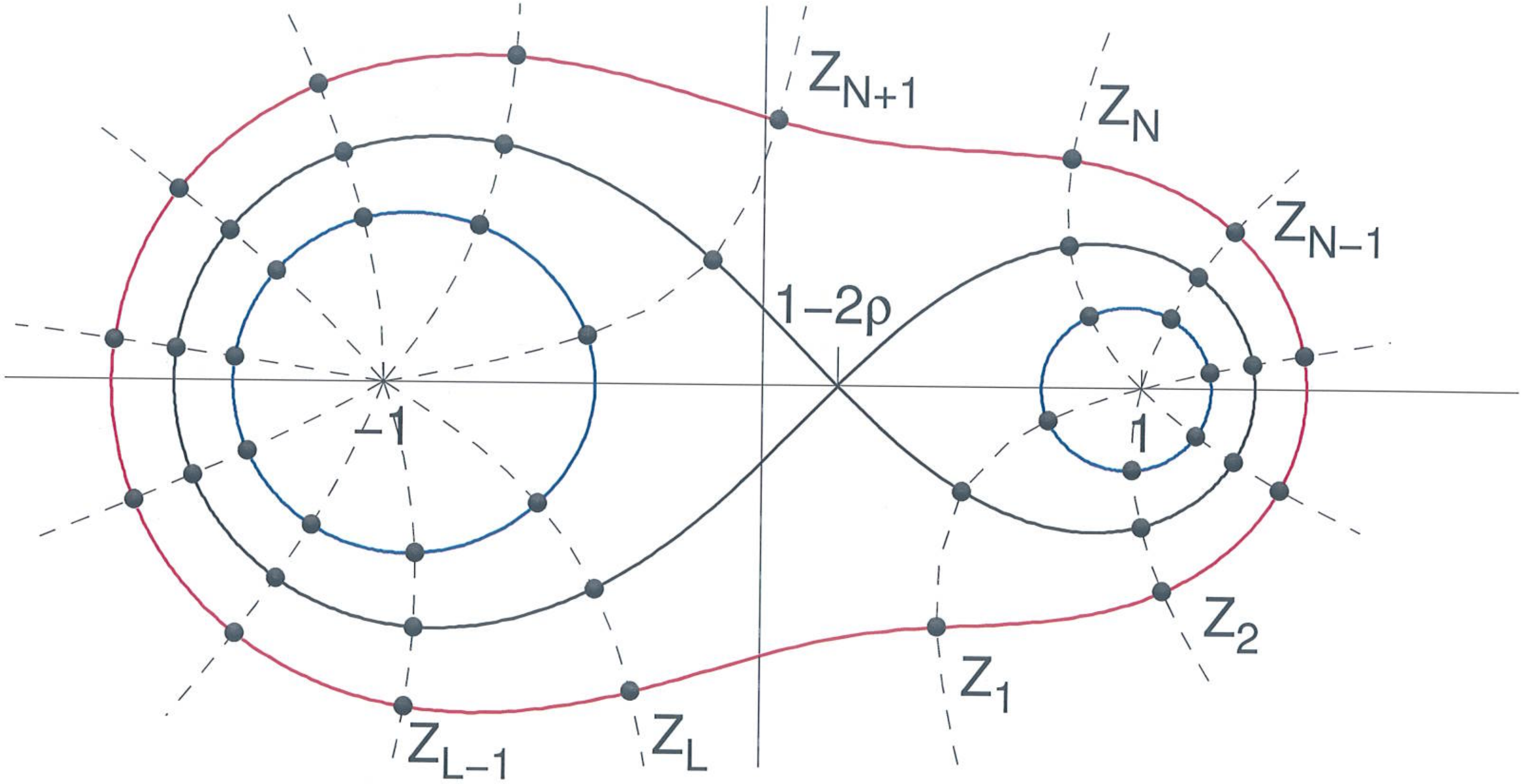
For a fixed value of the r.h.s. the roots lie on curves that satisfy

$$|1 - \zeta|^\rho \, |1 + \zeta|^{1-\rho} = const$$

where $\rho = N/L$ is the density.

Labelling the roots of the TASEP Bethe Equations

The loci of the roots (for $q = 0$) are remarkable curves: **The Cassini Ovals**



Procedure for solving the TASEP Bethe Equations

- For any given value of Y, *SOLVE* $(1 - z_i)^N (1 + z_i)^{L-N} = Y$. The roots are located on Cassini Ovals
- *CHOOSE N roots* $z_{c(1)}, \dots z_{c(N)}$ amongst the L available roots, with a *choice set* $c : \{c(1), \dots, c(N)\} \subset \{1, \dots, L\}$.
- *SOLVE* the self-consistent equation $\mathbf{A_c(Y) = Y}$ where

$$A_c(Y) = -2^L \prod_{j=1}^{N} \frac{z_{c(j)} - 1}{z_{c(j)} + 1} .$$

- *DEDUCE* from the value of Y, the $z_{c(j)}$'s and the energy corresponding to the choice set c :

$$2E_c(Y) = -N + \sum_{j=1}^{N} z_{c(j)} .$$

Calculation of the GAP

The first excited state is solution of a transcendental equation. For a density ρ:

$$E_1 = \qquad \underbrace{-2\sqrt{\rho(1-\rho)}\frac{6.509189337\ldots}{L^{3/2}}}_{\text{RELAXATION}} \pm \underbrace{\frac{2i\pi(2\rho-1)}{L}}_{\text{OSCILLATIONS}} .$$

- Non-diffusive: Largest relaxation time $T \sim L^z$ with $z = 3/2$ *(D. Dhar, L.H. Gwa and H. Spohn, D. Kim).*

- Oscillations $\rightarrow$ Traveling waves probed by dynamical correlations *(M. Barma, S. Majumdar, P. Krapivsky).*

- Classification of higher excitations *(J. de Gier and F.H.L. Essler, 2006).*