Lecture 1

Equilibrium vs Non-equilibrium

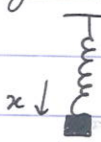
Basic Results on the ASEP

Integrable models and Statistical Physics:

When we learn physics, even at a very elementary level, a great emphasis is put on situations / examples / models where mathematical analysis can be carried out to full extent. If you try to remember what you have learnt, in each and every domain, you'll agree that what you've been taught was focused often on <u>exactly solvable cases</u> from which general inferences were drawn, and physical intuition was built / reasoning was learnt. Solvable (or integrable) systems are jewels or white stones in the path of a physicist, although they are extremely rare; more precisely they are non generic exceptions, i.e. exceptional.

Some <u>solvable systems</u>:

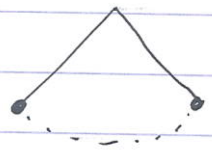
<u>Mechanics</u>:



1 d

harmonic oscillator $\ddot{x} + \omega^2 x = 0$ $\omega^2 = \frac{k}{m}$

Energy is conserved



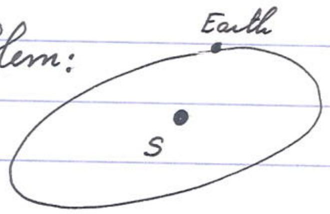
pendulum

$\ddot{\theta} + \omega^2 \sin\theta = 0$ $\omega^2 = \frac{g}{l}$

small angles $\sin\theta \to \theta$ harmonic, but in fact $\sin\theta$ can be solved (ELLIPTIC F.)

<u>Kepler Problem</u>:

Energy + angular momentum CONSERVED



elliptic / parabolic / hyperbolic motions

EXACTLY SOLVABLE

<u>Thermodynamics</u>: Perfect Gas

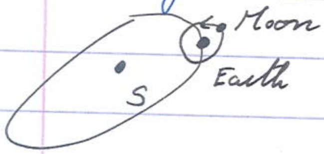
<u>Electromagnetism</u> / Optics: Field E by a charged sphere, cylinder wire (Gauss law) - Magnetic Field $\vec{B}$

Diffraction by an infinite slit
by a circular aperture (Bessel)

Quantum Mechanics : Harmonic oscillator
Hydrogen Atom | Schrödinger / Dirac Eq.
(There are more : Morse potential etc...)

Other remarkable examples : Friedman metrics in G-R
Chandrasekhar's black holes solutions
Each time, a beautiful mathematical structure, a GEM !

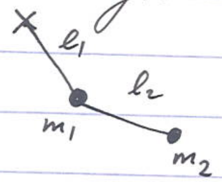
Note that you never solved the 3-body Pb



Poincaré proved that the 3-body Pb is not "integrable". The motion can not be described by simple, closed, formulae

→ chaotic behaviour.

Similarly, I doubt that you worked through the double-pendulum



Non-integrable models can be studied using "qualitative" techniques (topology), probabilistic methods, computers ... they seem to be generically different from the mathematical point of view.

The focus of these lectures will be on using integrability techniques for Non-eq. stat. mechanics and exact solution

Statistical Mechanics :

Aim: Deducing the macroscopic laws starting from the knowledge of microscopic dynamics: a fantastic reduction of complexity $10^{23} \to$ a few degrees of freedom to describe the world around us.

This come with a bonus : "universality"

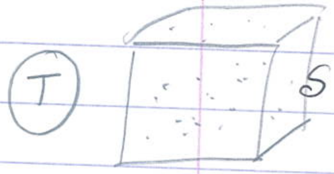
Physical systems, that differ deeply at the microscopic level, fall ~~exhibit~~ into "classes" (chapters in of books) that display similar behaviour, homeomorphic trends, at our scale.

Water and alcohols are chemically * but as far as fluid mechanics is concerned, they behave alike in certain range
the same Navier-Stokes equations, just change $\left\{ \begin{array}{l} \eta \\ \rho \\ \kappa : \text{thermal conductiv.} \end{array} \right.$

Magic: Reynolds Number $\rightarrow$ engineering models
air $\equiv$ water

Universality extends far beyond the similarity between materials e.g. phase transitions display universal features ~~are~~ fall into universality classes that encompass extremely different phenomena liquid/gas, paramagnetic/ferromagn., liquid crystals, superconductors.... e-weak symmetry breaking etc....

What do classical thermodynamics & equilibrium statistical mechanics teach us?



Let S be a system at thermal equilibrium at temperature T

Underlying a given macroscopic state of S, there is a HUGE number of microscopic configurations ($\sim 2^{\text{Avogadro}}$) that we do not perceive.

These microscopic configurations $\mathcal{C}$ can occur with a certain probability [if you consider an ensemble of identical systems S], and the knowledge of this probability distribution is the only information you need to investigate the thermodynamical behaviour of S a the macroscopic level.

Maxwell, Boltzmann, Gibbs:

$$p(\mathcal{C}) = \frac{e^{-E(\mathcal{C})/kT}}{Z}$$

energy, temperature

$k \simeq 1{,}38 \times 10^{-23}$

Z is a normalisation constant that ensure $\sum_{\text{all } \mathcal{C}} p(\mathcal{C}) = 1$

$$Z = \sum_{\text{all } \mathcal{C}} e^{-E(\mathcal{C})/kT}$$

$$\boxed{F = -kT \log Z}$$

(Boltzmann)

Free Energy (Thermod.) — Statistical mechanics

$F \xrightarrow{\text{use Maxwell's relations}} \begin{cases} \text{Pressure} & \text{Compressibility} \\ \text{Entropy} & \text{Heat capacities} \\ \text{Eq. of STATE} & \end{cases}$

change of state (F non analytic, Maxwell's construction)

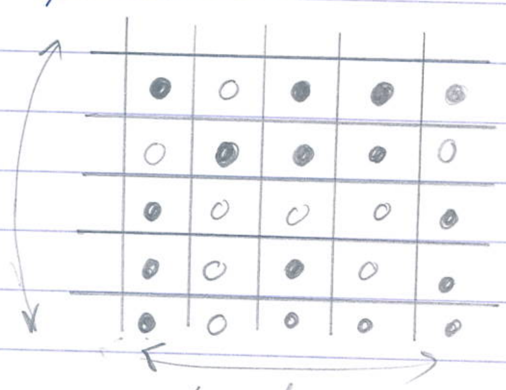
A macroscopic ($\sim 10^{23}$) observable O takes a value given by the average w.r.t. the canonical measure

$$\langle O \rangle = \sum_{C} O(C) \frac{e^{-E(C)/kT}}{Z}$$

Fluctuations do exist $O \neq \langle O \rangle$, but can only be observed at special conditions (criticality) or ~~also~~ for small systems

$$\mathrm{var}(O) = \langle O^2 \rangle - \langle O \rangle^2 = \sum_{C} O^2(C) \frac{e^{-E/kT}}{Z} - \left(\sum O \frac{e^{-E/kT}}{Z} \right)^2$$

Paradigm of Equilibrium Stat. Mech: **ISING MODEL**



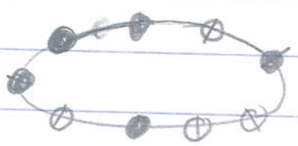
periodic BC

classical model

spins ● − down

○ + up

2dim $L \times L$ 2^{L^2} conf.



1d

Energy of a configuration (in kT-units) $\beta E = -J \sum_{\langle i,j \rangle} S_i S_j - h \sum_i S_i$, with $\beta = 1/kT$, $h = H/kT$

$S_i S_j = +1$ if they are alike

-1 if they are different

$$E(\mathcal{C}) = -J\,[\text{pairs of neighbours alike} - \text{pairs of neighbours} \neq] - h\,(\#\circ - \#\bullet) = \underline{-2J \times \text{pairs alike} - 2h\,\#\circ + \text{cste}}$$

neighbours $\times\times$ $\begin{matrix}\times\\\times\end{matrix}$

$$\#\circ + \#\bullet = L^d$$
$$\#\text{pairs} = L^d \times \frac{2d}{2} = d\,L^d$$

Variants of the model: Fix the number of $\bullet$ and $\circ$ a priori ("fixed magnetisation")

↳ Let the number of $\bullet$ and $\circ$ fluctuate: original Ising model

{ in 1d (Ising-Lenz)
{ and in 2d ($h=0$), Onsager, the free energy of the Ising model can be calculated exactly.

An integrable model that paved the way for modern statistical mechanics and phase transition theory.

- **Free energy of the 1d Ising model** (periodic lattice)

f = free energy per site

$$\frac{F}{L} = -kT \ln\left[e^{J}\cosh h + \left(e^{2J}\sinh^2 h + e^{-2J}\right)^{\frac{1}{2}}\right]$$

$-\frac{f}{kT} =$

(No phase transition at a finite temperature)

- **Free energy of the 2d Ising model in zero field** (Onsager '44)

$F/L^2 \equiv f$ (free energy / site)

$$\boxed{-\frac{f}{kT} = \frac{1}{2\pi}\int_0^{\pi} d\theta\, \ln\left[2\left(\cosh^2 2J + \sqrt{\sinh^4 2J - 2\sinh^2 2J\cos 2\theta + 1}\right)\right]}$$

TSVP→

Phase transition at a critical temperature $T_c > 0$ below T_c ferromagnetism spontaneous magn. above T_c: paramagnetic

→ A "mathematical" model displays a phase transition i.e. phase transitions do not require any "extra φ" beyond equilibrium statistical mechanics.

→ critical exponents in 2d.

I emphasize: this is *equilibrium* statistical mechanics configurations do not change, there is a no time-evolution. we have an energy function $\mathcal{C} \to E(\mathcal{C})$ and a sampling at microscopic level according to the canonical law. The rest is COMBINATORICS: there is no dynamics.

This is not physical: an equilibrium state is a dynamical state, at the microscopic level: things move, fluctuate (cf the Brownian Motion). The systems keeps evolving from one microconfiguration to another one, spins flip, particles move: WE SHOULD ADD DYNAMICAL RULES TO OUR MODELS.

HOW? Write a (quantum-mechanical) Hamiltonian → Heisenberg spin-chains.

A simpler model: COIN sound, physical, evolution rules a define an elementary dynamics.

This is the path we shall follow here and we shall use *stochastic*; *Markovian*, evolution rules.

Continuous time Markov dynamics:

- at time t, suppose that the system is in the micro configuration $\mathcal{C}$
- Between t and $t+dt$, the system can evolve $\mathcal{C} \to \mathcal{C}'$ with a probability:

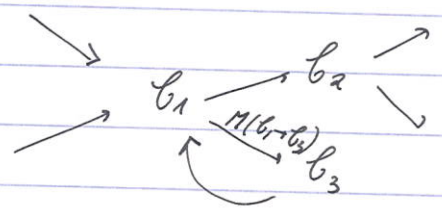
$$\underline{M(\mathcal{C} \to \mathcal{C}')}\, dt$$
(Transition RATE)

Crucial assumption: this transition rate (that may vary with time) DOES NOT DEPEND on the previous transitions that occured in the system; there is *no memory*: transitions at time t depend on what is there at t and *NOT* on the previous history: *MARKOV HYPOTHESIS*.

This is a dynamics in the configuration space.

CODING IN A MARKOV MATRIX:

GRAPH in config. space



Probability to stay in $\mathcal{C}$ between t & $t+dt$

$$= 1 - dt \sum_{\mathcal{C}'\neq\mathcal{C}} M(\mathcal{C}\to\mathcal{C}')$$

Useful to define $\underbrace{M(\mathcal{C},\mathcal{C})}_{\substack{\text{diagonal term}\\(<0)}} = -\sum_{\mathcal{C}\to\mathcal{C}'} M(\mathcal{C}\to\mathcal{C}')$

~~Markov~~ Master) Equation: $P_t(\mathcal{C})$ = prob. to be in conf. $\mathcal{C}$ at time t

$$\frac{dP_t(\mathcal{C})}{dt} = \sum_{\mathcal{C}'\neq\mathcal{C}} M(\mathcal{C}'\to\mathcal{C})\,P_t(\mathcal{C}') - P_t(\mathcal{C}) \overbrace{\sum_{\mathcal{C}'\to\mathcal{C}} M(\mathcal{C}\to\mathcal{C}')}^{\equiv -M(\mathcal{C}\to\mathcal{C})}$$

$$= \sum_{\mathcal{C}'} \left\{ M(\mathcal{C}'\to\mathcal{C})\,P_t(\mathcal{C}') - M(\mathcal{C}\to\mathcal{C}')\,P_t(\mathcal{C}) \right\}$$

$$\equiv \sum_{\mathcal{C}'} M(\mathcal{C}'\to\mathcal{C})\,P_t(\mathcal{C}')$$

Write it as a Matrix

$$M = \begin{array}{c|c} & \mathcal{C}' \\ \hline \mathcal{C} & \begin{pmatrix} & \circledast & \\ & \bullet & \end{pmatrix} \end{array}$$

a column contains all the rates to go from $\mathcal{C}'$ to any other configuration

$M(\mathcal{C}'\to\mathcal{C}') = -\sum_{\mathcal{C}\neq\mathcal{C}'} M(\mathcal{C}'\to\mathcal{C})$

$M(\mathcal{C}'\to\mathcal{C})$

CONVENTION:

$$M(\mathcal{C},\mathcal{C}') \equiv M(\mathcal{C}'\to\mathcal{C})$$

line (row) — column

The diagonal term = −(exit rate)

The sum of a column is 0 = $\boxed{(1\dots1)\,M = 0}$

Off diagonal terms ≥ 0

diagonal terms < 0

Markov Eq: $\boxed{\dfrac{dP_t}{dt} = M.P_t}$ $\qquad P_t = \begin{pmatrix} \vdots \\ P_t(\mathcal{C}) \\ \vdots \end{pmatrix}$ probability vector

We assume that M is <u>irreducible</u> (no subset of configurations stable by the dynamics i.e you can go from any $\mathcal{C}$ to $\mathcal{C}'$ in a finite # steps) → PERRON-FROBENIUS THM.

$\lambda \in$ Spectrum(M) : $\mathrm{Re}(\lambda) \leq 0$ $\Big\{$ True for finite systems. NOT ALWAYS FOR INFINITE SYST

There is a unique eigenvector of M with 0 eigenvalue; it is called the stationary state

$$\boxed{M P_{stat} = 0}$$

All the entries of P_{stat} are > 0

$$\underline{P_t(\mathcal{C}) \to P_{stat}(\mathcal{C})} \text{ qd } t \to \infty$$

P_{stat} satisfies:
$$\sum_{\mathcal{C}' \neq \mathcal{C}} M(\mathcal{C}' \to \mathcal{C})\, P_{stat}(\mathcal{C}') = P_{stat}(\mathcal{C}) \sum_{\mathcal{C}' \neq \mathcal{C}} M(\mathcal{C} \to \mathcal{C}')$$

HOW CAN WE CONSTRUCT a dynamics?

$M(\mathcal{C} \to \mathcal{C}')$: $N^2 - N$ rates to choose; arbitrarily?

assumptions: local changes (not always = SWENDSEN-WANG) such as spin flips, or spin exchanges

$\bullet \rightleftharpoons \circ$ $\qquad (\bullet\ \circ) \rightleftharpoons (\bullet\ \circ)$ chemical reactions.

EQUILIBRIUM DYNAMICS:

Suppose we wish to model a system at <u>thermal</u> equilibrium
What kind of Markov dynamics?

<u>PHYSICAL LAWS</u>: CONSTRAINTS on the RATES $\qquad$ BOLTZMANN + (ONSAGER)

(i) $P_{stat}(\mathcal{C}) = e^{-E(\mathcal{C})/kT} / Z$ $\qquad$ (N constraints)

(ii) DETAILED BALANCE

$$\boxed{\frac{M(\mathcal{C} \to \mathcal{C}')}{M(\mathcal{C}' \to \mathcal{C})} = e^{\frac{E(\mathcal{C}) - E(\mathcal{C}')}{kT}}}$$

ONSAGER, 19..

Fixes half of the rates.

Explanation of detailed balance:

Equilibrium dynamics is INVARIANT under time-reversal (movie projected backwards)

(Bayes)

$$\text{Proba}\,(\mathcal{C}_1 \text{ at } t+dt \text{ and } \mathcal{C}_0 \text{ at } t) = P(\mathcal{C}_1, t+dt \,|\, \mathcal{C}_0, t)\, P(\mathcal{C}_0, t)$$

in the equilibrium state

$$= \underbrace{M(\mathcal{C}_0 \to \mathcal{C}_1)}_{\text{Markov}}\, dt\, \underbrace{P_{eq}(\mathcal{C}_0)}_{\text{equilibrium is time-invariant}}$$

time-reversal, this should be equal to ("reversed movie")

$$\text{Proba}\,(\mathcal{C}_0 \text{ at } T_{final}-t \text{ and } \mathcal{C}_1 \text{ at } T_{final}-(t+dt)) = P(\mathcal{C}_0, T_F-t \,|\, \mathcal{C}_1, T_F-t-dt)\, P(\mathcal{C}_1, T_F-t-dt)$$

$$= M(\mathcal{C}_1 \to \mathcal{C}_0)\, dt\, P_{eq}(\mathcal{C}_1)$$

$$\frac{M(\mathcal{C}_0 \to \mathcal{C}_1)}{M(\mathcal{C}_1 \to \mathcal{C}_0)} = \frac{P_{eq}(\mathcal{C}_1)}{P_{eq}(\mathcal{C}_0)} \qquad \text{Detailed-balance}$$

Equilibrium systems obey detailed balance.

Note that even if one keeps locality + detailed balance a lot of freedom remains

a possible choice $M(\mathcal{C} \to \mathcal{C}') = e^{\frac{\beta}{2}[E(\mathcal{C}') - E(\mathcal{C})]}$

(Metropolis)

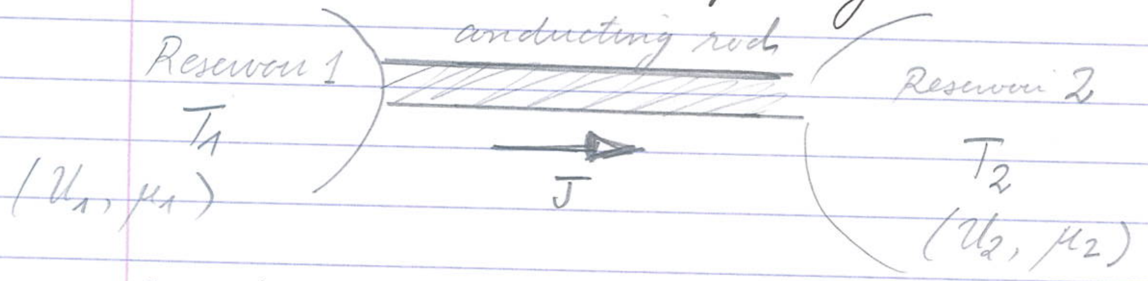
(But there are many other possibilities). Glauber (flips) Kawasaki (exchange)

AT EQUILIBRIUM, there are NO FLUXES or CURRENTS through the system. No net transport of matter, energy, spin etc... all currents must vanish on average, this is a consequence of time-reversal invariance

Fluctuations can occur.

Non-equilibrium Statistical Mechanics

Suppose we wish to describe the following situation:



In the stationary state, a non-vanishing current J exists. Time-reversal is broken: this is a non-equilibrium steady-state.

Classical Thermodynamics, Eq. Stat. Mech. are not able to describe this situation: $P(\mathcal{C}) = ??$

$T_1 = T_2$ we know
$T_1 \neq T_2$ only phenomenological laws.

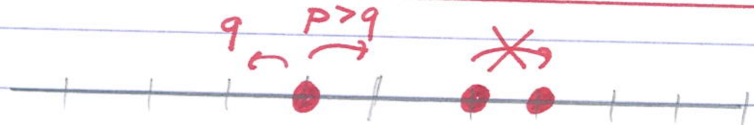
Is there an equation of state?
What plays the role of free energy, entropy?

No Principles available: looking for a theory.

Build suitable models: "rich" enough to teach us some physics but "simple" enough to be solvable.

The Ising Model of non-equilibrium: or "fruit fly", "yeast"

$\boxed{\text{The asymmetric simple exclusion process}}$



Particles hopping on a lattice.

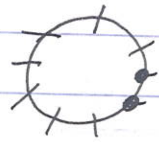
AT MOST one particle per site: exclusion rule INTERACTION N-body Pb

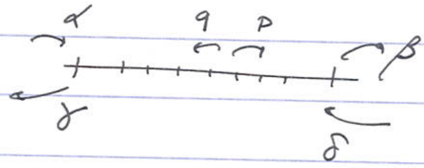
$p > q$: asymmetry a current is driven through the system: BREAKING DETAILED BALANCE a non-eq. system

- Interacting Brownian particles; Lattice gases ~'70, '80
- Transport through narrow-pore: molecular motors '80-'90 →
- Realistic models of traffic flow (a building block) '2000

Mathematical Q: SRS Varadhan: "The complex story of simple exclusion", 1996

Various boundary conditions:

(i) Periodic  L sites, N particles

(ii) Open boundaries with reservoirs 

(iii) Infinite lattice ($\mathbb{Z}$) 

Exercise: 1) Write the Markov Matrix for the periodic ASEP $(L=3, N=2)$ and $(L=4, N=2)$.
Find the stationary pdf.

2) For the periodic ASEP prove that

$$P_{stat}(\mathcal{C}) = \frac{1}{\binom{L}{N}}$$

3) The case of an infinite lattice:

- invariant measure $\quad \bigotimes_{i\in\mathbb{Z}} \nu_\rho^i \qquad \nu_\rho^i$ = Bernoulli on site i $\begin{cases} \text{OCCUPIED with } \rho \\ \text{EMPTY with } 1-\rho \end{cases}$

- Important result: SEP on $\mathbb{Z}$ $\quad \langle X_t^2 \rangle = 2\,\frac{1-\rho}{\rho}\sqrt{\frac{t}{\pi}}$ $\quad p = q = 1$ $\quad$ (?) vérifier

for $p>q$ $\quad \dfrac{\langle X_t \rangle}{t} \xrightarrow[a.s]{} (p-q)\underbrace{(1-\rho)}_{\text{excluded volume}}$ $\quad$ (average speed)

$$\lim_{t\to\infty} \frac{\langle X_t^2 \rangle - \langle X_t \rangle^2}{t} = (p-q)(1-\rho)$$

average over the initial conditions and the dynamics

The ASEP with open boundaries:

A model for the initial situation: a rod with 2 reservoirs

The stationary measure is unknown = no principles.
Let's consider the TASEP case: DEHP '93.

Explanation of the MATRIX ANSATZ.

Copier les tableaux de chiffres.

Astuce pour calculer Z_L (exercice d'ALEA) $\frac{1}{1-zC} = \frac{1}{1-\eta E}\,\frac{1}{1-\eta D}$

$$(1-zC) = (1-\eta D)(1-\eta E) = 1 - \underbrace{(\eta+\eta^2)}_{=z} C$$

Diagramme de cf: cas ASEP et PASEP

See details on next page.

ν_ρ^i : mesure de Bernoulli sur un site i $\begin{cases} \text{OCCUPÉ avec proba } \rho \\ \text{VIDE avec } 1-\rho \end{cases}$

densité moyenne $\langle \tau_i(t=0) \rangle = \rho$

Mesure initiale : $\bigotimes_{i \in \mathbb{Z}} \nu_\rho^i$ on peut montrer qu'elle est invariante par la dynamique

on moyenne par rapport à cette mesure :

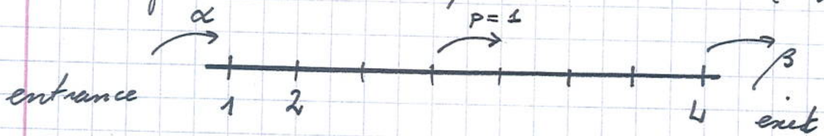
Position moyenne d'une particule marquée : $\lim_{t\to\infty} \frac{\langle X_t \rangle}{t} = (p-q)(1-\rho)$ ici inutile — égalité vraie $\forall t$ — VOLUME EXCLU

Fluctuations : $\lim_{t\to\infty} \frac{\langle X_t^2\rangle - \langle X_t\rangle^2}{t} = (p-q)(1-\rho) > 0$ pour $p>q$

nécessaire

BUT FOR $p=q=\frac{1}{2}$: $\boxed{\langle X_t^2 \rangle \simeq \frac{2(1-\rho)}{\rho}\left(\frac{pt}{\pi}\right)^{\frac{1}{2}}}$ ANOMALOUS DIFFUSION

En Physique : SYSTÈMES GRANDS mais finis, on s'intéresse aux "effets de taille finie" (SCALINGS p.ex. comment $D(L)$?)



2^L configurations

{For simplicity sake : consider only the TASEP case (yet)}

on cherche le noyau de la matrice de Markov $2^L \times 2^L$

La mesure microscopique N'EST PAS uniforme : solution D.E.H.P. '93

OBSERVATIONS EMPIRIQUES : $\alpha = \beta = 1$ $p=1$, $q=0$

Des entiers simples ! La plus petite proba divise les autres (quantum).

L = 1

0	1
1	1
Total :	2

L = 2

00	1
01	1
10	2
11	1
Total :	5

L = 3

000	1
001	1
010	2
011	1
100	3
101	2
110	3
111	1
Total :	14

L = 4

0000	1
0001	1
0010	2
0011	1
0100	3
0101	2
0110	3
0111	1
1000	4
1001	3
1010	5
1011	2
1100	6
1101	3
1110	4
1111	1
total :	42

remarques :

* ajouter un 0 à gauche → idem
* ajouter un 1 à droite → idem
* les dénominateurs : CATALANS !
* "Règle d'or" :

$$\boxed{\text{"} \ldots 10 \ldots \equiv \ldots 1 \ldots + \ldots 0 \ldots}$$

→ Assez d'information pour calculer tous les Po

Il y a des corrélations / les probabilités ne sont pas factorisées $P(010) \neq p_0 p_1 p_0$ scalaires.

Scalaires → Opérateurs
$0 \to E$
$1 \to D$

CODER les RÉCURRENCES par une algèbre :

$$\boxed{P(\tau) = \frac{1}{Z_L}\langle W|\prod_{i=1}^{L}\left(\tau_i D + (1-\tau_i) E\right)|V\rangle}$$

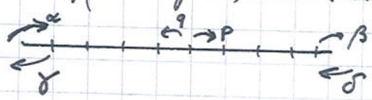
D, E opérateurs $\quad |V\rangle$ vecteur, $\quad \langle W|$ co-vecteur

$$\boxed{\begin{aligned} DE &= D + E \\ D|V\rangle &= \frac{1}{\beta}|V\rangle \\ \langle W|E &= \frac{1}{\alpha}\langle W| \end{aligned}}$$

traduit exactement les récurrences empiriques

cas général → DÉFORMATION de l'algèbre quadratique

$$\boxed{\begin{aligned} pDE - qED &= D + E \\ (\beta D - \delta E)|V\rangle &= |V\rangle \\ \langle W|(\alpha E - \gamma D) &= \langle W| \end{aligned}}$$



Mesure stationnaire → une trace sur l'algèbre quadratique idoine.

Quelques calculs explicites _ Diagramme de Phase :

1. Esquisse d'un preuve : (mécanisme)

1100 : 1. $p(1100) \overset{?}{=} \alpha\, p(0100) + \beta p(1101)$

$= \alpha\langle W|EDEE|V\rangle + \beta\langle W|DDED|V\rangle - \langle W|D\overbrace{DE}^{D+E}E|V\rangle$

$= 0$ ✓

le cas général fonctionne pareil (exercice: → 4 cas à distinguer)

2. L'algèbre existe (représentations explicites) :

• TASEP : exercice : si D et E de dimension finie alors elles commutent

Poser $D = 1 + \delta \quad E = 1 + \varepsilon \qquad \boxed{\delta\varepsilon = 1}$

→ dimension finie commutation nécessaire, mézalor : TOUS LES POIDS SONT ÉGAUX

$\langle W|DE|V\rangle = \langle W|D+E|V\rangle = \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\langle W|V\rangle$ et si commutation $= \langle W|ED|V\rangle = \frac{1}{\alpha}$

Cela implique $\boxed{\alpha + \beta = 1}$ $\quad D = \frac{1}{\beta}$ et $E = \frac{1}{\alpha}$ $\quad$ POIDS : $(\alpha\beta)^L \frac{1}{\alpha^{\#0}}\frac{1}{\beta^{\#1}}$

→ dès que $\alpha + \beta \neq 1$ il faut une représentation de dimension infinie

Typiquement δ et ε sont des SHIFTS sur $\ell(\mathbb{N})$

$$\delta = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \\ & & & & \ddots \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \ddots \end{pmatrix} \qquad |V\rangle = K\begin{pmatrix} 1 \\ \frac{1-\beta}{\beta} \\ \left(\frac{1-\beta}{\beta}\right)^2 \\ \vdots \end{pmatrix}$$

$\langle W| = K\left(1, \frac{1-\alpha}{\alpha}, \left(\frac{1-\alpha}{\alpha}\right)^2, \ldots\ldots\right)$

K tel que $\langle W|V\rangle = 1$ donc $K = \sqrt{\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{\alpha\beta}}$

- Partially asymetric case (PASEP)

$$D = \frac{1+\delta}{p-q} \qquad E = \frac{1+\varepsilon}{p-q}$$

Cours I

$$\boxed{\delta\varepsilon - x\varepsilon\delta = 1-x} \quad \text{avec } x = \frac{q}{p}$$

$$\delta = \begin{pmatrix} 0 & \sqrt{1-x} & & & \\ 0 & 0 & \sqrt{1-x^2} & & \\ 0 & 0 & 0 & \sqrt{1-x^3} & \\ & & & & \ddots \end{pmatrix} \qquad \varepsilon = {}^t\delta .$$

3. Quelques calculs explicites (TASEP) : **DIAGRAMME de PHASE.**

Normalisation : $Z_L = \langle W | C^L | V \rangle$ avec $C = D + E$

Profil de densité : $\rho_i \equiv \langle \tau_i \rangle = \dfrac{\langle W | C^{i-1} D C^{L-i} | V \rangle}{\langle W | C^L | V \rangle}$

Courant : $i \quad i+1$ $\quad J_{i \to i+1} = \langle \tau_i (1-\tau_{i+1}) \rangle = \dfrac{\langle W | C^{i-1} DE C^{L-i-1} | V \rangle}{\langle W | C^L | V \rangle}$

$$= \langle W | C^{L-1} | V \rangle \big/ \langle W | C^L | V \rangle$$

on remarque : dans l'état stationnaire le courant est uniforme

$$\boxed{J = \frac{\langle W | C^{L-1} | V \rangle}{\langle W | C^L | V \rangle} = \frac{Z_{L-1}}{Z_L}}$$

RM/ J = courant d'entrée $= \alpha \langle 1 - \tau_1 \rangle$
J = courant de sortie $= \beta \langle \tau_L \rangle$

Comment calculer ces quantités ?
- avec l'algèbre (ré-ordonner).
- avec une représentation explicite.
- avec des fonctions génératrices.

Soit U un produit quelconque de D et de E ; on peut toujours ré-ordonner sous la forme : $U = \sum_{n,m} a_{nm} E^n D^m$ et alors

$$\langle W | U | V \rangle = \sum_{n,m} a_{nm} \frac{1}{\alpha^n} \frac{1}{\beta^m} \langle W | V \rangle$$

il s'agit donc de trouver les coeff a_{mn} dans la base de PBW

La quantité de base à calculer c'est la normalisation $Z_L \overset{\text{i.e.}}{\Longleftrightarrow} U = C^L$

empiriquement :

$$C = D + E$$
$$C^2 = D^2 + ED + E^2 + D + E$$
$$C^3 = D^3 + ED^2 + E^2D + E^3 + 2(D^2 + ED + E^2) + 2(D+E) \quad \text{etc..}$$

On devine la structure : $\displaystyle C^N = \sum_{p=0}^{N} B_{N,p} \left(D^p + ED^{p-1} + E^2 D^{p-2} + \dots + E^p \right)$

Ce qui donne : il faut remarquer que les polynômes du type $Q_p = D^p + ED^{p-1} + E^2 D^{p-2} + \dots + E^p$ sont FERMÉS par multiplication par DE

$$DE\, Q_p = (\underbrace{D+E}_{= E+D})\, Q_p = \underbrace{(ED^p + \dots E^{p+1}) + D^{p+1}}_{= EQ_p \ldots} + \overbrace{DE(D^{p-1} + ED^{p-2} + \dots E}^{= DQ_p}$$

ainsi : $DE\, Q_p = Q_{p+1} + DE\, Q_{p-1}$

i.e. $\quad \underline{\underline{C Q_p = Q_{p+1} + C Q_{p-1}}} \qquad Q_0 = 1, \; Q_1 = C$

donc $\quad C Q_p = Q_{p+1} + Q_p + Q_{p-1} + \dots Q_1$

$$B_{N+1,p+1} = B_{N,p} + (B_{N,p+1} + B_{N,p+2} + \dots + B_{N,N})$$

i.e $B_{N+1,p+1} = B_{N,p} + B_{N+1,p+2}$ $\qquad B_{N,p}$ défini pour $0 \leq p \leq$

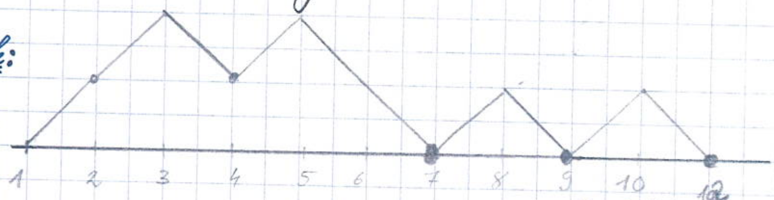
BALLOT NUMBERS : $\boxed{B_{N,p} = \frac{p\,(2N-1-p)!}{N!\,(N-p)!}}$

$$\boxed{Z_L = \sum_{p=0}^{L} \frac{p(2L-1-p)!}{L!\,(L-p)!} \left(\sum_{q=0}^{p} \frac{1}{\alpha^q} \frac{1}{\beta^{p-q}} \right)}$$

Remarque : Lien avec les chemins de Dyck

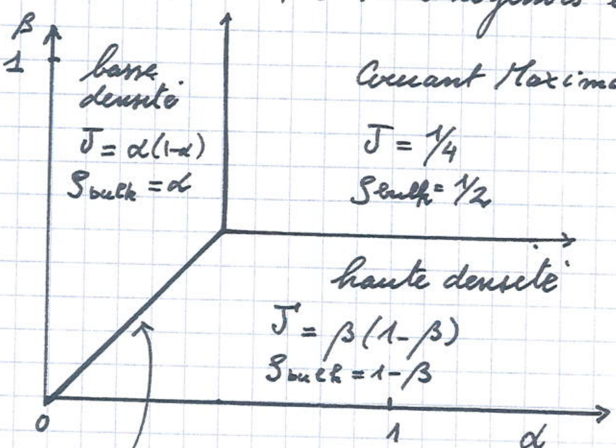
Chemin de Dyck : au-dessus de l'axe des abscisses $x=0$.

de longueur $2N$ avec P retours à $x=0$ (comptant le dernier) } compté par $B_{N,p}$



ici $2N = 12$
$P = 3$
$B_{6,3} = 3 \times \frac{8!}{6!\,3!} = 28$

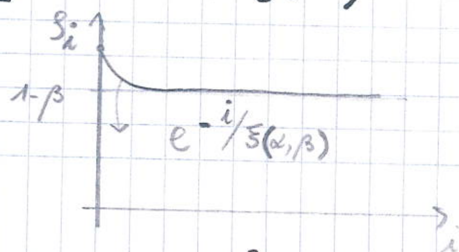
on s'intéresse à la limite $L \to \infty$: 3 régimes en fonction de α et β



phase de choc : "ρ est discontinue"

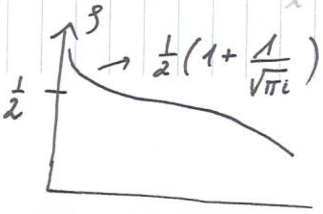
Aspect du profil de densité : $\left(\begin{array}{c} [1,L] \to [0,1] \\ i \mapsto \rho_i \end{array} \right)$

Haute densité



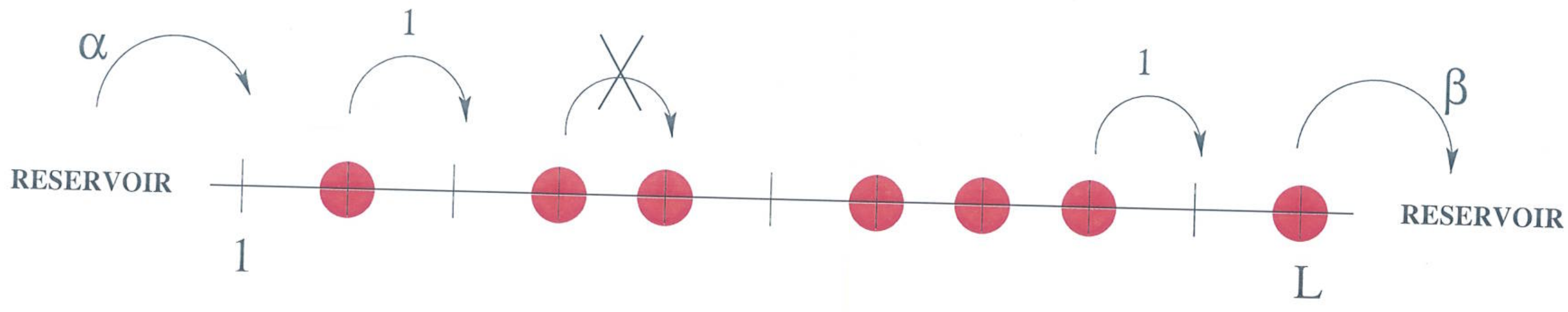
$\xi(\alpha,\beta)$ longueur de corrélation.

Maximal current phase



choc line $\quad \rho_i = \alpha + \frac{i}{L}(\overbrace{1-\beta-\alpha}^{1-2\alpha})$ linear profile, combination of shocks

The Matrix Ansatz (DEHP,1993)



A configuration $\mathcal{C}$ of the exclusion process is specified by a sequence of 0's and 1's (empty and occupied sites). Here we have the configuration 01011011101 on eleven sites

Instead of 0 and 1, we can use any two different symbols to represent the configuration: Let us choose the letter **D** for occupied sites and the letter **E** for empty sites. The previous configuration is written EDEDDEDDDED or $EDED^2ED^3ED$.

In general, a configuration $\mathcal{C}$ is given by

$$\mathcal{C} = \prod_{i=1}^{L} \left(\tau_i D + (1-\tau_i) E\right) .$$

where $\tau_i = 1$ (or 0) if the site i is occupied (or empty).

The TASEP algebra

The stationary probability of a configuration $\mathcal{C}$ is given by

$$P(\mathcal{C}) = \frac{1}{Z_L} \langle\alpha| \prod_{i=1}^{L} \left(\tau_i D + (1-\tau_i) E\right) |\beta\rangle \,.$$

where $\tau_i = 1$ (or 0) if the site i is occupied (or empty).
The normalization constant is $Z_L = \langle\alpha| \left(D+E\right)^L |\beta\rangle = \langle\alpha| C^L |\beta\rangle$ where $C = D + E = DE$.

The operators D and E, the vectors $\langle\alpha|$ and $|\beta\rangle$ satisfy

$$\begin{aligned} D\,E &= D + E \\ D\,|\beta\rangle &= \frac{1}{\beta}\,|\beta\rangle \\ \langle\alpha|\,E &= \frac{1}{\alpha}\langle\alpha| \end{aligned}$$

Average Stationary Current:

$$J = \langle \tau_i (1 - \tau_{i+1}) \rangle = \frac{\langle\alpha| C^{i-1}\, D\, E\, C^{L-i-1} |\beta\rangle}{\langle\alpha| C^L |\beta\rangle} = \frac{\langle\alpha| C^{L-1} |\beta\rangle}{\langle\alpha| C^L |\beta\rangle} = \frac{Z_{L-1}}{Z_L}$$

Equal-time Steady State Correlations

More generally, the Matrix Ansatz gives access to all equal time correlations in the steady-state.

Density Profile:

$$\rho_i = \langle \tau_i \rangle = \frac{\langle \alpha | C^{i-1} \, D \, C^{L-i} | \beta \rangle}{\langle \alpha | C^L | \beta \rangle}$$

Multi-body correlations:

$$\langle \tau_{i_1} \tau_{i_2} \ldots \tau_{i_k} \rangle = \frac{\langle \alpha | C^{i_1-1} \, D \, C^{i_2-i_1-1} \, D \ldots D \, C^{L-i_k} | \beta \rangle}{\langle \alpha | C^L | \beta \rangle}$$

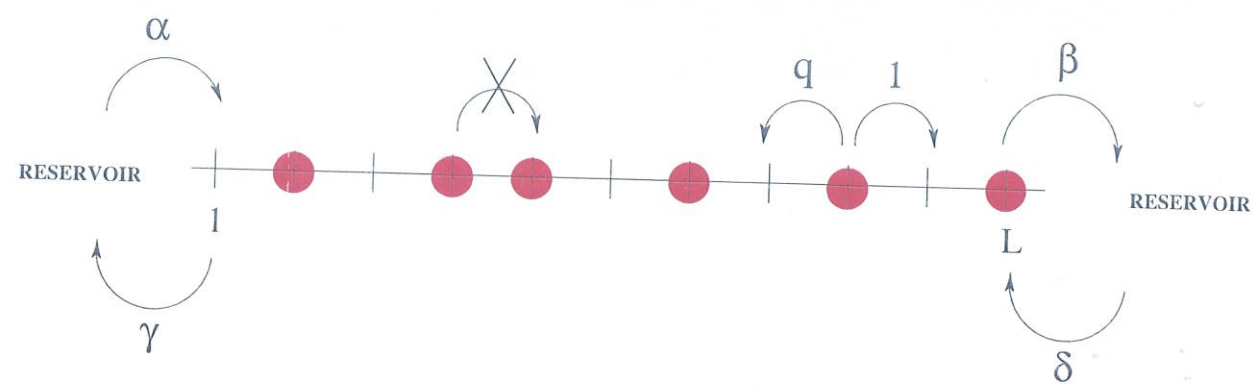
The expressions look formal but it is possible to derive explicit formulae: either by using purely combinatorial/algebraic techniques or via a specific representation (e.g., C can be chosen as a discrete Laplacian).

$$\langle \alpha | C^L | \beta \rangle = \sum_{p=1}^{L} \frac{p\,(2L-1-p)!}{L!\,(L-p)!} \, \frac{\beta^{-p-1} - \alpha^{-p-1}}{\beta^{-1} - \alpha^{-1}}$$

Phase diagram
Density Profile

A very large body of knowledge has been developed around this Matrix Ansatz: see the review of R. Blythe and M. R. Evans.

Matrix Ansatz for ASEP



The stationary probability of a configuration $\mathcal{C}$ is given by

$$P(\mathcal{C}) = \frac{1}{Z_L} \langle W| \prod_{i=1}^{L} (\tau_i D + (1-\tau_i) E) |V\rangle$$

where $\tau_i = 1$ (or 0) if the site i is occupied (or empty) and the normalization constant is $Z_L = \langle W| (D+E)^L |V\rangle$

The operators D and E, the vectors $\langle W|$ and $|V\rangle$ satisfy

$$\begin{aligned} D\,E - qED &= (1-q)(D+E) \\ (\beta\, D - \delta\, E)\,|V\rangle &= |V\rangle \\ \langle W|(\alpha\, E - \gamma\, D) &= \langle W| \end{aligned}$$

Representations of the quadratic algebra

The algebra encodes combinatorial recursion relations between systems of different sizes.

Infinite dimensional Representation:

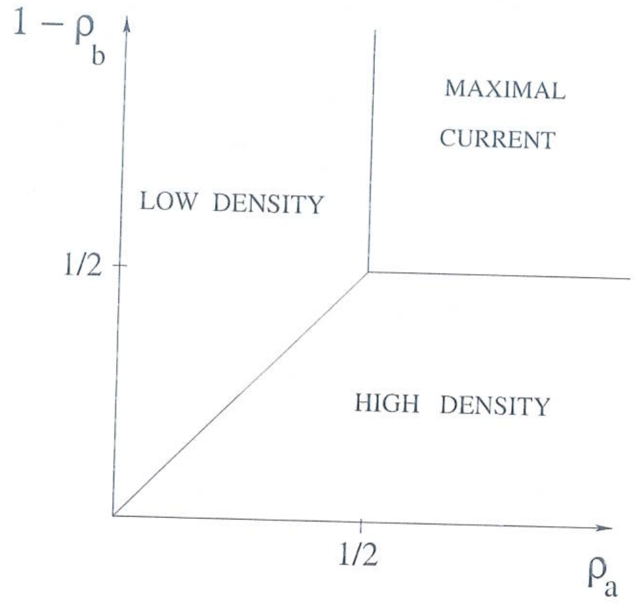
$D = 1 + d$ where d is a q-destruction operator.

$E = 1 + e$ where e is a q-creation operator.

$$d = \begin{pmatrix} 0 & \sqrt{1-q} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{1-q^2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{1-q^3} & \cdots \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad e = d^{\dagger}$$

The matrix Ansatz allows one to calculate Stationary State Properties (currents, correlations, fluctuations) and to derive the Phase Diagram in the infinite size limit.

The Phase Diagram



$\rho_a = \frac{1}{a_+ + 1}$: effective left reservoir density.

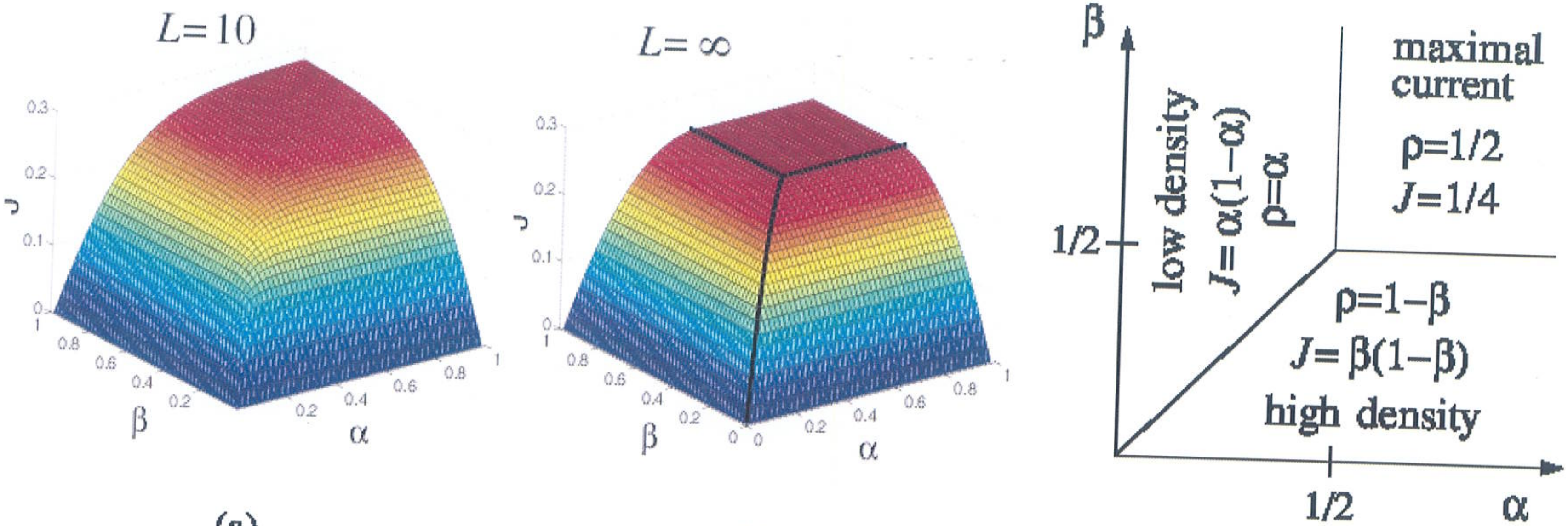
$\rho_b = \frac{b_+}{b_+ + 1}$: effective right reservoir density.

$$a_\pm = \frac{(1-q-\alpha+\gamma) \pm \sqrt{(1-q-\alpha+\gamma)^2 + 4\alpha\gamma}}{2\alpha}$$

$$b_\pm = \frac{(1-q-\beta+\delta) \pm \sqrt{(1-q-\beta+\delta)^2 + 4\beta\delta}}{2\beta}$$

Time-dependent Properties

The Matrix Ansatz allows us to calculate steady state properties in particular equal-time correlations, as for example the average current through the system in the long time limit.



(a) (b) (c)

How do we access to time-dependent properties?

- How does the system relax to its stationary state?
- What do the fluctuations of the current look like? What about its probability distribution?