Cloaking strategies for transient heat and mass transfer

Harsha Hutridurga

Imperial College London

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MY INTERESTS

- Homogenization of differential equations
- Strong advection problems
- Kinetic theory of gases
- Mathematics of metamaterials

INTRICATE CONNECTION TO

- Material sciences
- Environmental sciences
- Ecology
- Neurodegenerative diseases

- **Advection-diffusion-reaction models**
  periodically oscillating coefficients

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b}\left(\frac{x}{\varepsilon}\right)\cdot\nabla u^\varepsilon - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = 0$$

$$-\frac{1}{\varepsilon}A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\cdot\mathbf{n}(x) = \partial_t v^\varepsilon = \frac{\kappa}{\varepsilon^2}\Big(f(u^\varepsilon) - v^\varepsilon\Big)$$

  Adsorption model: coupling bulk field u^ε and surface field v^ε

- **Reaction-diffusion systems**

$$\frac{\partial u_i^\varepsilon}{\partial t} - \operatorname{div}\left(D_i\left(\frac{x}{\varepsilon}\right)\nabla_x u_i^\varepsilon\right) = \frac{1}{\varepsilon^2}\mathcal{R}_i(u^\varepsilon).$$

  Reversible chemical reaction between ℓ species:

$$\delta_1\mathcal{A}_1 + \cdots + \delta_\ell\mathcal{A}_\ell \quad \underset{\alpha\kappa}{\overset{\kappa}{\rightleftarrows}} \quad \beta_1\mathcal{A}_1 + \cdots + \beta_\ell\mathcal{A}_\ell$$

$$\text{Mass action kinetics} \quad \Longrightarrow \mathcal{R}_i(u) = \kappa\left(\prod_{i=1}^{\ell} u_i^{\delta_i} - \alpha\prod_{i=1}^{\ell} u_i^{\beta_i}\right).$$

[Joint work with] G.ALLAIRE (**École Polytechnique**)

- **Advection-diffusion models**
  incompressible advective field: $\operatorname{div} \mathbf{b}(x) = 0$ and $\mathbf{b} \cdot \mathbf{n}(x) = 0$.

$$\partial_t u^\varepsilon + \frac{1}{\varepsilon}\mathbf{b}(x) \cdot \nabla u^\varepsilon - \Delta u^\varepsilon = 0$$
$$\nabla u^\varepsilon \cdot \mathbf{n}(x) = 0$$

- Jacobian matrix associated with the flow.
- Asymptotic depends on the behaviour of $J(\tau)$.
- Bounded Jacobian yielding full diffusion.
- Unbounded Jacobians resulting in initial layer.

$$\partial_t u_0 - \Delta u_0 \in \mathcal{N}_{\mathbf{b}}^{\perp}$$
$$u_0(t, \cdot) \in \mathcal{N}_{\mathbf{b}}$$
$$u_0(0, x) = \mathcal{P}u^{\mathrm{in}}(x).$$

[Joint work with] T.HOLDING (Warwick); J.RAUCH (Michigan)

- **Kinetic Fokker-Planck equation**
probability density function: $f(t,x,v)$ on $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi(x) \cdot \nabla_v f = \sigma(x) \nabla_v \cdot \Big(\nabla_v f + v f \Big)$$

Degenerate: $\sigma(x)$ cab vanish in subset of non-zero measure in $\mathbb{T}^d$.

- **Geometric control condition:** $\exists\, C > 0$ and $T^* < +\infty$

$$\int_0^{T^*} \sigma(X_{x,v}(s))\, \mathrm{d}s \geq C \qquad \text{for a.e. } (x,v) \in \mathbb{T}^d \times \mathbb{R}^d.$$

- **Decay estimate:** there exist $C, \lambda > 0$ s.t.

$$\left\| f(t,x,v) - \Pi_g f^{in}(x,v) \right\|_{L^2(F^{-1})} \leq C e^{-\lambda t} \left\| f^{in}(x,v) - \Pi_g f^{in}(x,v) \right\|_{L^2(}$$

[Joint work with] DIETERT (Paris VII); HERAU (Nantes); MOUHOT (Cambridge)

- **Diffusion approximation & homogenization**
  Linear Boltzmann for probability density function:

$$
\begin{aligned}
\varepsilon \partial_t f^{\varepsilon,\eta}(t,x,v) &+ v \cdot \nabla_x f^{\varepsilon,\eta}(t,x,v) \\
&= \frac{\sigma^\eta(x)}{\varepsilon} \left(\int_{\mathcal{V}} f^{\varepsilon,\eta}(t,x,w)\, \mathrm{d}\mu(w) - f^{\varepsilon,\eta}(t,x,v) \right)
\end{aligned}
$$

  Interplay between the small parameters ε and η.

- **Initial boundary value problem:**

$$
\varepsilon\, \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon} \nabla_v \cdot \Big(\nabla_v f^\varepsilon + v f^\varepsilon \Big)
$$

  Diffusion approximation with specular type boundary condition.

[Joint work with] C.BARDOS (**Paris VII**); L.CESBRON (**Cambridge**)

- **Multi-species Boltzmann equations:** for $f_i^\varepsilon(t,x,v)$

$$\varepsilon\,\partial_t f_i^\varepsilon + v\cdot\nabla_x f_i^\varepsilon = \frac{1}{\varepsilon}\sum_{j=1}^{I} Q_{ij}(f_i^\varepsilon, f_j^\varepsilon)$$

  Study the $\varepsilon \to 0$ asymptotic.

- **Maxwell-Stefan cross diffusion model:**

$$\partial_t c_i + \nabla_x \cdot F_i = 0$$

$$-\nabla_x c_i = \sum_{j\neq i} \frac{c_j F_i - c_i F_j}{Đ_{ij}}$$

- Temperature dependance in Maxwell-Stefan.
- Local (in time) existence of solution to the corrected model.

[Joint work with] F.SALAVARANI (Paris Dauphine)

COMPETITION DIFFUSION IN ECOLOGY

- For population densities $u^\varepsilon(t,x)$, $v^\varepsilon(t,x)$ and non-negative $w^\varepsilon(t,x)$:

$$\partial_t u^\varepsilon - \nabla \cdot \Big(A^\varepsilon(x) \nabla u^\varepsilon \Big) + \frac{u^\varepsilon}{\varepsilon}\Big(v^\varepsilon + \lambda \left(1 - w^\varepsilon\right) \Big) = 0$$

$$\partial_t v^\varepsilon - \nabla \cdot \Big(B^\varepsilon(x) \nabla v^\varepsilon \Big) + \alpha \frac{v^\varepsilon}{\varepsilon}\Big(u^\varepsilon + \lambda w^\varepsilon \Big) = 0$$

$$\partial_t w^\varepsilon + \frac{u^\varepsilon}{\varepsilon}\left(w^\varepsilon - 1\right) + \frac{w^\varepsilon v^\varepsilon}{\varepsilon} = 0$$

  Study the $\varepsilon \to 0$ asymptotic.

- **Two-phase Stefan problem with latent heat.**
- Interplay b/w strong competition & high frequency oscillations.
- Oscillations are sometimes bad for competition.

[Joint work with] C.VENKATARAMAN (**St Andrews**)

THERMAL CLOAKING

- Suppose $\Omega \subset \mathbb{R}^d$ such that

$$B_2 \subset \Omega$$

- An observer able to detect conductivity of the material.
- At first instance, let Ω be a uniform conducting material. Here, the observer would say the conductivity is uniform.
- Is it possible to construct some "metamaterial" in $B_2 \setminus B_1$ such that irrespective of the conductivity in B_1, observer only able to detect uniform conductivity?
- Study the time-dependent problem.

[Joint work] CRASTER (**Imperial**); GUENNEAU (**Marseille**); PAVLIOTIS (**Imperial**)

- **Homogeneous model**

$$\begin{cases} \partial_t u_{\text{hom}}(t,x) = \Delta u_{\text{hom}}(t,x) + f(x) & \text{in } (0,\ell)\times\Omega, \\ \nabla u_{\text{hom}} \cdot \mathbf{n}(x) = g(x) & \text{on } (0,\ell)\times\partial\Omega, \\ u_{\text{hom}}(0,x) = u^{\text{in}}(x) & \text{in } \Omega. \end{cases}$$

- **Cloaked model**

$$\begin{cases} \rho_{\text{cl}}(x)\dfrac{\partial u_{\text{cl}}}{\partial t} = \nabla\cdot\Big(A_{\text{cl}}(x)\nabla u_{\text{cl}}\Big) + f(x) & \text{in } (0,\ell)\times\Omega, \\ \nabla u_{\text{cl}} \cdot \mathbf{n}(x) = g(x) & \text{on } (0,\ell)\times\partial\Omega, \\ u_{\text{cl}}(0,x) = u^{\text{in}}(x) & \text{in } \Omega, \end{cases}$$

- Construct ρ_{cl} and A_{cl} such that $u_{\text{cl}}(t,x) \approx u_{\text{hom}}(t,x)$ for $x \in \Omega\setminus B_2$

Proposition

Consider an invertible map $\mathbb{F} : \Omega \mapsto \Omega$ *such that* $\mathbb{F}(x) = x$ *for each* $x \in \Omega \setminus B_2$. *Then* $u(t,x)$ *is a solution to*

$$\rho(x)\frac{\partial u}{\partial t} = \nabla \cdot \Big(A(x) \nabla u \Big) + f(x) \qquad \text{for } (t,x) \in (0,\ell) \times \Omega$$

if and only if $v = u \circ \mathbb{F}^{-1}$ *is a solution to*

$$\mathbb{F}^*\rho(y)\frac{\partial v}{\partial t} = \nabla \cdot \Big(\mathbb{F}^* A(y) \nabla v \Big) + \mathbb{F}^* f(y) \qquad \text{for } (t,y) \in (0,\ell) \times \Omega$$

$$\mathbb{F}^*\rho(y) = \frac{\rho(x)}{\det(D\mathbb{F})(x)}; \qquad \mathbb{F}^* f(t,y) = \frac{f(t,x)}{\det(D\mathbb{F})(x)};$$

$$\mathbb{F}^* A(y) = \frac{D\mathbb{F}(x) A(x) D\mathbb{F}^\top(x)}{\det(D\mathbb{F})(x)} \qquad x = \mathbb{F}^{-1}(y)$$

Moreover we have for all $t > 0$, $u(t,\cdot) = v(t,\cdot) \qquad$ *in* $\Omega \setminus B_2$.

consider a Lipschitz map $\mathcal{F}_\varepsilon : \Omega \mapsto \Omega$ defined as

$$\mathcal{F}_\varepsilon(x) := \begin{cases} x & \text{for } x \in \Omega \setminus B_2 \\ \left(\frac{2-2\varepsilon}{2-\varepsilon} + \frac{|x|}{2-\varepsilon}\right) \frac{x}{|x|} & \text{for } x \in B_2 \setminus B_\varepsilon \\ \frac{x}{\varepsilon} & \text{for } x \in B_\varepsilon \end{cases}$$

Note that $\mathcal{F}_\varepsilon$ maps B_ε to B_1 and the annulus $B_2 \setminus B_\varepsilon$ to $B_2 \setminus B_1$.

CLOAKING COEFFICIENTS

$$\rho_{\text{cl}}(x) = \begin{cases} 1 & \text{for } x \in \Omega \setminus B_2, \\ \mathcal{F}_\varepsilon^* 1 & \text{for } x \in B_2 \setminus B_1, \\ \eta(x) & \text{for } x \in B_1 \end{cases}$$

$$A_{\text{cl}}(x) = \begin{cases} \text{Id} & \text{for } x \in \Omega \setminus B_2, \\ \mathcal{F}_\varepsilon^* \text{Id} & \text{for } x \in B_2 \setminus B_1, \\ \sigma(x) & \text{for } x \in B_1 \end{cases}$$

Theorem

Let $u_{\mathrm{cl}}(t,x)$ *be the solution to the thermal cloak problem with the cloaking coefficients* $\rho_{\mathrm{cl}}(x), A_{\mathrm{cl}}(x)$*. Let* $u_{\mathrm{hom}}(t,x)$ *be the solution to the homogeneous conductivity problem. Suppose the data are such that*

$$f \in \mathrm{L}^2(\Omega), \quad \operatorname{supp} f \subset \Omega \setminus B_2, \quad g \in \mathrm{L}^2(\partial\Omega), \quad u^{\mathrm{in}} \in \mathrm{H}^1(\Omega).$$

Then, there exists a time instant $T < \infty$ *such that for all* $t \geq T$ *we have*

$$\left\| u_{\mathrm{cl}}(t,\cdot) - u_{\mathrm{hom}}(t,\cdot) \right\|_{\mathrm{H}^{\frac{1}{2}}(\partial\Omega)} \leq \varepsilon^d \left(\left\| u^{\mathrm{in}} \right\|_{\mathrm{H}^1} + \|f\|_{\mathrm{L}^2(\Omega)} + \|g\|_{\mathrm{L}^2(\partial\Omega)} \right).$$

$$\|w\|_{\mathrm{H}^{\frac{1}{2}}(\partial\Omega)} := \left(\int_{\partial\Omega} |w(x)|^2 \, \mathrm{d}\sigma(x) + \int_{\partial\Omega} \int_{\partial\Omega} \frac{|w(x) - w(y)|^2}{|x-y|^d} \, \mathrm{d}\sigma(x) \, \mathrm{d}\sigma(y) \right)$$

CRUCIAL OBSERVATION

$$\rho^\varepsilon(x) = \begin{cases} 1 & \text{for } x \in \Omega \setminus B_\varepsilon, \\ \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right) & \text{for } x \in B_\varepsilon \end{cases}$$

$$A^\varepsilon(x) = \begin{cases} \mathrm{Id} & \text{for } x \in \Omega \setminus B_\varepsilon, \\ \frac{1}{\varepsilon^{d-2}}\sigma\left(\frac{x}{\varepsilon}\right) & \text{for } x \in B_\varepsilon \end{cases}$$

Computing push-forwards yield

$$\rho_{\mathrm{cl}}(y) = \mathcal{F}_\varepsilon^* \rho^\varepsilon(y); \qquad A_{\mathrm{cl}}(y) = \mathcal{F}_\varepsilon^* A^\varepsilon(y).$$

Change-of-variable principle implies

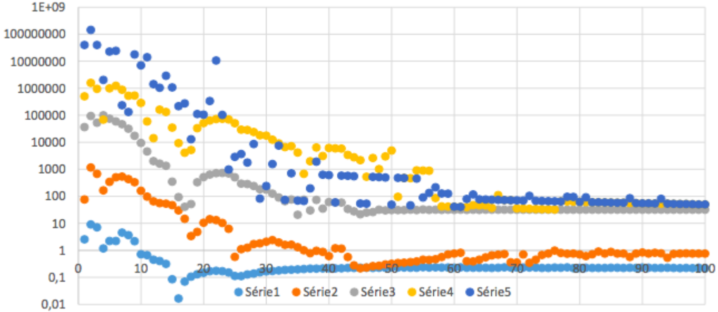
$$u_{\mathrm{cl}}(t,x) = u^\varepsilon(t,x) \qquad \text{for all } x \in \Omega \setminus B_2$$

with $u^\varepsilon(t,x)$ being the solution to

$$\rho^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial t} = \nabla \cdot \Big(A^\varepsilon(x)\nabla u^\varepsilon\Big) + f(x) \qquad \text{in } (0,\ell)\times\Omega$$

SOME NUMERICS

$$\mathcal{G}_\varepsilon(t) := \frac{\left\| u^\varepsilon(t,\cdot) - u_{\rm hom}(t,\cdot) \right\|_{\mathrm{H}^{\frac{1}{2}}(\partial\Omega)}}{\varepsilon^d \left(\|u^{\rm in}\|_{\mathrm{H}^1} + \|f\|_{\mathrm{L}^2(\Omega)} + \|g\|_{\mathrm{L}^2(\partial\Omega)} \right)}$$



Cyan (serie 1) is for ε=10^{-1}

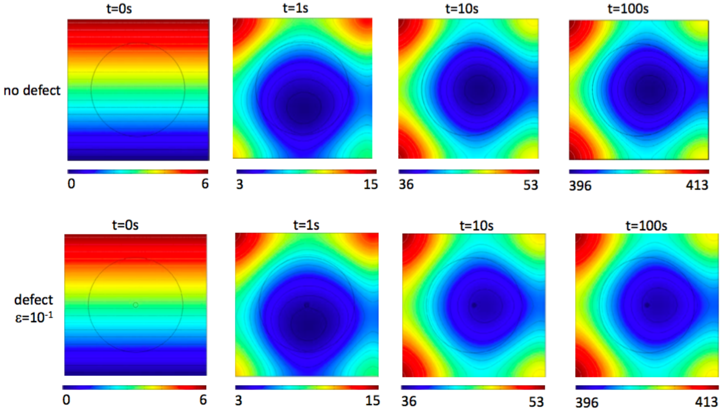
Orange (serie 2) is for ε=10^{-2}

Grey (serie 3) is for ε=10^{-3}

Yellow (serie 4) is for ε=10^{-4}

Dark blue (serie 5) is for ε=10^{-5}

SOME NUMERICS



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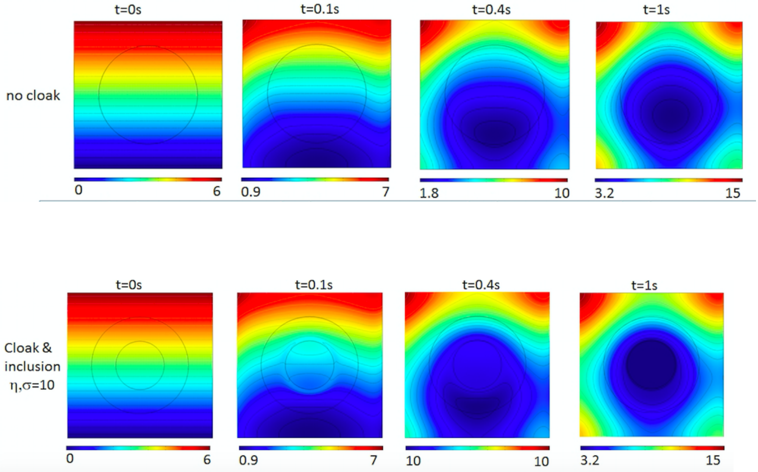

THANK YOU